Example : 1

(i) Find the 7^{th} term in the expansion of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$

(ii) Find the coefficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$

Solution

(i) In the expansion of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$

The general terms is $T_{r+1} = {}^9C_r \left(\frac{4x}{5}\right)^{9-r} \left(-\frac{5}{2x}\right)^r$

For 7^{th} term (T_7), Put r = 6

$\Rightarrow \quad T_7 = T_{6+1} = {}^9C_6 \left(\frac{4x}{5}\right)^{9-6} \left(-\frac{5}{2x}\right)^6$

$\Rightarrow \quad T_7 = \frac{9 \times 8 \times 7}{3!} \left(\frac{4}{5}\right)^3 x^3 \left(-\frac{5}{2}\right)^6 \frac{1}{x^6}$

$\Rightarrow \quad T_7 = \frac{9 \times 8 \times 7}{3!} 5^3 \frac{1}{x^3}$

$\Rightarrow \quad T_7 = \frac{10500}{x^3}$

(ii) In $\left(ax^2 + \frac{1}{bx}\right)^{11}$ general term is $T_{r+1} = {}^{11}C_r\, a^{11-r}\, b^{-r}\, x^{22-3r}$

for term involving x^7 , 22 – 3r = 7

$\Rightarrow \quad r = 5$

Hence T_{5+1} or the 6^{th} term will contain x^7.

$T_6 = {}^{11}C_5\, (ax^2)^{11-5} \left(\frac{1}{bx}\right)^5 = \frac{11 \times 10 \times 9 \times 8 \times 7}{5!} \frac{a^6}{b^5} x^7 = \frac{462a^6}{b^5} x^7$

Hence the coefficient of x^7 is $\frac{462a^6}{b^5}$

Example : 2

Find the term independent of x in $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$

Solution

$T_{r+1} = {}^9C_r \left(\frac{3x^2}{2}\right)^{9-r} \left(-\frac{1}{3x}\right)^r = {}^9C_r \left(\frac{3x^2}{2}\right)^{9-r} \left(-\frac{1}{3x}\right)^r x^{18-3r}$

for term independent of x, 18 – 3r = 0

$\Rightarrow \quad r = 6$

Hence T_{6+1} or 7^{th} term is independent of x.

$T_7 = {}^9C_6 \left(\frac{3x^2}{2}\right)^{9-6} \left(-\frac{1}{3x}\right)^6 = \frac{9 \times 8 \times 7}{3!} \left(\frac{3}{2}\right)^3 \left(-\frac{1}{3}\right)^6 = \frac{7}{18}$

Example : 3

Find the coefficient of x^{11} in the expansion of $(2x^2 + x - 3)^6$.

Solution

$(2x^2 + x - 3)^6 = (x - 1)^6 (2x + 3)^6$

term containing x^{11} in $(2x^2 + x - 3)^6$

$(x - 1)^6 = {}^6C_0 x^6 - {}^6C_1 x^5 + {}^6C_2 x^4 - {}^6C_3 x^3 + ...$

$(2x + 3)^6 = {}^6C_0 (2x)^6 + {}^6C_1 (2x)^5 3 + {}^6C_2 (2x)^4 3^2 + ...$

term containing x^{11} in the product $(x - 1)^6 (2x + 3)^6 = [C_0 x^6] [{}^6C_1 (2x)^5 3] - [{}^6C_1 x^5] [{}^6C_0 (2x)^6]$

$= 32 (18 x^{11}) - 6 (64) x^{11} = 192 x^{11}$

$\Rightarrow$ the coefficient of x^{11} is 192

Example : 4

Find the relation between r and n so that coefficient of $3r^{th}$ and $(r + 2)^{th}$ terms of $(1 + x)^{2n}$ are equal.

Solution

In $(1 + x)^n$, $T_{r+1} = {}^{2n}C_r x^r$

$T_{3r} = {}^{2n}C_{3r-1} x^{3r-1}$

$T_{r+2} = {}^{2n}C_{r+1} x^{r+1}$

If the coefficient are equal then ${}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$

There are two possibilities

Case – 1

$3r - 1 = r + 1$

$\Rightarrow$ $r = 1$

$\Rightarrow$ $T_{3r} = T_3$ and $T_{r+2} = T_3$

$\Rightarrow$ T_{3r} and T_{r+2} are same terms

Case – 2

${}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$

$\Rightarrow$ ${}^{2n}C_{3r-1} = {}^{2n}C_{2n-(r+1)}$

$\Rightarrow$ $3r - 1 = 2n - (r + 1)$

$\Rightarrow$ $r = n/2$

Example : 5

Find the coefficient of x^3 in the expansion $(1 + x + x^2)^n$.

Solution

$(1 + x + x^2)^n = [1 + x (1 + x)]^n = {}^nC_0 + {}^nC_1 x (1 + x) + {}^nC_2 x^2 (1 + x)^2 + ...$

Coefficient of $x^3 = {}^nC_2$ [coeff of x in $(1 + x)^2$] + ${}^nC_3$ [coeff of x^0 in $(1 + x)^3$]

$$= {}^nC_2 (2) + {}^nC_3 (1) = \frac{2n(n-1)}{2} + \frac{n(n-1)(n-2)}{3!} = \frac{n(n-1)}{6} [6 + n - 2] = \frac{n(n-1)(n+4)}{6}$$

Example : 6

If ${}^nC_r$ is denoted as C_r, show that

(a) $$(C_0 + C_1) (C_1 + C_2) (C_2 + C_3) .... (C_{n-1} + C_n) = \frac{C_0C_1.....C_n(n+1)^n}{n!}$$

(b) $$\frac{C_1}{C_0} + 2\frac{C_2}{C_1} + 3\frac{C_3}{C_2} + ... + n\frac{C_n}{C_{n-1}} = \frac{n(n+1)}{2}$$

Solution

(a) LHS = $(C_0 + C_1) (C_1 + C_2) (C_2 + C_3) .... (C_{n-1} + C_n)$

Multiply and Divide by $C_0 C_1 C_2 ..... C_n = C_0 C_1 C_2 ....C_n \left(1+\frac{C_1}{C_0}\right)\left(1+\frac{C_2}{C_1}\right) ...... \left(1+\frac{C_n}{C_{n-1}}\right)$

using $\frac{C_r}{C_{r-1}} = \frac{n-r+1}{r} = C_0 C_1 C_2 C_3 ...... C_n \left(1+\frac{n-1+1}{1}\right) \times \left(1+\frac{n-2+1}{2}\right) + ... + \left(1+\frac{n-n+1}{n}\right)$

$= C_0 C_1 C_2 ...... C_n \left(\frac{n+1}{1}\right)\left(\frac{n+1}{2}\right) + ... + \left(\frac{n+1}{n}\right) = C_0C_1C_2C_3 ...... C_n \frac{(n+1)^n}{n!}$ = RHS

(b) LHS = $\frac{C_1}{C_0} + 2\frac{C_2}{C_1} + 3\frac{C_3}{C_2} + \ldots\ldots\ldots + n\frac{C_n}{C_{n-1}}$

using $\frac{C_r}{C_{r-1}} = \frac{n-r+1}{r} = \left(\frac{n-1+1}{1}\right) + 2\left(\frac{n-2+1}{2}\right) + \ldots\ldots. + n\frac{(n-n+1)}{n}$

= n + (n – 1) + (n – 2) + ... + 1

= Sum of first n natural numbers = $\frac{n(n+1)}{2}$ = RHS

Example : 7

Show that

(a) $C_0^2 + C_1^2 + C_2^2 + C_3^2 + \ldots\ldots\ldots + C_n^2 = \frac{(2n)!}{n!\,n!}$

(b) $C_0\,C_1 + C_1C_2 + C_2C_3 + \ldots\ldots C_{n-1}\,C_n = \frac{(2n)!}{(n-1)!\,(n+1)!}$

Solution

Consider the identities $(1 + x)^n = C_0 + C_1 x + C_2x^2 + \ldots\ldots\ldots\ldots + C_nx^n\ (1 + x)^n$

$= C_0x^n + C_1\,X^{n-1} + C_2\,x^{n-2} + \ldots\ldots. + C_n$

multiplying these we get another identity

$(1 + x)^n\,(x + 1)^n = (C_0 + C_1x = (C_0 + C_1x + C_2x^2 + \ldots\ldots\ldots + C_nx^n) = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots\ldots. + C_n)$

