Sampling Theorem and its Importance

- Sampling Theorem:

  "A bandlimited signal can be reconstructed exactly if it is sampled at a rate atleast twice the maximum frequency component in it."

  Figure 1 shows a signal $g(t)$ that is bandlimited.

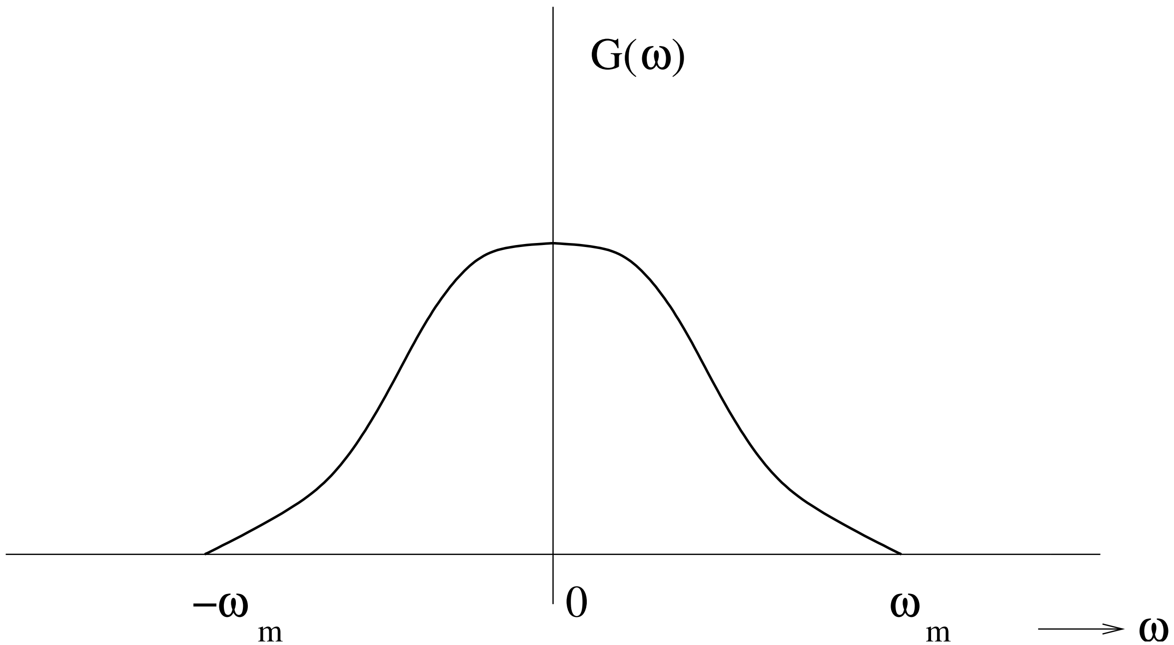

Figure 1: Spectrum of bandlimited signal $g(t)$

- The maximum frequency component of $g(t)$ is f_m. To recover the signal $g(t)$ exactly from its samples it has to be sampled at a rate $f_s \geq 2f_m$.
- The minimum required sampling rate $f_s = 2f_m$ is called

Nyquist rate.

Proof: Let $g(t)$ be a bandlimited signal whose bandwidth is f_m ($\omega_m = 2\pi f_m$).

