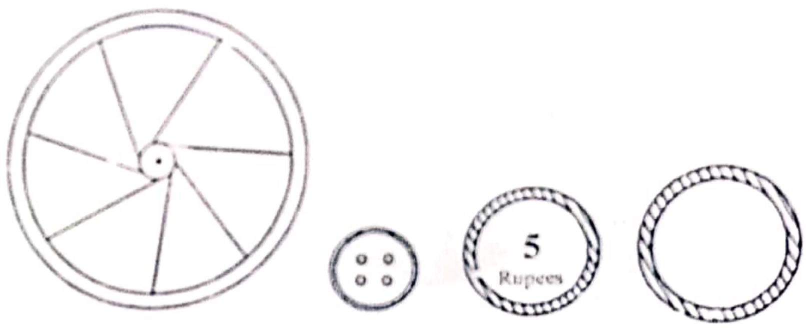
# 10 CIRCLES



## 10.1 INTRODUCTION

We may have come across many objects in daily life, which are round in shape, such as wheels of a vehicle, bangles, coins of 50 p, Rs.1, Rs.5, buttons of shirts and etc. In a clock you might have observed that the second's hand goes round the dial of the clock rapidly and its tip moves in a round path. This path traced by the tip of the second's. Is hand is called a circle. In this chapter, we will study about circles, other related terms and some properties of a circle.




Wheel Button Coin Bangle

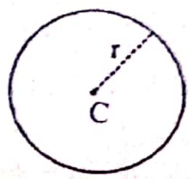
## 10.2 TERMS RELATED TO CIRCLE

### 10.2.1 Circle

**Definition:** The locus of a point which moves in a plane so that its distance from a fixed point in that plane remains constant is called a **circle**.

**In other words**, a **circle** is the set of all those points in a plane each of which is at a constant distance from a fixed point in the plane.

The fixed point is called the **centre** and the constant distance is called the **radius**.

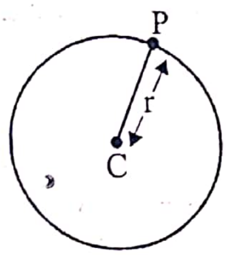


The radius of a circle is always positive. The adjoining figure shows a circle with C as its centre and r as its radius.

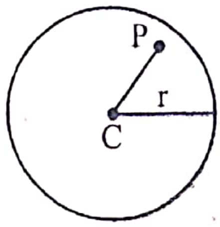
A circle may be denoted by O.

A circle divides the plane on which it lies into three parts. They are

**(i) The circle:** A point P lies on the circle if and only if its distance from the centre of the circle is equal to the radius of the circle. See fig (i)




(i)



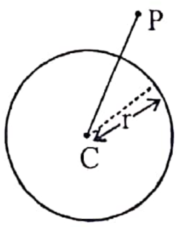

(ii)

**(ii) Interior of a circle:** A point P lies inside a circle if and only if its distance from the centre of the circle is less than the radius of the circle. In the figure(ii), CP < r, therefore, P lies inside the circle.

The set of all points P of the plane such that CP < r form the interior of the circle.

**Circular Region:** The set of all points P of the plane which either lie on the circle or inside the circle form the circular region.

**(iii) Exterior of a circle:** A point P lies outside a circle if and only if its distance from the centre of the circle is greater than the radius of the circle.

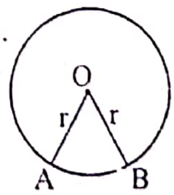



(iii)

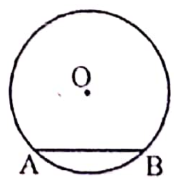
In the above figure, CP > r, therefore P lies outside the circle.

The set of all points P of the plane such that CP > r form the exterior of the circle.

2. **Radius:** A line segment joining the centre and a point on the circle is called its radius. The plural of radius is radii. In the given figure, OA and OB are radii of circle C (o, r).