(a) Compare coefficients of x^n on both sides

In LHS, coeff. of x^n = coeff of x^n in $(1 + x)^{2n} = {}^{2n}C_0$

In RHS, terms containing x^n are $C_0^{\,2}\,x^n + C_1^{\,2}\,x^n + C_2^{\,2}\,x^n + \ldots\ldots. + C_n^{\,2}\,x^n$

$\Rightarrow$ Coeff. of x^n on RHS = $C_0^{\,2} + C_1^{\,2} + C_2^{\,2} + \ldots\ldots. + C_n^{\,2}$

equating the coefficients $C_0^{\,2} + C_1^{\,2} + C_2^{\,2} + \ldots\ldots\ldots\ C_n^{\,2} = {}^{2n}C_n$

$C_0^2 + C_1^2 + C_2^2 + \ldots\ldots\ldots + C_n^2 = \frac{(2n)!}{n!\,n!}$

(b) Compare the coefficients of x^{n-1} on both sides

In LHS, coeff. of $x^{n-1} = {}^{2n}C_{n-1}$

In RHS, term containing x^{n-1} is $C_0C_1\,x^{n-1} + C_1C_2\,x^{n-1} + \ldots\ldots\ldots$

Hence coeff. of x^{n-1} in RHS = $C_0C_1 + C_1C_2 + C_2\,C_3 + \ldots\ldots.$

equation of the coefficients,

$C_0C_1 + C_1C_2 + \ldots\ldots. = C_{n-1}\,C_n = {}^{2n}C_{n-1} = \frac{(2n)!}{(n-1)!(n+1)!}$

Example : 8

Let $S_n = 1 + q + q^2 + q^3 + \ldots\ldots\ldots + q^n$

$S_n = 1 + \left(\frac{q+1}{2}\right)^2 + \left(\frac{q+1}{2}\right)^3 + \ldots\ldots. + \left(\frac{q+1}{2}\right)^n$

prove that ${}^{n+1}C_1 + {}^{n+2}C_2\,S_1 + {}^{n+1}C_3\,S_2 + \ldots\ldots\ldots + {}^{n+1}C_{n+1}\,S_n = 2^n\,S_n$

Solution

S_n = sum of (n + 1) terms of a G.P. = $\frac{1-q^{n+1}}{1-q}$

$$S_n = \frac{1-\left(\frac{q+1}{2}\right)^{n+1}}{1-\left(\frac{q+1}{2}\right)} = \frac{2^{n+1}-(q+1)^{n+1}}{(1-q)\,2^n}$$

Consider the LHS = $^{n+1}C_1 + {}^{n+1}C_2\left(\frac{1-q^2}{1-q}\right) + {}^{n+1}C_3\left(\frac{1-q^3}{1-q}\right) + \ldots\ldots + {}^{n+1}C_{n+1}\left(\frac{1-q^{n+1}}{1-q}\right)$

$$= \frac{1}{1-q}\,[{}^{n+1}C_1\,(1-q) + {}^{n+1}C_2\,(1-q^2) + \ldots\ldots\ldots + {}^{n+1}C_{n+1}\,(1-q^{n+1})]$$

$$= \frac{1}{1-q}\,[({}^{n+1}C_1 + {}^{n+1}C_2 + \ldots\ldots\ {}^{n+1}C_{n+1}) - ({}^{n+1}C_1 q + {}^{n+1}C_2 q^2 + \ldots\ldots + {}^{n+1}C_{n-1} q^{n+1})]$$

$$= \frac{1}{1-q}\,[(2^{n+1} - 1 - ((1+q)^{n+1} - 1)] = \frac{2^{n+1} - (1+q)^{n+1}}{1-q} = 2^n\, S_n = \text{RHS}$$

Example : 9

Show that $3^{2n+2} - 8n - 9$ is divisible by 64 if $n \in N$.

Solution

$3^{2n+2} - 8n - 9 = (1+8)^{n+1} - 8n - 9 = [1 + (n+1)\,8 + ({}^{n+1}C_2\, 8^2 + \ldots\ldots] - 8n - 9$

$= {}^{n+1}C_2\, 8^2 + {}^{n+1}C_3\, 8^3 + {}^{n+1}C^4\, 8^4 + \ldots\ldots$

$= 64[{}^{n+1}C_2 + {}^{n+1}C_3\, 8 + {}^{n+1}C_4 8^2 + \ldots\ldots]$

which is clearly divisible by 64

Example : 10

Find numerically greatest term in the expansion of $(2 + 3x)^9$, when $x = 3/2$

Solution

$$(2+3x)^9 = 2^9\left(1+\frac{3x}{2}\right)^9 = 2^9\left(1+\frac{9}{4}\right)^9$$

Let us calculate $m = \frac{x(n+1)}{x+1} = \frac{(9/4)(9+1)}{(9/4)+1} = \frac{90}{13} = 6\,\frac{12}{13}$

as m is not an integer, the greatest term in the expansion is $T_{[m]+1} = T_7$

$\Rightarrow$ the greatest term $= 2^0\,(T_7) = 2^9\ {}^9C_6\left(\frac{9}{4}\right)^6 = \frac{7\times 3^{13}}{2}$

Example : 11

If a_1, a_2, a_3 and a_4 are the coefficients of any four consecutive terms in the expansion of $(1+x)^n$, prove that

$$\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{2a_2}{a_2+a_3}$$

Solution

Let a_1 = coefficient of $T_{r+1} = {}^nC_r$ $\Rightarrow$ $a_2 = {}^nC_{r+1} = {}^nC_r$

$\Rightarrow$ $a_2 = {}^nC_{r+1}$, $a_3 = {}^nC_{r+2}$, $a_4 = {}^nC_{r+3}$

$\Rightarrow$ $\frac{a_1}{a_1+a_2} = \frac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}} = \frac{{}^nC_r}{{}^{n+1}C_{r+1}} = \frac{r+1}{n+1}$ and $\frac{a_3}{a_3+a_4} = \frac{{}^nC_{r+2}}{{}^nC_{r+2} + {}^nC_{r+3}} = \frac{{}^nC_{r+2}}{{}^{n+1}C_{r+3}} = \frac{r+3}{n+1}$

LHS $= \frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} + \frac{r+1}{n+1} = \frac{r+3}{n+1} = \frac{2(r+2)}{n+1}$

RHS $= \frac{2a_2}{a_2+a_3} = \frac{2\ {}^nC_{r+1}}{{}^nC_{r+1} + {}^nC_{r+2}} = \frac{2^n C_{r+1}}{{}^{n+1}C_{r+2}} = \frac{2(r+2)}{n+1}$

Hence R.H.S. = L.H.S

Example : 12

Prove that following ($C_r = {}^nC_r$)

(a) $C_1 + 2C_2 + 3C_3 + \ldots\ldots\ldots\; n\,C_n = n\,2^{n-1}$

(b) $C_1 - 2C_2 + 3C_3 + - \ldots\ldots\ldots = 0$

(c) $C_0 + 2C_1 + 3C_2 + \ldots\ldots + (n+1)\,C_n = (n+2)\,2^{n-1}$

Solution

Consider the identity : $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$

Differentiating w.r.t. x, we get another identity $n(1 + x)^{n-1}$

$$= C_1 + 2C_2\,x + 3\,C_3x^2 + \ldots\ldots\ldots + nC_n\,x^{n-1} \quad \ldots\ldots\ldots\ldots\ldots(i)$$

(a) substituting x = 1 in (i), we get :

$C_1 + 2\,C_2 + 3C_3 + \ldots\ldots\ldots + nC_n = n\,2^{n-1}$ ....(ii)

(b) Substituting x = –1 in (i), we get

$C_1 - 2C_2 + 3C_3 - 4C_4 + \ldots\ldots\ldots + nC_n\,(-1)^{n-1} = 0$

(c) LHS = $C_0 + 2C_1 + 3C_2 + \ldots.. + (n+1)C_n = (C_0 + C_1 + C_2 + \ldots\ldots) + (C_1 + 2C_2 + 3C_3 + \ldots\ldots + nC_n)$

$= 2^n + n\,2^{n-1} = (n+1)\,2^{n-1}$ [using (ii)]

This can also be proved by multiplying (i) by x and then differentiating w.r.t. x and then substituting x = 1.