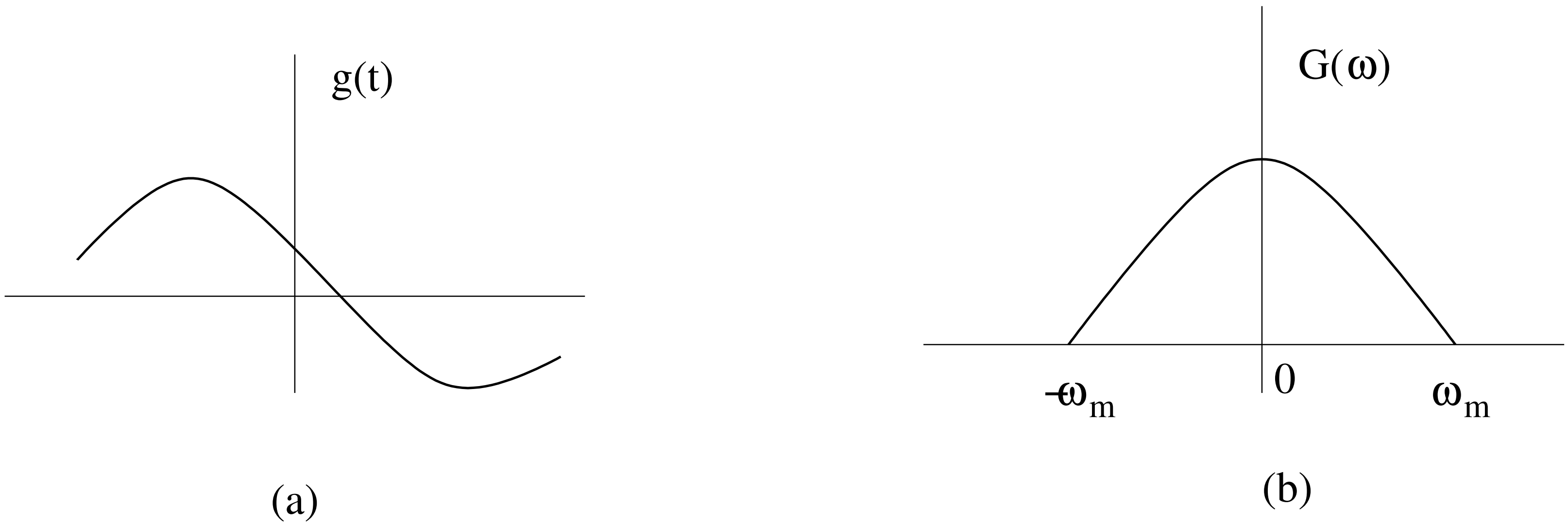

Figure 2: (a) Original signal $g(t)$ (b) Spectrum $G(\omega)$

$\delta_T(t)$ is the sampling signal with $f_s = 1/T > 2f_m$.

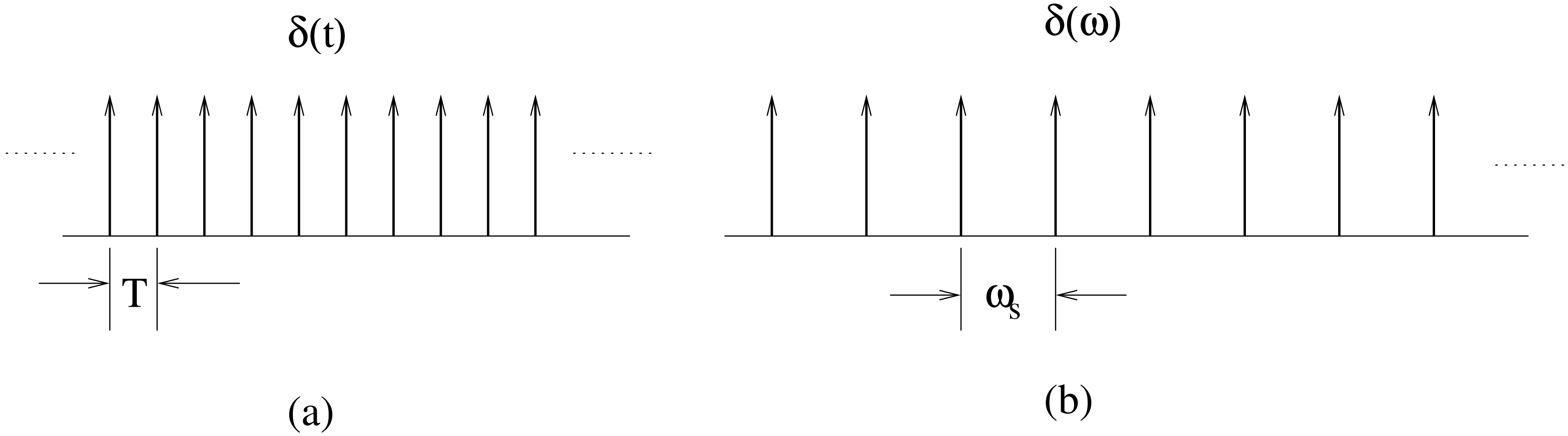

Figure 3: (a) sampling signal $\delta_T(t)$ (b) Spectrum $\delta_T(\omega)$

- Let $g_s(t)$ be the sampled signal. Its Fourier Transform $G_s(\omega)$ is given by

