Vectors and 3D Geometry

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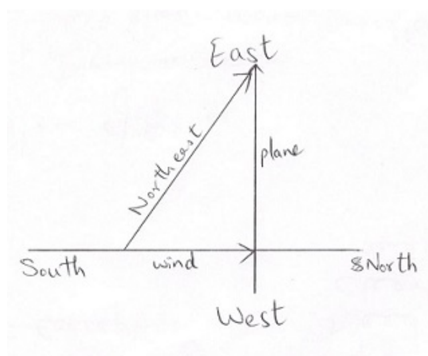
1 Why study Vectors?

Vectors are everywhere. Cars, aeroplanes, ships, tornado, wind, water guns, computer animation, skiing, balloons, soccer, sports, Other than mathematical computations, vectors are useful – in computer graphics; for better understanding of acceleration and velocity in physics; to pilots and sailors in navigation of planes and ships.

A plane moving in east direction with constant velocity and wind current flowing in north direction (very slowly) makes the plane move diagonally or north east. That's resultant vector



In this course, we learn about angle between a line and a plane in Cartesian as well as vector form. We'll look at some simple examples to understand better.

2 Angle between a line and a plane

2.1 Cartesian Form

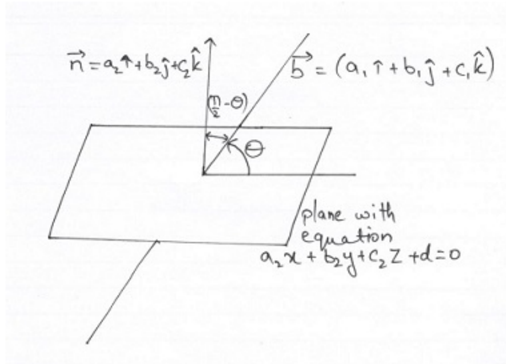
Theorem 2.1. *If θ is the angle between the line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and the plane $a_2x + b_2y + c_2z + d = 0$, then*

$$\sin\theta = \frac{(a_1a_2 + b_1b_2 + c_1c_2)}{(\sqrt{a_1^2 + b_1^2 + c_1^2})(\sqrt{a_2^2 + b_2^2 + c_2^2})}.$$

Proof. The direction ratios of the given line are a_1, b_1, c_1. So, the given line is parallel to $\overrightarrow{b} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$.

Let θ be the angle between the given line and the given plane.

Since given line is parallel to the vector $\overrightarrow{b}$, angle between line and plane is same as angle between vector and plane i.e., θ.



The normal to the given plane is parallel to $\overrightarrow{n} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$.
Then, $(\frac{\pi}{2} - \theta)$ is the angle between the given line and the normal to the given plane.
$\therefore$ $(\frac{\pi}{2} - \theta)$ is the angle between $\overrightarrow{b}$ and $\overrightarrow{n}$
By Definition of scalar product, $\cos(\frac{\pi}{2} - \theta) = \frac{\overrightarrow{b}.\overrightarrow{n}}{|\overrightarrow{b}|.|\overrightarrow{n}|}$.

$$\Rightarrow \sin\theta = \frac{(a_1\hat{i} + b_1\hat{j} + c_1\hat{k}).(a_2\hat{i} + b_2\hat{j} + c_2\hat{k})}{|a_1\hat{i} + b_1\hat{j} + c_1\hat{k}|.|a_2\hat{i} + b_2\hat{j} + c_2\hat{k}|}.$$

$$\Rightarrow \sin\theta = \frac{(a_1a_2 + b_1b_2 + c_1c_2)}{(\sqrt{a_1^2 + b_1^2 + c_1^2})(\sqrt{a_2^2 + b_2^2 + c_2^2})}.$$

□

2.1.1 Important Results

1. *Condition for the given line to be perpendicular to the given plane*

   The line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ is perpendicular to the plane $a_2x + b_2y + c_2z + d = 0$

   $\Leftrightarrow$ The line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ is parallel to the normal to the plane $a_2x+b_2y+c_2z+d = 0$.