Example : 13

Prove that

(a) $$\frac{C_0}{1} + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \ldots\ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$$

(b) $$3C_0 + 3^2\,\frac{C_1}{2} + 3^3\,\frac{C_2}{3} + 3^4\,\frac{C_3}{4} + \ldots\ldots + 3^{n+1}\,\frac{C_n}{n+1} = \frac{4^{n+1}-1}{n+1}$$

Solution

Consider the identity :

$(1 + x)^n = C_0 + C_1x + C_2\,x^2 + \ldots\ldots\ldots + C_nx^n$ ....(i)

(a) Integrating both sides of (i) within limits 0 to 1, we get

$$\int_0^1 (1+x)^n\,dx = \int_0^1 (C_0 + C_1x + \ldots\ldots C_nx^n)\,dx$$

$$\left.\frac{(1+x)^{n+1}}{n+1}\right]_0^1 = \left. C_0\,x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \ldots\ldots\ldots + \frac{C_nx^{n+1}}{n+1}\right]_0^1$$

$$\frac{2^{n+1}-1}{n+1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots\ldots\ldots + \frac{C_n}{n+1}$$

(b) Integrating both sides of (i) within limits – 1 to + 1, we get:

$$\int_{-1}^1 (1+x)^n\,dx = \int_{-1}^1 (C_0 + C_1x + \ldots\ldots + C_nx^n)\,dx$$

$$\left.\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^1 = \left. C_0\,x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \ldots\ldots\ldots\ldots + \frac{C_nx^{n+1}}{n+1}\right]_{-1}^{+1}$$

$$\frac{2^{n+1}-0}{n+1} = \left(C_0 + \frac{C_1}{2} - \frac{C_2}{3} + \ldots.. + \frac{C_n}{n+1}\right) - \left(-C_0 + \frac{C_1}{2} - \frac{C_2}{3} + \ldots\ldots\right)$$

$$\Rightarrow \quad \frac{2^{n+1}}{n+1} = 2C_0 + \frac{2C_2}{3} + \frac{2C_4}{5} + \ldots\ldots\ldots\ldots$$

$$\Rightarrow \quad \frac{2^n}{n+1} = C_0 + \frac{C_2}{3} + \frac{C_4}{5} + \ldots\ldots\ldots$$

Hence proved

Note : If the sum contains $C_0, C_1, C_2, C_3 ....C_n$ (i.e. all +ve coefficients), then integrate between limits 0 to 1. If the sum contains alternate plus and minus (+ – signs), then integrate between limits – 1 to 0. If the sum contains even coefficients (C_0, C_2, C_4 .....), then integrate between – 1 and +1.

Example : 15

$1^2 C_1 + 2^2 C_2 + 3^2 C_3 + ... + n^2 C_n = n(n + 1) 2^{n-2}$

Solution

Consider the identity :

$(1 + x)^n = C_0 + C_1x + C_2 x^2 + ... + C_n x^n$

Differentiating both sides w.r.t. x;

$n(1 + x)^{n-1} = C_1 + 2C_2 x + ... + nC_n x^{n-1}$

multiplying both sides by x.

$n x (1 + x)^{n-1} = C_1 x + 2 C_2 x^2 + ... + n C_n x^n$

differentiate again w.r.t. x;

$nx (n - 1) (1 + x)^{n-2} + n (1 + x)^{n-1} = C_1 + 2^2 C_2 x + ... + n^2 C_n x$

substitute x = 1 in this identity

$n(n - 1) 2^{n-2} + n 2^{n-1} = C_1 + 2^2 C_2 + 3^2 C_3 + ... + n^2 C_n$

$\Rightarrow$ $n 2^{n-2} (n + 1) = C_1 + 2^2 C_2 + ... + n^2 C_n$

Hence proved

Example : 16

If ${}^{2n}C_r = C_r$, prove that : $C_1^2 - 2C_2^2 + 3C_3^2 - + ... - 2n C_{2n}^2 = (-1)^{n-1} nC_n$.

Solution

Consider

$(1 - x)^{2n} = C_0 - C_1 x + C_2x^2 - + ... + C_{2n} x^{2n}$ ....(i)

and

$(x + 1)^{2n} = C_0 x^{2n} + C_1 x^{2n-1} + C_2 x^{2n-2} + ... + C_{2n-1} x + C_{2n}$ ....(ii)

We will differentiate (i) w.r.t. x and then multiply with (ii)

Differentiating (i), we get :

$- 2n (1 - x)^{2n-1} = - C_1 + 2 C_2 x - 3 C_3x^2 + ... + 2n C_{2n} x^{2n-1}$

$\Rightarrow$ $2n (1 - x)^{2n-1} = C_1 - 2 C_2 x + 3 C_3 x^2 - + ... - 2n C^{2n} x^{2n-1}$

new multiplying with (ii)

$2n (1 - x)^{2n-1} (x + 1)^{2n} = (C_0 x^{2n} + C_1 x^{2n-1} + ... + C_{2n}) \times (C_1 - 2 C_2 x + 3 C_3 x^2 - + ... - 2n C_{2n} x^{2n-1})$

Comparing the coefficients of x^{2n-1} on both sides; coefficient in RHS

$= C_1^2 - 2 C_2^2 + 3 C_3^2 - + ... - 2n C_{2n}^2$

Required coeff. in LHS $=$ coeff. of x^{2n-1} in $2n (1 - x)^{2n-1} (1 + x)^{2n-1} (1 + x)$

$=$ coeff. of x^{2n-1} in $2n (1 - x^2)^{2n-1}$ + coeff. of x^{2n-1} in $2nx (1 - x^2)^{2n-1}$

$=$ coeff. of x^{2n-1} in $2n (1 - x^2)^{2n-1}$ + coeff. of x^{2n-2} in $2n (1 - x^2)^{2n-1}$

Now the expansion of $(1 - x^2)^{2n-1}$ contains only even powers of x.

Hence coefficients in LHS :

$= 0 + 2n$ [coeff. of x^{2n-2} in $(1 - x^2)^{2n-1}$]

$= 2n [{}^{2n-1}C_{n-1} (-1)^{n-1}]$

$$= 2n \left(\frac{(2n-1)!}{(n-1)!\,n!}(-1)^{n-1}\right)$$

$= n\ {}^{2n}C_n (-1)^{n-1}$

Now equating the coefficients in RHS and LHS, we get $C_1^2 - 2C_2^2 + 3C_3^2 - + ... 2n C_{2n}^2 = (-1)^{n-1} n\,{}^{2n}C_n$

Example : 17

Find the sum of series :

$$\sum_{r=0}^{n} (-1)^r\ {}^nC_r \left(\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + ...m \text{ terms}\right)$$

Solution

$$\sum_{r=0}^{n} (-1)^r \; {}^nC_r \left(\frac{1}{2}\right)^r = \sum_{r=0}^{n} {}^nC_r \left(-\frac{1}{2}\right)^r = \text{expansion of } \left(1-\frac{1}{2}\right)^n$$

$$\sum_{r=0}^{n} (-1)^r \; {}^nC_r \left(\frac{3^r}{2^{2r}}\right) = \sum_{r=0}^{n} {}^nC_r \left(-\frac{3}{4}\right)^r = \text{expansion of } \left(1-\frac{3}{4}\right)^n$$

$$\sum_{r=0}^{n} (-1)^r \; {}^nC_r \left(\frac{7^r}{2^{3r}}\right) = \sum_{r=0}^{n} {}^nC_r \left(-\frac{7}{8}\right)^r = \text{expansion of } \left(1-\frac{7}{8}\right)^n \text{ and so on ....}$$

Now adding all these we get ;

$$\text{Required Sum } = \sum_{r=0}^{n} (-1)^r \; {}^nC_r \left(\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+\text{........m terms}\right)$$

$$= \left(1-\frac{1}{2}\right)^n + \left(1-\frac{3}{4}\right)^n + \left(1-\frac{7}{8}\right)^n + \text{..... m terms}$$

$$= \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{8^n} + \text{....... m terms of GP}$$

$$= \frac{\frac{1}{2^n}\left(1-\frac{1}{2^{mn}}\right)}{1-\frac{1}{2^n}} = \frac{2^{mn}-1}{(2^n-1)2^{mn}}$$

Example : 18

If $(1 + x)^n = C_0 + C_1 x + C_2x^2 + \text{...........} + C_n x^n$ then show that the sum of the products of the C_is taken two at a time represented by : $\sum\sum_{0 \le i < j \le n} C_iC_j$ is equal to $2^{2n-1} - \frac{(2n)!}{2n!\, n!}$

Solution

The square of the sum of n terms is given by :

$$(C_0 + C_1 + C_2 + \text{.......} C_n)^2 = (C_0^{\,2} + C_1^{\,2} + C_2^{\,2} + \text{..........} + C_n^{\,2}) + 2 \sum\sum_{0 \le i < j \le n} C_iC_j$$

substituting $C_0 + C_1 + C_2 + \text{.........} + C_n = 2^n$

and $C_0^{\,2} + C_1^{\,2} + C_2^{\,2} + \text{........} + C_n^{\,2} = {}^{2n}C_n$

we get $(2^n)^2 = {}^{2n}C_n + 2 \sum\sum_{0 \le i < j \le n} C_iC_j \quad \Rightarrow \quad \sum\sum_{0 \le i < j \le n} C_iC_j = \frac{2^{2n} - {}^{2n}C_n}{2} = 2^{2n-1} - \frac{(2n)!}{2n!\, n!}$

Example : 19

If $(2 + \sqrt{3})^n = I + f$ where I and n are positive integers and $0 < f < 1$, show that I is an odd integer and $(1 - f)(I + f) = 1$.