$$
\begin{aligned}
\mathcal{F}(g_s(t)) &= \mathcal{F}\left[g(t)\delta_T(t)\right] \\
&= \mathcal{F}\left[g(t)\sum_{n=-\infty}^{+\infty}\delta(t-nT)\right] \\
&= \frac{1}{2\pi}\left[G(\omega) * \omega_0 \sum_{n=-\infty}^{+\infty}\delta(\omega-n\omega_0)\right] \\
G_s(\omega) &= \frac{1}{T}\sum_{n=-\infty}^{+\infty} G(\omega) * \delta(\omega-n\omega_0) \\
G_s(\omega) &= \mathcal{F}\left[g(t) + 2g(t)\cos(\omega_0 t) + 2g(t)\cos(2\omega_0 t) + \cdots\right] \\
G_s(\omega) &= \frac{1}{T}\sum_{n=-\infty}^{+\infty} G(\omega-n\omega_0)
\end{aligned}
$$

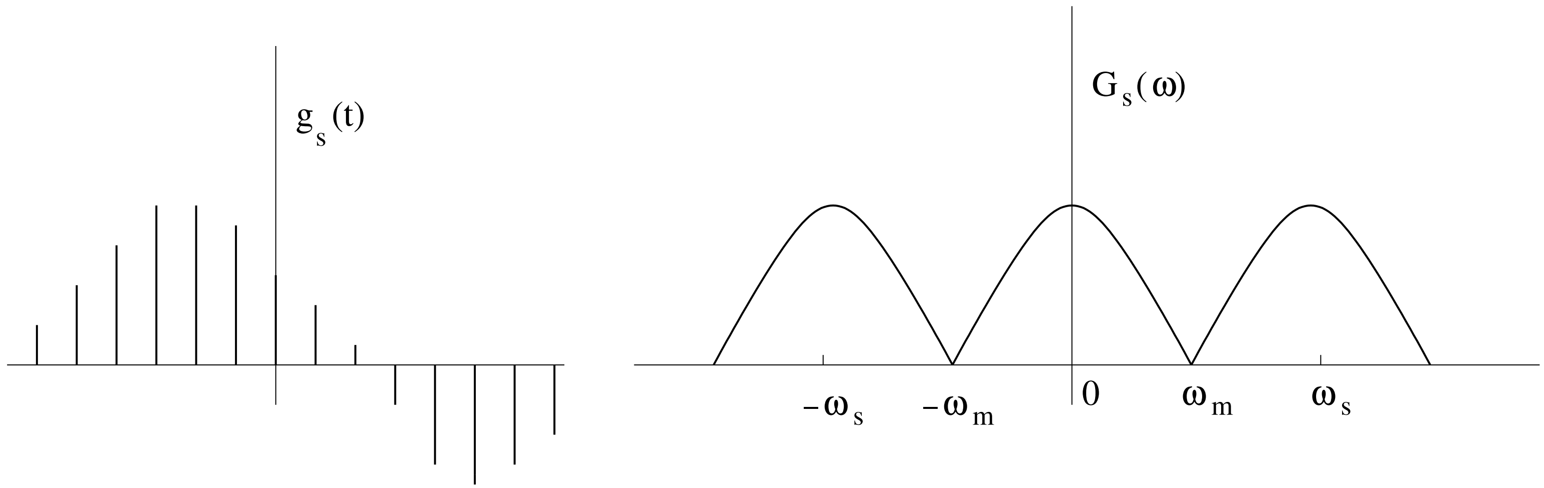

Figure 4: (a) sampled signal $g_s(t)$ (b) Spectrum $G_s(\omega)$

- If $\omega_s = 2\omega_m$, i.e., $T = 1/2f_m$. Therefore, $G_s(\omega)$ is given by

$$G_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} G(\omega - n\omega_m)$$

- To recover the original signal $G(\omega)$:
  1. Filter with a Gate function, $H_{2\omega_m}(\omega)$ of width $2\omega_m$.