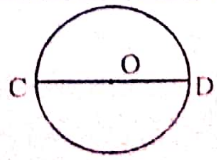



3. **Chord:** Line segment joining any two points on a circle is called a chord of the circle. In figure, AB is a chord.




4. **Diameter:** A chord which passes through the centre of a circle is called a diameter of the circle. In the figure, CD is a diameter. A diameter divides a circle into two equal parts, each part is called a semicircle.

Length of diameter = 2 × radius.

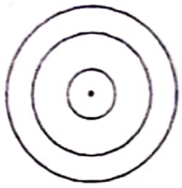



**Note:** (i) A circle has infinite number of chords.

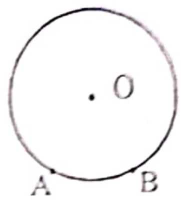
(ii) The longest chord is the diamter of a circle.

(iii) A circle also has infinite number of diameters.

5. **Concentric circles:** Circles having the same centre but with different radii are said to the concentric circles.



6. **Arc of a circle:** Any continuous part of a circle is called an arc of the circle. The arc of a circle is denoted by the symbol '$\frown$'




In the figure $\overset{\frown}{AB}$ denotes the arc AB of the circle with centre O.

Arc of a circle is divided into following categories:

**(i) Circumference:** The whole arc of a circle is called the circumference of the circle. The length of the circumference of a circle is the length of its whole arc.

Circumference of the circle

$= 2\pi r$, r is the radius of circle

$= \pi \times d$, d is the diameter of circle

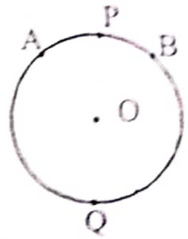
**(ii) Semicircle:** One half of the whole arc of a circle is called a semicircle of the circle. A diameter of a circle divides circle into two equal arcs. Each of these two arcs is called a semicircle.

Circumference of semicircle

$= \pi r + d = \pi r + 2r$

$= (\pi+2)r$ units.

**(iii) Minor arc & major arc:** If the arc is less than a semicircle, then it is called a minor arc.




If the arc is greater than a semicircle then it is called a major arc. If we move from A to B in the clockwise direction, it is minor arc AB. But if we move from A to B in the anti clockwise direction, it is major arc AB.

To avoid this confusion we sometimes put a point between the end points of the arc and thus in figure APB is a minor arc and AQB is a major arc.

$\overset{\frown}{AB}$ will stand for minor arc AB untill and otherwise stated.

7. **Degree measure of an arc:** Let AB be an arc of a circle with centre O. If $\angle AOB = \theta^\circ$ then degree measure of $\overset{\frown}{AB} = \theta^\circ$ or


$$m(\overset{\frown}{AB}) = \theta^\circ \Leftrightarrow \angle AOB = \theta^\circ$$

If $m(\overset{\frown}{AB}) = \theta^\circ$ then $m(\overset{\frown}{BA}) = (360 - \theta)^\circ$

Degree measure of a circle is 360°.

- The degree measure of a circle is 360°.
- An arc whose degree measure is less than 180° is called a **minor arc of the circle.**
- An arc whose degree measure is greater than 180° is called a **major arc of the circle.**
- Relation between the length of an arc and its degree measure

$$\text{Length of an arc} = \text{circumference} \times \frac{\text{Degree measure of an arc}}{360^\circ}$$

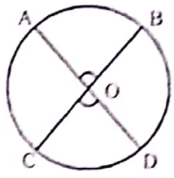
$$= 2\pi r \times \frac{\theta}{360^\circ}$$

**Congruent arcs:** Two arcs $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ of a circle are said to be congruent if they have the same degree measures.

$\overset{\frown}{AB} \cong \overset{\frown}{CD}$

$\Leftrightarrow\ m(\overset{\frown}{AB}) \cong m(\overset{\frown}{CD})$

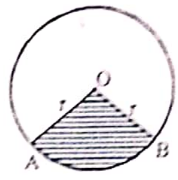
$\Leftrightarrow\ \angle AOB = \angle COD$




8. **Sector of a circle:** The region enclosed by an arc of a circle and its two bounding radii is called a sector of the circle.