Solution

$(2 + \sqrt{3})^n = f'$ where $0 < f' < 1$ because $2 - \sqrt{3}$ is between 0 and 1

Adding the expansions of $(2 + \sqrt{3})^n$ and $(2 - \sqrt{3})^n$, we get ; $1 + f + f' = (2 + \sqrt{3})^n + (2 - \sqrt{3})^n$

$= 2 [C_0 2^n + C_2 2^{n-2} (\sqrt{3})^2 + \text{.....}] =$ even integer ....(i)

$\Rightarrow$ $f + f'$ is also an integer

now $0 < f < 1$ and $0 < f' < 1$ $\Rightarrow$ $0 < f + f' < 2$

The only integer between 0 and 2 is 1

Hence $f + f' = 1$ ....(ii)

Consider (i)

$1 + f + f' =$ even integer

$\Rightarrow$ $I + 1 =$ even integer [using (ii)]

$\Rightarrow$ I = odd integer

also $(I + f)(I - f) = (I + f)(f') = (2 + \sqrt{3})^n (2 - \sqrt{3})^n = 1$

Example : 20

If $(6\sqrt{6} + 14)^{2n+1} = P$, prove that the integral part of P is an even integer and $P f = 20^{2n+1}$ where f is the fractional part of P.

Solution

Let I be the integral part of P

$\Rightarrow$ $P = I + f = (6\sqrt{6} + 14)^{2n+1}$

Let $f' = (6\sqrt{6} - 14)$ lies between 0 and 1, $0 < f' < 1$

subtracting f′ from I + f to eliminate the irrational terms in RHS of (i)

$I + f - f' = (6\sqrt{6} + 14)^{2n+1} - (6\sqrt{6} - 14)^{2n+1} = 2[{}^{2n+1}C_1 (6\sqrt{6})^{2n} (14) + {}^{2n+1}C_3 (6\sqrt{6})^{2n-2} (14)^3 + ...]$

= even integer ....(ii)

$\Rightarrow$ $f - f'$ is an integer

now $0 < f < 1$ and $0 < f' < 1$

$\Rightarrow$ $0 < f < 1$ and $-1 < -f' < 0$

adding these two, we get; $-1 < f - f' < 1$

$\Rightarrow$ $f - f' = 0$ ....(iii)

Consider (ii)

$1 + f - f'$ = even integer

$\Rightarrow$ I + 0 = even integer [using (iii)]

$\Rightarrow$ integral part of P is even

Also $P f = (I + f) f = (1 + f) f' = (6\sqrt{6} + 14)^{2n+1} (6\sqrt{6} - 14)^{2n+1} = 216 - 196)^{2n+1} = 20^{2n+1}$

Example : 21

Expand $\frac{2-x}{(1-x)(3-x)}$ in ascending powers of x and find x^r. Also state the range of x for which this e x - pression is valid.

Solution

Given expression = $\frac{2-x}{(1-x)(3-x)}$

On expressing RHS in the form of partial fractions, we get

Given expression = $\frac{1}{2(1-x)} + \frac{1}{2(3-x)}$

$\Rightarrow$ Given expression = $\frac{1}{2}(1-x)^{-1} + \frac{1}{6}\left(1-\frac{x}{3}\right)^{-1}$

Using the expansions of $(1 - x)^{-1}$, we get

Given expression = $\frac{1}{2}(1 + x + x^2 + x^3 + ...) + \frac{1}{6}\left(1+\frac{x}{3}+\frac{x^2}{9}+\frac{x^3}{27}+........\right)$

$\Rightarrow$ Given expansion = $\left(\frac{1}{2}+\frac{1}{6}\right) + \left(\frac{1}{2}+\frac{1}{18}\right)x + \left(\frac{1}{2}+\frac{1}{54}\right)x^2 + ... + \left(\frac{1}{2}+\frac{1}{63^r}\right)x^r + ...$

$\Rightarrow$ Given expression = $\frac{2}{3} + \frac{5}{9}x + \frac{14}{27}x^2 + ... + \frac{1}{2}\left(1+\frac{1}{3^{r+1}}\right)x^r + ...$

Coefficient of $x^r = \frac{1}{2}\left(1+\frac{1}{3^{r+1}}\right)x^r$

Since $(1 - x)^{-1}$ is valid for $x \in (-1, 1)$ and $(1 - x/3)^{-1}$ is valid for $x \in (-3, 3)$, the given expression is valid for $x \in (-1, 1)$ (i.e. take intersection of the two sets)

Hence $\dfrac{2-x}{(1-x)(3-x)}$ is valid for $-1 < x < 1$

Example : 22

If $y = \dfrac{3}{4} + \dfrac{3.5}{4.8} + \dfrac{3.57}{4.812} + \ldots\ldots\ldots$ till infinity, show that $y^2 + 2y - 7 = 0$

Solution

It is given that : $y = \dfrac{3}{4} + \dfrac{3.5}{4.8} + \dfrac{3.57}{4.812} + \ldots\ldots$ to ∞

On adding 1 to both sides, we get :

$1 + y = 1 + \dfrac{3}{4} + \dfrac{3.5}{4.8} + \dfrac{3.57}{4.812} + \ldots\ldots$ to ∞ ....(i)

Now we will find the sum of series on RHS or (i)

For this consider the expansion of $(1 + t)^n$, where n is negative or fraction :

$(1 + t)^n = 1 + nt + \dfrac{n(n-1)}{1.2}t^2 + \dfrac{n(n-1)(n-2)}{1.2.3}t^3 + \ldots\ldots$ to ∞ where $|t| < 1$ ....(ii)

On comparing (i) and (ii), we get

$nt = 3/4$ ....(iii)

$\dfrac{n(n-1)}{1.2}t^2 = \dfrac{3.5}{4.8}$ ....(iv)

and $(1 + t)^n = 1 + y$

Consider (iv) : $\dfrac{n(n-1)}{1.2}t^2 = \dfrac{3.5}{4.8}$

$\Rightarrow \quad \dfrac{(n-1)t}{2} = \dfrac{5}{8}$ [using (iii)]

$\Rightarrow \quad (n-1)\,t = \dfrac{5}{4}$

$\Rightarrow \quad nt - t = \dfrac{5}{4}$

$\Rightarrow \quad \dfrac{3}{4} - t = \dfrac{5}{4}$ [using (iii)]

$\Rightarrow \quad t = -1/2$ and $n = -3/2$

$\Rightarrow$ Sum of series on RHS of (i) $= \left(1-\dfrac{1}{2}\right)^{-3/2}$

$\Rightarrow \quad 1 + y = (1 - 1/2)^{-3/2} \quad \Rightarrow \quad 2^{3/2} = 1 + y$

On squaring, we get $8 = (1 + y)^2$

$\Rightarrow \quad y^2 + 2y - 7 = 0$

Hence proved

Example : 23

Find the coefficient of $x_1^2\, x_2\, x_3$ in the expansion of $(x_1 + x_2 + x_3)^4$.

Solution

To find the required coefficient, we can use multinomial theorem in the question.