   $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

2. *Condition for the given line to be parallel to the given plane*

   The line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ is parallel to the plane $a_2x + b_2y + c_2z + d = 0$

   $\Leftrightarrow$ The line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ is perpendicular to the normal to the plane $a_2x + b_2y + c_2z + d = 0$.

   $\Leftrightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Example 1. Find the angle between the line $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-3}{2}$ and the plane $3x+4y+z+5 = 0$.

Solution 1. The given line is parallel to $\overrightarrow{b} = 3\hat{i} - \hat{j} + 2\hat{k}$.
And, the normal to the given plane is $\overrightarrow{n} = 3\hat{i} + 4\hat{j} + \hat{k}$. Let θ be the angle between the given line and the given plane.
Then, $\sin\theta = \frac{\overrightarrow{b}.\overrightarrow{n}}{|\overrightarrow{b}|.|\overrightarrow{n}|} = \frac{(3\hat{i}-\hat{j}+2\hat{k}).(3\hat{i}+4\hat{j}+\hat{k})}{|3\hat{i}-\hat{j}+2\hat{k}|.|3\hat{i}+4\hat{j}+\hat{k}|} = \frac{\{3\times3+(-1)\times4+2\times1\}}{\sqrt{3^2+(-1)^2+2^2}\sqrt{3^2+4^2+1^2}} = \sqrt{\frac{7}{52}}$.

$\Rightarrow \theta = \sin^{-1}\left(\sqrt{\frac{7}{52}}\right).$

Hence, the angle between the given line and the given plane is $\sin^{-1}\left(\sqrt{\frac{7}{52}}\right)$.

Example 2. Find the equation of the plane passing through the points $(0,0,0)$ and $(3,-1,2)$ and parallel to the line $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$.

Solution 2. Any plane through $(0,0,0)$ is

$$a(x-0) + b(y-0) + c(z-0) = 0 \Leftrightarrow ax + by + cz = 0 \tag{1}$$

If this plane passes through $(3,-1,2)$, we have $3a - b + 2c = 0$

Also, if the plane (2) is parallel to the given line then the normal to this plane is perpendicular to the given line.

$\therefore a \times 1 + b \times (-4) + c \times 7 = 0 \Leftrightarrow a - 4b + 7c = 0.$

Cross multiplying (2) and (2), we have $\frac{a}{(-7+8)} = \frac{b}{(2-21)} = \frac{c}{(-12+1)} = k(say)$

$\Leftrightarrow \frac{a}{1} = \frac{b}{-19} = \frac{c}{(-11)} = k$

$\Leftrightarrow a = k, b = -19k$ and $c = -11k$ in (2), we get the required equation of the plane as

$kx - 19ky - 11kz = 0 \Leftrightarrow x - 19y - 11z = 0.$

2.2 Vector Form

Theorem 2.2. *If θ be the angle between the line $\vec{r} = \vec{a} + \lambda\vec{b}$ and the plane $\vec{r}.\vec{n} = q$ then*

$$\sin\theta = \frac{\vec{b}.\vec{n}}{|\vec{b}|.|\vec{n}|}.$$

Proof. Clearly, the line $\vec{r} = \vec{a} + \lambda\vec{b}$ is parallel to $\vec{b}$, and the plane $\vec{r}.\vec{n} = q$ is normal to $\vec{n}$.

Let θ be the angle between the given line and the given plane. Then, the angle between $\vec{b}$ and $\vec{n}$ is $(\frac{\pi}{2} - \theta)$.

$\therefore \cos(\frac{\pi}{2} - \theta) = \frac{\vec{b}.\vec{n}}{|\vec{b}|.|\vec{n}|}$ and $\cos(\frac{\pi}{2} - \theta) = \sin\theta$.

Hence, the result. □

2.2.1 Important Results

1. Condition for the given line to be perpendicular to the given plane

   The line $\vec{r} = \vec{a} + \lambda\vec{b}$ is perpendicular to the plane $\vec{r}.\vec{n} = q$

   $\Leftrightarrow \vec{b}$ is perpendicular to the plane $\vec{r}.\vec{n} = q$

   $\Leftrightarrow \vec{b}$ is parallel to the normal $\vec{n}$ to the plane

   $\Leftrightarrow \vec{b} = t\vec{n}$, for some scalar t.