In the fig. OABO is the sector of the circle.

The sector corresponding to the minor arc is called minor sector and the sector corresponding to major arc is major sector. Minor sector is shaded.

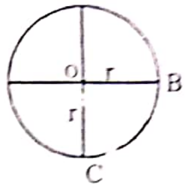



$$\text{Area of sector} = \pi r^2 \times \frac{\text{Degree measure arc}}{360^\circ}$$

9. **Segment of a circle:** The part of the circular region bounded by an arc and a chord including the arc and the chord, is called a segment of the circle.

The segment containing the minor arc is called the minor segment. Thus A PBA is the minor segment of the circle C (o,r) and the segment containing the major arc is called the major segment. Thus A QBA is the major segement of the circle C (o,r).

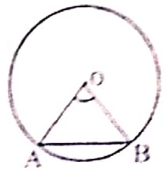

$$\text{Area of segment} = \text{Area of sector} - \frac{1}{2} r^2 \sin\theta$$




10. **Quadrant of a circle:** One fourth of a circle is called a quadrant. In the fig OBCO is a quadrant of the circle C (o,r)

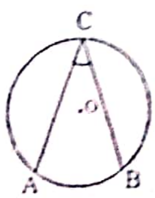
11. **Congruent circles:** Two circles C (o,r) and C (o,s) are said to be congruent only when r = s, i.e. Two circles are congruent if and only if they have equal radii.

12. **Angle subtended by a chord at the centre:** is the angle formed at the centre of the circle when the end points of the chord are joined to the centre. In fig. AB is a chord of the circle. AO and BO are radii of the circle, then $\angle AOB$ is the angle subtended by the chrod at the centre of the circle.




13. **Angle subtended by the chord at a point on the circle:** is the angle formed by joining the end points of the chord to a point on the circle.

In figure $\angle ACB$ is the angle subtended by the chord AB at the point C of the circle.




**Theorem 1:** Equal chords of a circle subtend equal angles at the centre.

**Theorem–2:**

(Converse of Theorem 1):

If the angles subtended by two chords of a circle at the centre are equal, then chords are equal.

**Theorem 3:** If two arcs of a circle are congruent then the corresponding chords are euqal.

**Theorem 4:**

(Converse of Theorem 3):

If two chords of a circle are equal then their corresponding arcs are congruent.

**Theorem 5:** The perpendicular from the centre of a circle to a chord bisects the chord.

**Theorem–6**

(Converse of Theorem 5):

The line joining the centre of a circle to the mid point of a chord is perpendicular to the chord.

**Corollary:** The perpendicular bisectors of two chords of a circle intersect at its centre.

**Theorem 7:** Equal chords of a circle are equidistant from the centre.

**Theorem 8:**

(Converse of Theorem 7):

Chords of a circle that are equidistant from the centre are equal.

## 10.3 CIRCLE THROUGH GIVEN POINTS

(i) Circle through one given point- Many circles can be drawn through one point.

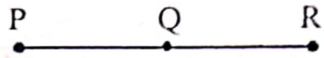
(ii) Circle through two given points- Many circles can be drawn passing through two given points.

(iii) Circle through three given points-

(a) If points are collinear-

Let P, Q, R be three given points

Then no circle can be drawn passing through these three points.




(b) If points are non-collinear

We can draw only one circle passing through three given points.

**Theorem 9:** There is one and only one circle passing through three given non collinear points.

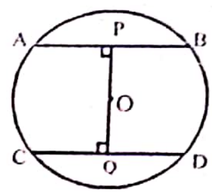
**Corollary:** Of any two chords of a circle, the one which is larger is nearer to the centre.