The coefficient of $x_1^2 x_2 x_3$ in the expansion of $(x_1 + x_2 + x_3)^4 = \dfrac{4!}{2!\,1!\,1!} = 12$

Hence coefficient of $x_1^2 x_2 x_3 = 12$
Note : Also try to solve this question without the use of multinomial theorem

Example : 24
Find the coefficient of x^7 in the expansion of $(1 + 3x - 2x^3)^{10}$.
Solution
Using the multinomial theorem, the general term of the expansion is :

$$T_{p,q,r} = \frac{10!}{p!\,q!\,r!}(1)^p (3x)^q (-2x^3)^r,$$

where $p + q + r = 10$. Find the coefficient of x^7 , we must have $q + 3r = 7$.
Consider q + 3r = 7
From the above relationship, we can find the possible values which p, q and r can take
Take r = 0
$\Rightarrow$ q = 7 and p = 3
$\Rightarrow$ $(p, q, r) \equiv (3, 7, 0)$ ....(i)
Take r = 1
$\Rightarrow$ q = 4 and p = 5
$\Rightarrow$ $(p, q, r) \equiv (5, 4, 1)$ ....(ii)
Take r = 2
$\Rightarrow$ q = 1 and p = 7
$\Rightarrow$ $(p, q, r) \equiv (7, 1, 2)$ ....(iii)
If we take r > 2, we get q < 0, which is not possible.
Hence (i), (ii) and (iii) and the only possible combination of values which p, q and r can take.

Using (i), (ii) and (iii), coefficient of $x^7 = \dfrac{10!}{1!\,3!\,7!}3^7 + \dfrac{10!}{5!\,4!\,1!}3^4(-2)^1 + \dfrac{10!}{7!\,2!\,1!}3^1(-2)^2 = 62640$

Hence coefficient of $x^7 = 62640$

Example : 25
Find the coefficient of x^{50} in the expansion : $(1 + x)^{1000} + 2x(1 + x)^{999} + 3x^2(1 + x)^{998} + ... + 1001x^{1000}$.
Solution
It can be easily observed that series is an Arithmetic-Geometric series with common difference = 1, common ratio = x/(1+x) and number of terms = 1001
Let $S = (1 + x)^{1000} + 2x(1 + x)^{999} + 3x^2(1 + x)^{998} + ... + 1001x^{1000}$ ....(i)
Multiple both sides by x/(1 + x) to get
$xS/(1 + x) = x(1 + x)^{999} + 2x^2(1 + x)^{998} + 3x^3(1 + x)^{997} + ... 1000x^{1000} + 1001x^{1001}/(1 + x)$ ....(ii)
Shift (ii) by one term and subtract it from (i) to get :
$S/(1 + x) = (1 + x)^{1000} + x(1 + x)^{999} + x^2(1 + x)^{998} + ... x^{1000} - 1001x^{1001}/(1 + x)$
$\Rightarrow$ $S = (1 + x)^{1001} + x(1 + x)^{1000} + x^2(1 + x)^{999} + ... x^{1000}(1 + x) - 1001x^{1001}$
Now the above series, upto the term $x^{1000}(1 + x)$, is G.P. with first term = $(1 + x)^{1001}$, common ratio = x/(1 + x) and number of terms = 1001

$$\Rightarrow \quad S = \frac{(1+x)^{1001}\left[1-\left(\frac{x}{1+x}\right)^{1001}\right]}{1-\frac{x}{1+x}} - 1001\,x^{1001}$$

$\Rightarrow$ $S = (1 + x)^{1002} - x^{1001}(1 + x) - 1001x^{1001}$
Coefficient of x^{50} in the series S = coeff. of x^{50} in $(1 + x)^{1002}$ ($\because$ other terms can not produce x^{50})
$\Rightarrow$ Coefficient of x^{50} in the series S = ${}^{1002}C_{50}$
Hence the coefficient of x^{50} in the given series = ${}^{1002}C_{50}$

Example : 26

Find the total number of terms in the expansion of $(x + y + z + w)^n$, $n \in N$.

Solution

Consider the expansion :

$(x + y + z + w)^n = (x + y)^n + {}^nC_1 (x + y)^{n-1} (z + w) + {}^nC_2 (x + y)^{n-2} (z + w)^2 + ... + {}^nC_n (z + w)^n$

Number of terms on the RHS = (n + 1) + n.2 + (n – 1) . 3 + ... + (n + 1)

$$= \sum_{r=0}^{n} (n-r+1)(r+1) = \sum_{r=0}^{n} (n+1) + \sum_{r=0}^{n} nr - \sum_{r=0}^{n} r^2$$

$$= (n+1)\sum_{r=0}^{n} 1 + n\sum_{r=0}^{n} r - \sum_{r=0}^{n} r^2 = (n+1)(n+1) + \frac{n(n)(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)}{6}[6(n+1) + 3n^2 - 2n^2 - n] = \frac{n+1}{6}[n^2 + 5n + 6] = \frac{(n+1)(n+2)(n+3)}{6}$$

Using multinomial theorem :

$$(x + y + z + w)^n = \sum_{r=0}^{n} \frac{n!\, x^{n_1} y^{n_2} z^{n_3} w^{n_4}}{n_1!\, n_2!\, n_3!\, n_4!}$$, where n_1, n_2, n_3 and n_4 can have all possible values for 0, 1, 2, ..., n subjected to the condition $n_1 + n_2 + n_3 + n_4 = n$ ....(i)

Therefore, the number of distinct terms in the multinomial expansion is same as the non-negative integral solutions of (i)

$\Rightarrow$ Number of distinct terms = Number of non-negative integral solutions

$\Rightarrow$ Number of distinct terms = coefficient of x^n in the expansion $(1 + x + x^2 + ...+ x^n)^4$

$$= \text{coefficient of } x^n \text{ in } \left(\frac{1-x^{n+1}}{1-x}\right)^4$$

$$= \text{coefficient of } x^n \text{ in } (1 - x^{n+1})^4 (1 - x)^{-4} = {}^{n+4-1}C_{4-1} = {}^{n+3}C_3$$

$\Rightarrow$ Number of distinct terms = $\frac{(n+1)(n+2)(n+3)}{6}$

Example : 27

Let n be a positive integer and $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n} x^{2n}$.

Show that $a_0^2 - a_1^2 - + ... + a_{2n}^2 = 2_n$.

Solution

Consider the given identity : $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n} x^{2n}$ ....(i)

Replace x by –1/x in this identity to get :

$$\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x}\ \frac{a_2}{x^2} - + ... + \frac{a_{2n}}{x^{2n}}$$

$\Rightarrow$ $(1 - x + x^2)^n = a_0 x^{2n} - a_1 x^{2n-1} + a_2 x^{2n-2} - + ... + a_{2n}$ ....(ii)

Multiply (i) and (ii) and also compare coefficient of x^{2n} on both sides to get :

$a_0^2 - a_1^2 + a_2^2 - + ... + a_{2n}^2$ = coefficient of x^{2n} in $(1 + x + x^2)^n (1 - x + x^2)^n$

$\Rightarrow$ LHS = coefficient of x^{2n} in $(1 + x^2 + x^4)^n$

$\Rightarrow$ LHS = coefficient of x^{2n} in $a_0 + a_1x^2 + a_2x^4 + ... + a_nx^{2n} + ... + a_{2n}x^{4n}$ [replace x by x^2 in (i)]

$\Rightarrow$ LHS = a_n

Hence $a_0^2 - a_1^2 + a_2^2 - +........ + a_{2n}^2 = a_n$

Example : 28

If $\sum_{r=0}^{2n} a_r (x-2)^r = \sum_{r=0}^{2n} b_r (x-3)^r$ and $a_k = 1$ for all $k \geq n$, show that $b_n = {}^{2n+1}C_{n+1}$.