2. Scale it by T.

$$G(\omega) = TG_s(\omega)H_{2\omega_m}(\omega).$$

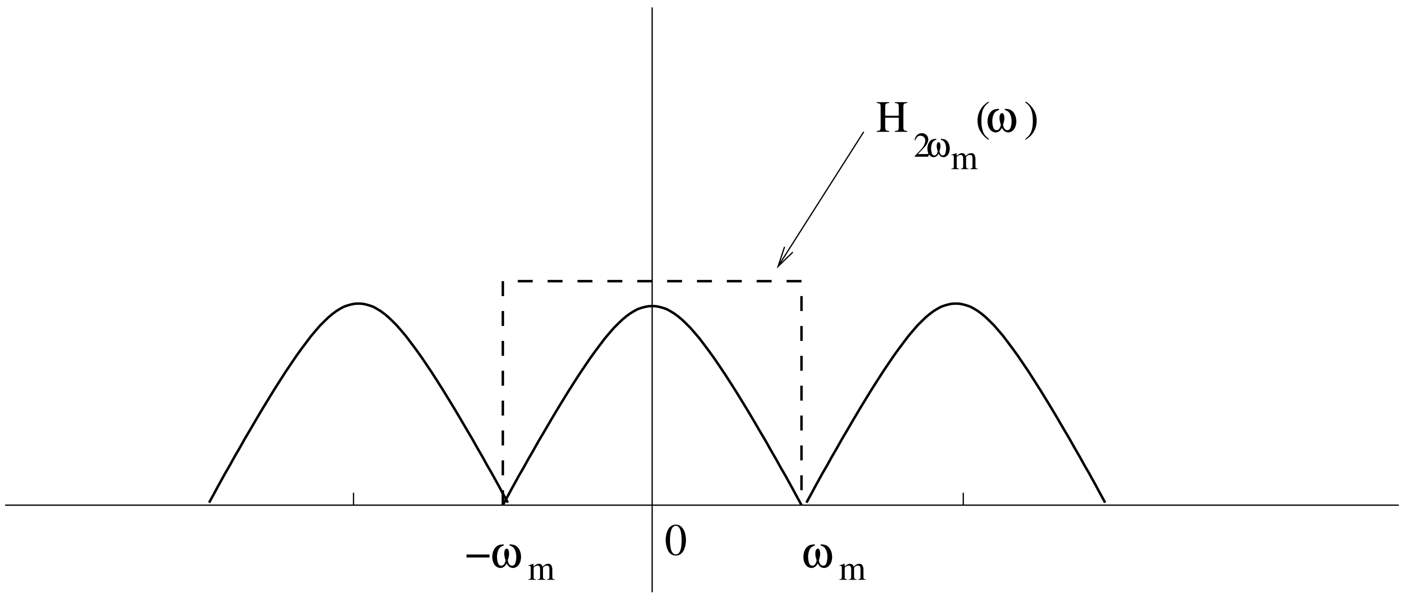

Figure 5: Recovery of signal by filtering with a filter of width $2\omega_m$

- Aliasing
  - Aliasing is a phenomenon where the high frequency components of the sampled signal interfere with each other

because of inadequate sampling $\omega_s < 2\omega_m$.