**OR**

**Converse:** Of any two chords of a circle, the one which is nearer to the centre is longer.

***Illustration 1***

***In given fig. AB and CD are two parallel chords of a circle with centre O and radius 5 cm such that AB = 8 cm and CD = 6 cm. If OP $\perp$ AB and OQ $\perp$ CD, determine the length of PQ.***




***Solution***

We know that the perpendicular from the centre of a circle to a chord bisects the chord.

$$\therefore AP = \frac{1}{2} \times AB = 4 \text{ cm}$$

$$CQ = \frac{1}{2} \times CD = 3 \text{ cm}$$

Join OA and OC

Then OA = O C = 5 cm.

From angled $\Delta OPA$,

$OP^2 = OA^2 - AP^2 = 5^2 - 4^2$

$= 25 - 16 = 9$

OP = 3 cm.

From the rt. angled $\Delta OQC$

$OQ^2 = OC^2 - CQ^2$

$= 5^2 - 3^2$

$= 25 - 9 = 16$

O Q = 4 cm.

Since $OP \perp AB$, $OQ \perp CD$ and $AB \parallel CD$ the points P, O, Q are collinear.

$\therefore PQ = OP + OQ$

$= 3 + 4$ cm

$= 7$ cm.

***Illustration 2***

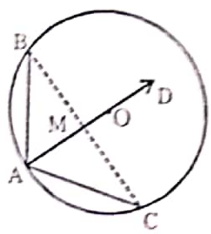
***Two chords AB and AC of a circle are equal. Prove that the centre of the circle lies on the angle bisector of ∠BAC.***

***Solution***

Given AB and AC are two equal chords of a circle C (o, r) and AD is the bisector of $\angle BAC$.

**To prove:** O lies on AD.

**Construction:** Join BC, meeting AD at M.




Proof: In $\Delta BAM$ and $\Delta CAM$, we have

AB = AC (given)

$\angle BAM = \angle CAM$ (given)

AM = AM (common)

$\therefore \Delta BAM \cong \Delta CAM$ by SAS rule.

=> BM = CM and $\angle BMA = \angle CMA$ by cpct.

=> BM = CM and $\angle BMA = \angle CMA = 90^\circ$.

[$\therefore \angle BMA + \angle CMA = 180^\circ$.

& $\angle BMA = \angle CMA$]

$\therefore \angle BMA = \angle CMA = 90^\circ$.

=> AM is the perpendicular bisector of chord BC.

=> AD is the $\perp$ bisector of the chord BC. But, the perpendicular bisector of the chord always passes through the centre of the circle.

$\therefore$ AD passes through the centre O of the circle.

=> O lies on AD.

***Illustration 3***

***In an equilateral triangle prove that the centroid and circumference coincide.***

***Solution***

Given: ABC is an equilateral triangle

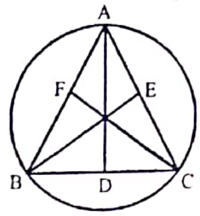
***To prove***: The centroid and circumcentre of $\Delta ABC$ coincide.

***Construction***: Draw medians AD, BE and CF which intersect at O (centroid).

Proof: In $\Delta BEC$ and $\Delta BFC$

$\angle BCE = \angle CBF$ (each = $60^\circ$)

BC = BC (common)




$$CE = BF \left[\frac{1}{2} AC = \frac{1}{2} AB\right]$$

$\therefore\ \Delta BEC \cong \Delta CFB$ (SAS)

$\Rightarrow$ BE = CF (cpct)

Similarly AD = BE

Thus AD = BE = CF

$$\Rightarrow \frac{2}{3} AD = \frac{2}{3} BE = \frac{2}{3} CF$$

$\Rightarrow$ OA = OB = OC [centroid divides the medians in the ratio 2 : 1]

$\Rightarrow$ O is equidistant from the three vertices.