Solution

Let $y = x - 3 \quad \Rightarrow \quad y + 1 = x - 2$

So given expression reduces to :

$$\sum_{r=0}^{2n} a_r (y+1)^r = \sum_{r=0}^{2n} b_r (y)^r$$

$\Rightarrow \quad a_0 + a_1 (y + 1) + ... + a_{2n} (y + 1)^{2n} = b_0 + b_1 y + ... + b_{2n} y^{2n}$

Using $a_k = 1$ for all $k \geq n$, we get

$\Rightarrow \quad a_0 + a_1 (y + 1) + ... a_{n-1} (y + 1)^{n-1} + (y + 1)^n + ... + (y + 1)^{2n}$

$= b_0 + b_1 y + ... + b_n y^n + ...+ b_{2n} y^{2n}$

Compare coefficient of y^n on both sides, we get :

${}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + ... + {}^{2n}C_n = b_n$

Using the formula, ${}^nC_r = {}^nC_{n-r}$, we get :

${}^nC_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + ... + {}^{2n}C_n = b_n$

Using, ${}^nC_0 = {}^{n+1}C_0$ for first term, we get :

${}^{n+1}C_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + ... + {}^{2n}C_n = b_n$

On combining the first two terms with use of the formula,

${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$, we get :

${}^{n+2}C_1 + {}^{n+2}C_2 + ... + {}^{2n}C_n = b_n$

If we combine terms on LHS like we have done in last step, finally we get :

${}^{2n}C_n = b_n \quad \Rightarrow \quad b_n = {}^{2n+1}C_{n+1} \quad$ (using ${}^nC_r = {}^nC_{n-r}$)

Hence $b_n = {}^{2n+1}C_{n+1}$

Example : 29

Prove that $\sum_{r=1}^{k} (-3)^{r-1}\ {}^{3n}C_{2r-1} = 0$, where k = 3n/2 and n is an even positive integer.

Solution

Let $n = 2m \quad \Rightarrow \quad k = 3m$

LHS = $\sum_{r=1}^{2m} (-3)^{r-1}\ {}^{6m}C_{2r-1} = {}^{6m}C_1 - 3\ {}^{6m}C_3 + 9\ {}^{6m}C_5 - .... + (-3)^{3m-1}\ {}^{6m}C_{6m-1}$ ....(i)

Consider $\quad (1 + x)^{6m} = {}^{6m}C_0 + {}^{6m}C_1 x + {}^{6m}C_2 x^2 + ... + {}^{6m}C_{6m} x^{6m}$ and

$(1 - x)^{6m} = {}^{6m}C_0 - {}^{6m}C_1 x + {}^{6m}C_2 x^2 + ...+ {}^{6m}C_{6m} x^{6m}$

On subtracting the above two relationships, we get

$(1 + x)^{6m} - (1 - x)^{6m} = 2 ({}^{6m}C_1 x + {}^{6m}C_3 x^3 + {}^{6m}C_5 x^5 + ... + {}^{6m}C_{6m-1} x^{6m-1})$

Divide both sides by 2x to get :

$$\frac{(1+x)^{6m} - (1-x)^{6m}}{2x} = {}^{6m}C_1 + {}^{6m}C_3 x^2 + ... + {}^{6m}C_{6m-1} x^{6m-2}$$

Put $x = \sqrt{3}i$ in the above identity to get :

$$\frac{(1+i\sqrt{3})^{6m} - (1-i\sqrt{3})^{6m}}{2\sqrt{3}i} = {}^{6m}C_1 - 3\ {}^{6m}C_3 + ... + (-3)^{3m-1}\ {}^{6m}C_{6m-1} \qquad ....(ii)$$

Comparing (i) and (ii), we get

$$\text{LHS} = \frac{2^{6m}\left[\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{6m} - \left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)^{6m}\right]}{2\sqrt{3}i}$$

$\Rightarrow$ LHS = $\dfrac{2^{6m}[(\cos 2\pi m + i\sin 2\pi m) - (\cos 2\pi m - i\sin 2\pi m)]}{2\sqrt{3}i}$ (using De morvie's Law)

$\Rightarrow$ LHS = $\dfrac{2^{6m} 2i\sin 2\pi m}{2\sqrt{3}i} = \dfrac{2^{6m}\sin 2\pi m}{\sqrt{3}} = 0$ (because sin 2 πm = 0)

Example : 30

Show by expanding $[(1 + x)^n - 1]^m$ where m and n are positive integers, that

$^mC_1\ ^nC_m - {}^mC_2\ ^{2n}C_m + {}^mC_3\ ^{3n}C_m \ldots\ldots\ldots = (-1)^{m-1}\, n^m$.

Solution

Consider : $[(1 + x)^n - 1]^m$ and expand $(1 + x)^n$ binomially

$\Rightarrow$ $[(1 + x)^n - 1]^m = [1 + {}^nC_1 x^2 + \ldots\ldots\ldots + {}^nC_n\, x^n) - 1]^m$

$\Rightarrow$ $[(1 + x)^n - 1]^m = [{}^nC_1\, x + {}^nC_2\, x^2 + \ldots\ldots. + {}^nC_n\, x^n]^m$

$\Rightarrow$ $[(1 + x)^n - 1]^m = x^m\, [{}^nC_1 + {}^nC_2\, x + \ldots\ldots..+ {}^nC_n\, x^{n-1}]^m$ ....(i)

Now consider : $[(1 + x)^n - 1]^m = (-1)^m\, [1 - (1 + x)^n]^m$

$\Rightarrow$ $[(1 + x)^n - 1)^m = (-1)^m\, [1 - {}^mC_1\, (1 + x)^n + {}^mC_2\, (1 + x)^{2n} - \ldots\ldots\ldots\ldots..]$ ....(ii)

Comparing (i) and (ii), we get :

$x^m\, [{}^nC_1 + {}^nC_2 x + \ldots\ldots\ldots + {}^nC_n x^{n-1}]^m\, [1 - {}^mC_1\, (1 + x)^n + {}^mC_2\, (1 + x)^{2n} - \ldots\ldots..]$

Compare coefficient of x^m on both sides to get :

$n^m = (-1)^m\, [-{}^mC_1\ \ ^nC_m + {}^mC_2\ ^{2n}C_m - {}^mC_3\ ^{3n}C_m + \ldots\ldots\ldots.]$

$\Rightarrow$ $^mC_1\ ^nC_m - {}^mC_2\ ^{2n}C_m + {}^mC_3\ ^{3n}C_m - + \ldots\ldots\ldots = (-1)^{m-1}\, n^m$

Hence proved

Example : 31

Show that $\displaystyle\sum_{r=1}^{n} (-1)^{r-1}\frac{C_r}{r} = \sum_{r=1}^{n}\frac{1}{r}$

Solution

Consider : $(1 - x)^n = C_0 - C_1 x + C_2 x^2 - \ldots\ldots\ldots. + (-1)^n\, C_n\, x^n$

$\Rightarrow$ $1 - (1 - x)^n = C_1 x - C_2\, x^2 + C_3\, x^3 + \ldots\ldots.. + (-1)^{n-1}\, C_n x^n$ $(\because C_0 = 1)$

Divide both sides by x to get :

$$\frac{1-(1-x)^n}{x} = C_1 - C_2\, x + C_3 x^2 + \ldots\ldots\ldots + (-1)^{n-1}\, C_n\, x^{n-1}$$

Integrate both sides between limits 0 and 1 to get :

$$\int_0^1 \frac{1-(1-x)^n}{x} = \int_0^1 [C_1 - C_2 x + C_3 x^2 + \ldots\ldots + (-1)^{n-1} C_n x^{n-1}]\, dx$$

$$\Rightarrow \int_0^1 \frac{1-(1-x)^n}{1-(1-x)} = C_1 x - C_2\, \frac{x^2}{2} = C_3\, \frac{x^3}{3} - \ldots\ldots\ldots + (-1)^{n-1} C_n\, \left.\frac{x^n}{n}\right]_0^1$$

It can be easily observed that integrand on the LHS is the summation of n terms of G.P. whose first term is 1 and common ratio is (1 – x).

$$\Rightarrow \int_0^1 [1 + (1-x) + (1-x)^2 + \ldots\ldots\ldots + (1-x)^{n-1}]\, dx = C_1 - \frac{C_2}{2} + \frac{C_3}{3} - + \ldots\ldots.. + \frac{(-1)^{n-1}C_n}{n}$$

$$\Rightarrow \left. x - \frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} - \ldots\ldots\ldots.. - \frac{(1-x)^n}{n}\right]_0^1 = C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \ldots\ldots. + \frac{(-1)^{n-1}C_n}{n}$$

$$\Rightarrow 1 + \frac{1}{2} = \frac{1}{3} + \ldots\ldots\ldots.. + \frac{1}{n} = C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \ldots\ldots\ldots = \frac{(-1)^{n-1}C_n}{n}$$

$$\Rightarrow \sum_{r=1}^{n} (-1)^{r-1} \frac{C_r}{r} = \sum_{r=1}^{n} \frac{1}{r}$$. Hence proved