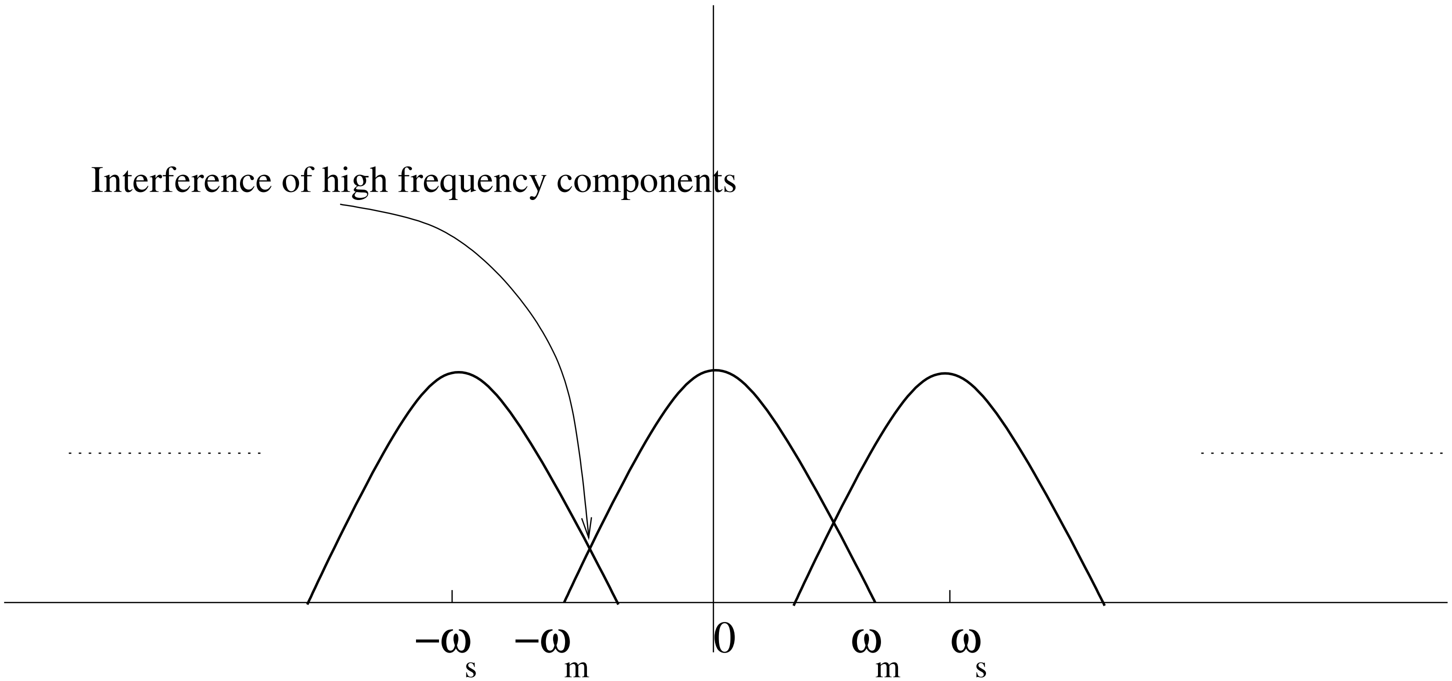

Figure 6: Aliasing due to inadequate sampling

Aliasing leads to distortion in recovered signal. This is the reason why sampling frequency should be atleast twice the bandwidth of the signal.

- Oversampling
  - In practice signal are oversampled, where f_s is significantly higher than Nyquist rate to avoid aliasing.

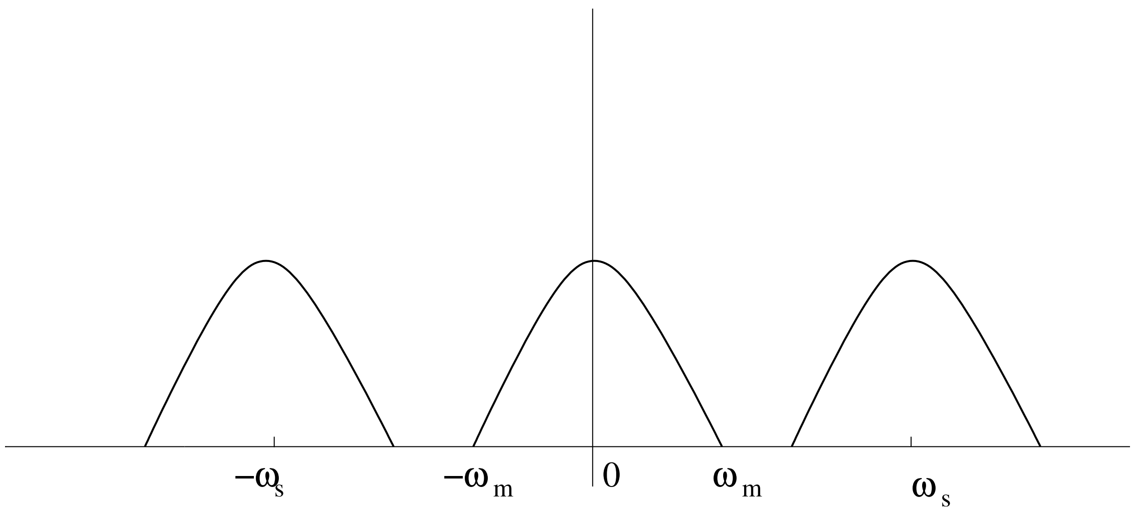

Figure 7: Oversampled signal-avoids aliasing

Problem: Define the frequency domain equivalent of the *Sampling Theorem* and prove it.