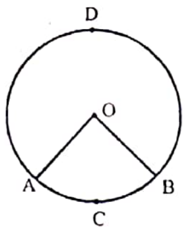
$\Rightarrow$ O is circumcentre of ABC.

Hence, the centroid and circumcentre coincide.

Hence proved.

## 10.4 ANGLE SUBTENDED BY AN ARC AT THE CENTRE

In the fig. C is the centre of the circle and AB is an arc. $\angle AOB$ is called the angle subtended by the arc AB.




If the arc is minor arc i.e. $\overset{\frown}{ACB}$ then the angle subtended by the arc at the centre will be less than $180^\circ$.

If the arc is major i.e. $\overset{\frown}{ADB}$, then the angle subtended by the arc at the cente will be greater than $180^\circ$ and less than $360^\circ$.

*** If two chords of a circle are equal, then their corresponding arcs are congruent and conversely if two arcs of a circle are congruent then their corresponding chords are equal.**

*** Congruent arcs (or equal arcs) of a circle subtend equal angles at the centre.**

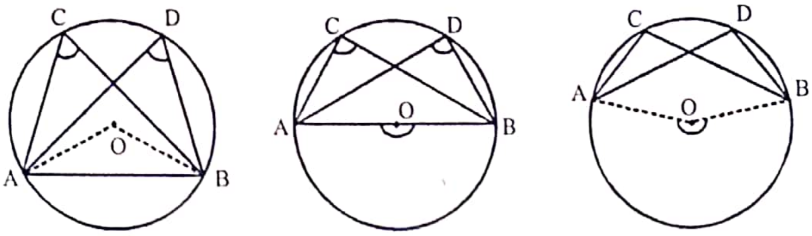
**Conversely:** Equal angles are subtended by congruent arcs (or equal arcs) at the centre.

**Theorem–10:** The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.

**Theorem–11:** The angle in semicircle is a right angle.

**Converse:** The arc of a circle which subtendes a right angle at any point on the remaining part of the circle is a semicircle.

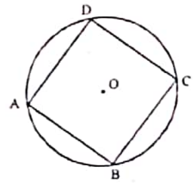
**Theorem–12:** Angle in the same segment of a circle are equal.



(i) (ii) (iii)

**Theorem–13:** If a line segment joining two points subtends equal angles at two other points lying on the same side of the line segment then the four points are concyclic i.e. lie on the same circle

## 10.5 CYCLIC QUADRILATERAL



A Quadilateral whose vertices lie on a circle is called a cyclic quadrilateral. The points which lie on a circle are called concyclic points. Here A, B, C, D are concyclic points. Also, here ABCD is called cyclic quadrilateral.

**Theorem–14:**

The sum of either pair of opposite angles of a cyclic quadrilateral is 180°.

***OR***

The opposite angles of a cyclic quadrilateral are supplementary.

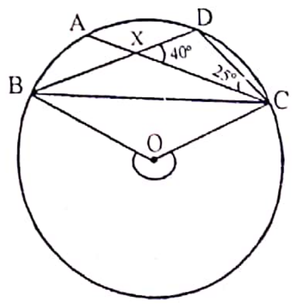
**Theorem–15:**

If the sum of any pair of opposite angles of a quadrilateral is 180°, then the quadrilateral is cyclic.

**Theorem–16:** If one side of a cyclic quadrialteral is produced, then the exterior angle is equal to the interior opposite angle.