2. Condition for a given line to be parallel to a given plane

   The line $\overrightarrow{r} = \overrightarrow{a} + \lambda \overrightarrow{b}$ is parallel to the plane $\overrightarrow{r}.\overrightarrow{n} = q$

   $\Leftrightarrow \overrightarrow{b}$ is parallel to the plane $\overrightarrow{r}.\overrightarrow{n} = q$

   $\Leftrightarrow \overrightarrow{b}$ is perpendicular to the normal $\overrightarrow{n}$ to the plane

   $\Leftrightarrow \overrightarrow{b}.\overrightarrow{n} = 0.$

3. Distance between a line and a plane, parallel to each other

   If the line $\overrightarrow{r} = \overrightarrow{a} + \lambda \overrightarrow{b}$ is parallel to the plane $\overrightarrow{r}.\overrightarrow{n} = q$ then the distance between them is $\frac{|\overrightarrow{a}.\overrightarrow{n} - q|}{|\overrightarrow{n}|}$, which is the same as the same as the distance of a point from the plane.

Example 3. Find the angle between the line $\overrightarrow{r} = (\hat{i} + \hat{j} - 3\hat{k}) + \lambda(2\hat{i} + 2\hat{j} + \hat{k})$ and the plane $\overrightarrow{r} - (6\hat{i} - 3\hat{j} + 2\hat{k}) = 5$.

Solution 3. We know that the angle θ between the line $\overrightarrow{r} = \overrightarrow{a} + \lambda \overrightarrow{b}$ and the plane $\overrightarrow{r}.\overrightarrow{n} = q$ is given by

$$\sin\theta = \frac{\overrightarrow{b}.\overrightarrow{n}}{|\overrightarrow{b}|.|\overrightarrow{n}|}$$

Here, $\overrightarrow{b} = 2\hat{i} + 2\hat{j} + \hat{k}$ and $6\hat{i} - 3\hat{j} + 2\hat{k}$.

Therefore,

$$\sin\theta = \frac{(2\hat{i} + 2\hat{j} + \hat{k}).(6\hat{i} - 3\hat{j} + 2\hat{k})}{|2\hat{i} + 2\hat{j} + \hat{k}|.|6\hat{i} - 3\hat{j} + 2\hat{k}|}.$$

$$= \frac{(2 \times 6 + 2 \times (-3) + 1 \times 2)}{(\sqrt{2^2 + 2^2 + 1^2}).(\sqrt{6^2 + (-3)^2 + 2^2})} = \frac{8}{21}$$

$\Rightarrow \theta = \sin^{-1}\left(\frac{8}{21}\right)$.

Example 4.

Find the value of m for which the line $\overrightarrow{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(2\hat{i} + \hat{j} + 2\hat{k})$ is parallel to the plane $\overrightarrow{r}.(3\hat{i} - 2\hat{j} + m\hat{k}) = 12$.

Solution 4. We know that the line $\overrightarrow{r} = \overrightarrow{a} + \lambda \overrightarrow{b}$ is parallel to the plane

$$\overrightarrow{r}.\overrightarrow{n} = q \Leftrightarrow \overrightarrow{b}.\overrightarrow{n} = 0.$$

Here, $\overrightarrow{b} = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\overrightarrow{n} = 3\hat{i} - 2\hat{j} + m\hat{k}$. Therefore, the given line is parallel to the given plane $\Leftrightarrow \overrightarrow{b}.\overrightarrow{n} = 0$

$\Leftrightarrow (2\hat{i} + \hat{j} + 2\hat{k}).(3\hat{i} - 2\hat{j} + m\hat{k}) = 0$

$\Leftrightarrow 2 \times 3 + 1 \times (-2) + 2 \times m = 0 \Leftrightarrow 2m = -4 \Leftrightarrow m = -2.$

Hence, $m = -2$.