Example : 32

Show that $\frac{C_0}{1} - \frac{C_1}{5} + \frac{C_2}{9} - \frac{C_3}{13} + \ldots\ldots\ldots + (-1)^n \frac{C_n}{4n+1} = \frac{4^n n!}{1.5.9\ldots\ldots(4n+1)}$

Solution

On observing the LHS of the relationship to be proved, we can conclude that the expansion of $(1 - x^4)^n$ must be used to prove LHS equals RHS Hence,

$(1 - x^4)^n = C_0 - C_1x^4 + C_2x^8 - C_3x^{12} + \ldots\ldots\ldots + (-1)^n C_n x^{4n}$

Integrating both sides between limits 0 and 1, we get :

$$\int_0^1 (1-x^4)^n = \frac{C_0}{1} - \frac{C_1}{5} + \frac{C_9}{9} - \frac{C_{13}}{13} + \ldots\ldots\ldots + (-1)^n \frac{C_n}{4n+1} \qquad \ldots\ldots\ldots(i)$$

Let $I_n = \int_0^1 (1-x^4)^n \, dx$ .... (ii)

apply by-parts taking $(1 - x^4)^n$ as the I part and dx as the II part ,

$$\Rightarrow I_n = (1 - x^4)^n x]_0^1 - \int_0^1 n(1-x^4)^{n-1} (-4x^3)\, x\, dx$$

$$\Rightarrow I_n = 4n \int_0^1 x^4(1-x^4)^{n-1}dx = 4n \int_0^1 [1-(1-x^4)](1-x^4)^{n-1}\, dx$$

$$\Rightarrow I_n = 4n \int_0^1 (1-x^4)^{n-1}\, dx - 4n \int_0^1 (1-x^4)^n\, dx$$

$$\Rightarrow I_n = 4n\, I_{n-1} - 4n\, I_n$$

$$\Rightarrow I_n = \frac{4n}{4n+1}\, I_{n-1}$$

Replace n by 1, 2, 3, 4, ..., n–1 in the above identity and multiply all the obtained relations,

$$\Rightarrow I_n = \frac{4n}{4n+1} \cdot \frac{4(n-1)}{4n-3} \cdot \frac{4(n-2)}{4n-7} \ldots\ldots\ldots\ldots \frac{4}{5}\, I_0 \qquad \ldots\ldots\ldots\ldots(iii)$$

Finding I_0

I_0 can be obtained by substituting n = 0 in (ii) i.e.

$$I_0 = \int_0^1 (1-x^4)^0\, dx = \int_0^1 dx = 1$$

Substitute the value of I_0 in (iii) to get :

$$I_n = \frac{4n}{4n+1} \cdot \frac{4(n-1)}{4n-3} \cdot \frac{4(n-2)}{4n-7} \ldots\ldots\ldots\ldots \frac{4}{5}$$

$$\Rightarrow I_n = \frac{4^n\, n!}{1.5.9.13\ldots\ldots\ldots(4n+1)}$$

Using (i)

$$\frac{C_0}{1} - \frac{C_1}{5} + \frac{C_2}{9} - \frac{C_3}{13} + \ldots\ldots.. + (-1)^n \frac{C_n}{4n+1} = \frac{4^n\, n!}{1.5.9.....(4n+1)}$$

Hence proved

Example : 33

Show that $x^n - y^n$ is divisible by $x - y$ if n is natural number.

Solution

Let $P(n) = x^n - y^n$ is divisible by $x - y$

We consider P(1)

P(1) : $x^1 - y^1$ is divisible by $x - y$

$\Rightarrow$ P(1) is true

Now let us assume p(k) to be true

i.e. we are given P(k) : $x^k - y^k$ is divisible by $x - y$

Let $x^k - y^k = (x - y)\, m,\ m \in I$

Consider P(k + 1) :

P(k + 1) : $x^{k+1} - y^{k+1}$ is divisible by $x - y$;

Now $x^{k+1} - y^{k+1} = x^{k+1} - x^k y + x^k y - y^{k+1}$

$= x^k (x - y) + y (x^k - y^k)$

$= x^k (x - y) + y (x - y)m$

$= (x - y) (x^k + my)$

Hence P(k + 1) is true whenever P(k) is true.

Hence according to the principle of Mathematical Induction, P(n) is true for all natural numbers.

Example : 34

Show that $5^{2n+2} - 24n - 25$ is divisible by 576.

Solution

Let P(n) : $5^{2n+2} - 24 - 25$ is divisible by 576

P(1) : $5^{2(1)+2} - 24\,(1) - 25$ is divisible by 576

P(1) : 576 is divisible by 576

$\Rightarrow$ P(1) is true

P(k) : $5^{2k+2} - 24k - 25 = 576m,\ m \in N$

P(k + 1) : $5^{2k+4} - 24\,(k + 1) - 25$ is divisible by 576

Consider $5^{2k+4} - 24\,(k + 1) - 25$

$= 5^{2k+4} - 24\,(k + 1) - 25$

$= 5^{2k+2} \,.\, 5^2 - 24k - 49$

$= 25\,(24k + 25 + 576m) - 24k - 49$ [using P(k)]

$= (576)\, 25m - 576k + 576$

$= 576\,(25m - k + 1)$

$\Rightarrow$ $5^{2k+4} - 24\,(k + 1) - 25$ is divisible by 576

Hence P(k + 1) is true whenever p(k) is true

Hence according to the principle of Mathematical Induction P(n) is true for all natural numbers.

Example : 35

Show that $2^n > n$ for all natural numbers

Solution

Let P(n) : $2^n > n$

P(1) : $2^1 > 1$

$\Rightarrow$ P(1) is true

P(k) : $2^k > k$

Assume that p(k) is true

P(k + 1) : $2^{k+1} > k + 1$

consider P(k) : $2^k > k$

$\Rightarrow$ $2^{k+1} > 2k$

$\Rightarrow \quad 2^{k+1} > k + k$

But we have $k \geq 1$

Adding $2^{k+1} + k > k + k + 1$

$2^{k+1} > k + 1$

Hence P(k + 1) is true whenever P(k) is true

Hence according to the principle of Mathematical Induction, P(n) is true for all natural numbers.

Example : 36

Prove by the method of Induction that : $\frac{1}{3.7} + \frac{1}{7.11} + \frac{1}{11.15} + ... + \frac{1}{(4n-1)(4n+3)} = \frac{n}{3(4n+3)}$

Solution

Let P(n) : $\frac{1}{3.7} + \frac{1}{7.11} + \frac{1}{11.15} + ... + \frac{1}{(4n-1)(4n+3)} = \frac{n}{3(4n+3)}$

P(1) : $\frac{1}{3.7} = \frac{1}{3(4+3)}$

P(1) : $\frac{1}{21} = \frac{1}{21}$

$\Rightarrow$ P(1) is true

P(k) $\frac{1}{3.7} + \frac{1}{7.11} + ... + \frac{1}{(4k-1)(4k+3)} = \frac{k}{3(4k+3)}$

Assume that P(k) is true

P(k + 1) : $\frac{1}{3.7} + \frac{1}{7.11} + ... + \frac{1}{(4k-1)(4k+3)} + \frac{1}{(4k+3)(4k+7)} = \frac{k+1}{3(4k+7)}$

LHS $= \left(\frac{1}{3.7} + \frac{1}{7.11} + ... + k \text{ terms}\right) + \frac{1}{(4k+3)(4k+7)}$

$= \frac{k}{3(4k+3)} + \frac{1}{(4k+3)(4k+7)}$ [using P(k)]

$= \frac{k(4k+7)+3}{3(4k+3)(4k+7)}$

$= \frac{(4k+3)(k+1)}{3(4k+3)(4k+7)} = \frac{(k+1)}{3(4k+7)}$ = RHS of P(k + 1)

Hence P(k + 1) is true whenever P(k) is true

Hence according to the principle of Mathematical Induction, P(n) is true for all natural numbers.

Example : 37

Using Mathematical Induction, show that $n(n^2 - 1)$ is divisible by 24 if n is an odd positive integer.