***Illustration 4***

***In the figure, let BD and CA intersect at X. If the ∠DXC = 40° and ∠XCD = 25°, then find ∠BAC and reflex ∠BOC.***




***Solution***

In $\Delta XDC$

$\angle XDC = 180° - (40° + 25°)$ [Angle sum property of a $\Delta$]

$\Rightarrow \angle XDC = 115°$

$\angle BAC = \angle XDC = 115°$ [Angle in the same segment]

$\angle BOC = 2 \times 115°$ [Angle at the centre is double to the angle in the alternate segment]

Reflex $\angle BOC = 230°$

***Illustration 5***

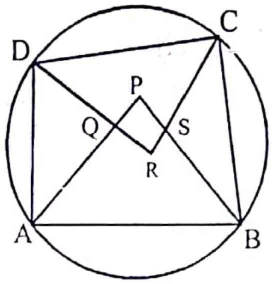
***Prove that the quadrilateral formed by angle bisectors of a cyclic quadrilateral is also cyclic.***

***Solution***

Given: A cyclic quadrilateral ABCD in which AP, BP, CR and DR are the bisectors of $\angle A$, $\angle B$, $\angle C$ and $\angle D$ respectively, forming quad. PQRS.

To prove: PQRS is a cyclic quadrilateral.

Proof: Since the sum of the angles of traingle is 180°, from $\Delta$ ABP and $\Delta$ CDR, we have




$$\left.\begin{aligned}\angle APB + \angle PAB + \angle PBA = 180^\circ \\ \angle CRD + \angle RCD + \angle RDC = 180^\circ\end{aligned}\right\}$$

$$\Rightarrow \begin{cases}\angle APB + \frac{1}{2}\angle A + \frac{1}{2}\angle B = 180^\circ & .....(i) \\ \angle CRD + \frac{1}{2}\angle C + \frac{1}{2}\angle D = 180^\circ & .....(ii)\end{cases}$$

Adding (i) and (ii), we get

$\angle APB + \angle CRD + \frac{1}{2}(\angle A + \angle B + \angle C + \angle D) = 360^\circ$

$\Rightarrow \angle APB + \angle CRD = 180^\circ$ $[\because \angle A + \angle B + \angle C + \angle D = 360^\circ]$

$\angle QPS + \angle QRS = 180^\circ$.

Thus, two opposite angles of quad. PQRS are supplementary.

Hence, PQRS is cyclic.

***Illustration 6***

***A pair of opposite sides of a cyclic quadrilateral are equal. Prove that its diagonals are also equal.***

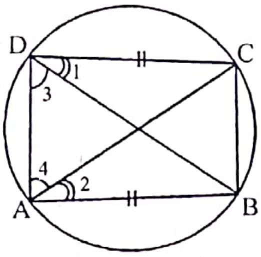
***Solution***

We are given a cyclic quadrilateral ABCD in which AB = DC. (Figure)

We want to prove that AC = BD.

$\because$ $\angle 1 = \angle 2$ .....(Angles in the same segment of a circle)

and $\angle 3 = \angle 4$ .....(Angles subtended by equal chords in a circle)




$\therefore$ $\angle 1 + \angle 3 = \angle 2 + \angle 4$

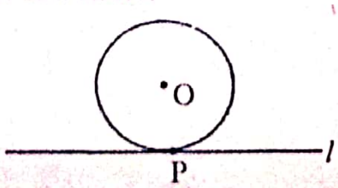
or $\angle ADC = \angle BAD$

But these are the angles subtended by the diagonals AC and BD in the same circle.

$\therefore$ AC = BD.

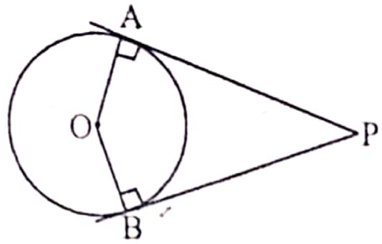
## 10.6 TANGENT OF A CIRCLE

A straight line which touches a circle at only one point is called tangent of a circle. Here line $l$ is tangent of a circle. The point at which the tangent line meets the circle is called the point of contact. In figure P is called the point of contact for tangent of a circle.

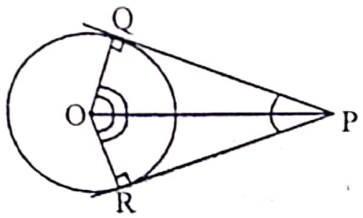


## Tangent

**Theorem:** The tangent at any point of a circle is perpendicular to the radius through the point of contact.
**Theorem:** The lengths of tangents drawn from an external point to a circle are equal. In figure PA = PB.



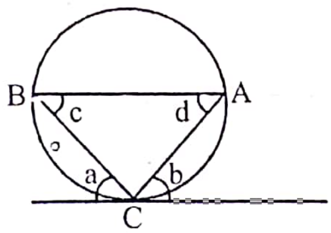

**Theorem:** If two tangents are drawn from an external point then–
* They subtend equal angles at the centre and
* They are equally inclined to the line segment joining the centre to that point




→ PQ = PR → $\angle QOP = \angle ROP$
→ $\angle QPO = \angle RPO$ → $\Delta POQ$ and $\Delta POR$ are congruent.

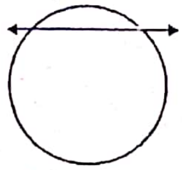
### 10.6.1 Angle between tangent and chord

If a chord is drawn through the point of contact of a tangent to a circle then the angles which this chord makes with the given tangent are equal respectively to the angles formed in the corresponding alternate segments.
i.e. from the figure
$\angle a = \angle d$ & $\angle b = \angle c$



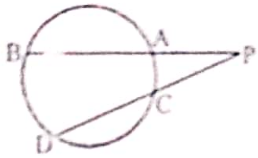

### 10.6.2 Secant of a circle

A line which intersects a circle at two distinct points is called a secant.



Secant of circle

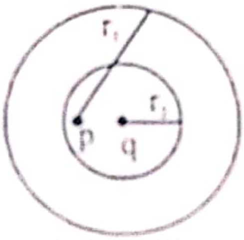
**Theorem:** If PAB is a secant to a circle, intersecting the circle at A and B and PT is a tangent segment then– $PA \times PB = PT^2$
* If PAB and PCD are two secants of the circle, intersecting the circle at A, B, C and D then $PA \times PB = PC \times PD$.




### 10.6.3 Common Tangent

To find the number of common tangents for different cases in circle.

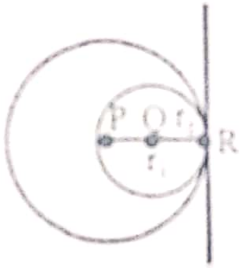
**Case 1: When one circle is inside the other**




If the centres of two circles are P and Q & their radii are $r_1$ and $r_2$ respectively then $PQ < r_1 - r_2$.
No. of common tangent = 0 (zero)

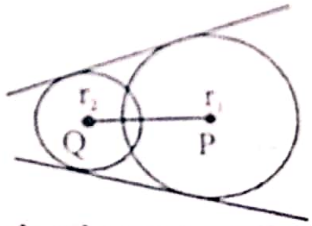
**Case 2: When one circle touch other circle inside it**

If the centres of two circles are P and Q and their radii are $r_1$ and $r_2$ then $PQ = r_1 - r_2$.
No. common tangent = 1 (one)



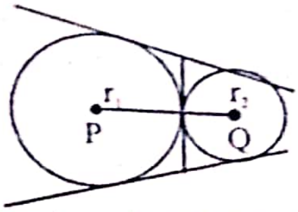

**Case 3: When two circles intersect each other at two points**

If the centres of two circles are P & Q and their radii are $r_1$ and $r_2$ respectively then $r_1 - r_2 < PQ < r_1 + r_2$
No. of common tangents = 2 (two)




**Case 4: When two circles touch each other externally**

If the centres of two circles are P & Q and their radii are $r_1$ and $r_2$ respectively then $PQ = r_1 + r_2$.
No. of common tangents = 3 (three)




**Case 5: When two circles neither touch each other nor intersect each other.**

If the centres of two circles are P and Q and their radii are $r_1$ and $r_2$ respectively then $PQ > r_1 + r_2$.
No. of common tangents = 4 (four)