Solution

To prove a statement for odd numbers only, it is required to show that

(a) P(1) is true

(b) P(k + 2) is true whenever p(k) is true

P(1) : $1 (1^2 - 1)$ is divisible by 24

$\Rightarrow$ P(1) is true

P(k) : k(k2 – 1) is divisible by 24 if k is odd

Assume that P(k) is true

Let $k (k^2 - 1) = 24m$ where $m \in N$

P(k + 2) : $(k + 2) [(k + 2)^2 - 1]$ is divisible by 24, if k is odd

Consider $(k + 2) [(k + 2)^2 - 1]$

$= (k + 2) (k^2 + 4k + 3)$

$= k^3 + 6k^2 + 11k + 6$
$= (24m + k) + 6k^2 + 11k + 6$
$= (24m + 6k^2 + 12k + 6$ [using P(k)]
$= 24m + 6 (k + 1)^2$
$= 24m + 6 (2p)^2$ [$\because$ k is odd]
$= 24m + 24p^2$
$= 24 (m + p^2)$

Hence P(k + 2) is true whenever P(k) is true
Hence according to the principle of Mathematical Induction, P(n) is true for all natural numbers.

Example : 38

Prove that $\cos x \cos 2x \cos 4x .... \cos 2^{n-1} x = \frac{\sin 2^n x}{2^n \sin x}$

Solution

P(1) : $\cos x = \frac{\sin 2x}{2 \sin x}$

P(1) : cos x = cos x (using sin 2x = 2 sin x cos x)

$\Rightarrow$ P(1) is true

P(k) : $\cos x \cos 2x \cos 4x .... \cos 2^{k-1} x = \frac{\sin 2^k x}{2^k \sin x}$

Let P(k) be true. Consider P(k + 1)

P(k = 1) : $\cos x \cos 2x \cos 4x .... \cos 2^{k-1} x \cos 2^k x = \frac{\sin 2^{k+1} x}{2^{k+1} \sin x}$

LHS = $\left(\frac{\sin 2^k x}{2^k \sin x}\right) \cos 2^k x = \frac{2 \sin 2^k x \cos 2^k x}{2^{k+1} \sin x} = \frac{\sin 2^{k+1} x}{2^{k+1} \sin x}$ = RHS

Hence P(k + 1) is true whenever P(k) is true

$\therefore$ by mathematical induction P(n) is true $\forall$ n $\in$ N

Example : 39

By the method of induction, show that $(1 + x)^n \geq 1 + nx$ for n N, $x > -1$, $x \neq 0$

Solution

Let P(n) : $(1 + x)^n \geq 1 + nx$

$\Rightarrow$ P(1) : $(1 + x)^1 \geq 1 + x$ which is true

Let P(k) be true $\Rightarrow$ $(1 + x)^k \geq 1 + kx$ ....(i)

Consider P(k + 1) : $(1 + x)^{k+1} \geq 1 + (k + 1)x$

From (i) : $(1 + x)^k \geq 1 + kx$

$\Rightarrow$ $(1 + x)^{k+1} \geq (1 + kx) (1 + x)$ (as $(1 + x) > 0$)

$\Rightarrow$ $(1 + x)^{k+1} \geq 1 + (k + 1)x + kx^2$

as kx^2 is positive, it can be removed form the smaller side.

$\Rightarrow$ $(1 + x)^{k+1} \geq 1 + (k + 1)x$

$\Rightarrow$ P(k + 1) is true

Hence P(1) is true and P(k + 1) is true whenever P(k) is true

$\Rightarrow$ By induction, P(n) is true for all n $\in$ N

Example : 40

Prove that $x(x^{n-1} - na^{n-1}) + a^n (n - 1)$ is divisible by $(x - a)^2$ for n > 1 and n $\in$ N

Solution

Let P(n) : $x(x^{n-1} - na^{n-1}) + a^n (n - 1)$ is divisible by $(x - a)^2$

As n > 1, we will start from P(2)

For n = 2, the expression becomes

$= x(x - 2a) + a^2 (2 - 1) = (x - a)^2$ which is divisible by $(x - a)^2$

$\Rightarrow$ P(2) is true

Let P(k) be true

$\Rightarrow$ x $(x^{k-1} - ka^{k-1}) + a^k (k - 1)$ is divisible by $(x - a)^2$

For n = k + 1, the expression becomes = $x[x^k - (k + 1) a^k] + a^{k+1}k = x^{k+1} - kxa^k - xa^k + ka^{k+1}$

$= [x^{k+1} - kx^2a^{k-1} + xa^k (k - 1)] + kx^2 a^{k-1} - xa^k(k - 1) - kxa^k - xa^k + ka^{k+1}$

$= x[x(x^{k-1} - ka^{k-1}) + a^k (k - 1)] + ka^{k-1} (x^2 - 2ax + a^2)$

= divisible by $(x - a)^2$ from P(k) + $ka^{k-1} (x - a)^2$

Hence the complete expression is divisible by $(x - a)^2$

$\Rightarrow$ P(K + 1) is true

Hence P(2) is true and P(k + 1) is true whenever P(k) is true

$\Rightarrow$ By induction, P(n) is true for all n > 1, $n \in N$

Alternate Method : Let $f(x) = x(x^{n-1} - na^{n-1}) + a^n (n - 1)$

It can be show that $f(a) = f'(a) = 0$

$\Rightarrow$ f(x) is divisible by $(x - a)^2$

Example : 41

For any natural number n > 1, prove that $\frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots\ldots + \frac{1}{2n} > \frac{13}{24}$

Solution

Let P(n) : $\frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots\ldots + \frac{1}{2n} > \frac{13}{24}$

for n = 2, $\frac{1}{2+1} + \frac{1}{2+2} > \frac{13}{24} \Rightarrow \frac{7}{12} > \frac{13}{24}$ which is true

$\Rightarrow$ P(2) is true

Let P(k) be true

$$\Rightarrow \quad \frac{1}{k+1} + \frac{1}{k+2} + \ldots\ldots\ldots\ldots + \frac{1}{2k} > \frac{13}{24}$$

Consider P(k + 1) :

$$\Rightarrow \quad \frac{1}{k+2} + \frac{1}{k+3} + \ldots\ldots\ldots + \frac{1}{(k+1)+(k+1)} > \frac{13}{24}$$

Using P(k) we have :

$$\Rightarrow \quad \frac{1}{k+1} + \frac{1}{k+2} + \ldots\ldots\ldots + \frac{1}{2k} > \frac{13}{24}$$

adding $\frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1}$ on both sides, we get

$$\Rightarrow \quad \frac{1}{k+2} + \frac{1}{k+3} + \ldots\ldots\ldots + \frac{1}{2k+1} + \frac{1}{2k+2} > \frac{13}{24} + \frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1}$$

$$\Rightarrow \quad \frac{1}{k+2} + \ldots\ldots\ldots + \frac{1}{2k+1} = \frac{1}{2k+2} > \frac{13}{24} + \frac{(2k+2)+(2k+1)-2(2k+1)}{2(k+1)(2k+1)}$$

$$\Rightarrow \quad \frac{1}{k+2} + \ldots\ldots\ldots\ldots + \frac{1}{2k+1} + \frac{1}{2k+2} > \frac{13}{24} + \frac{1}{2(k+1)(2k+1)}$$

as $\frac{1}{2(k+1)(2k+1)}$ is positive, it can be removed the smaller side

$$\Rightarrow \quad \frac{1}{k+2} + \ldots\ldots\ldots + \frac{1}{2k+1} + \frac{1}{2k+2} > \frac{13}{24}$$

$\Rightarrow$ P(k + 1) is true

Hence P(2) is true and P(k + 1) is true whenever P(k) is true

$\Rightarrow$ By induction, P(n) is true for all n > 1, $n \in N$

Example : 42

If n > 1, prove that $n! < \left(\frac{n+1}{2}\right)^n$

Solution

Let P(n) : $n! < \left(\frac{n+1}{2}\right)^n$

for n = 2, $2! < \left(\frac{3}{2}\right)^2$ which is true

$\Rightarrow$ P(2) is true

Let P(k) be true

$\Rightarrow$ $k! < \left(\frac{k+1}{2}\right)^k$

P(k + 1) : $(k+1)! < \left(\frac{k+2}{2}\right)^{k+1}$ ....(i)

using P(k), we have

$k! < \left(\frac{k+1}{2}\right)^k$

$\Rightarrow$ $(k+1)! < \frac{(k+1)^{k+1}}{2^k}$ ....(ii)

Let us try to compare the RHS of (i) and (ii).

Let us assume that $\frac{(k+1)^{k+1}}{2^k} < \left(\frac{k+2}{2}\right)^{k+1}$ ....(iii)

$\Rightarrow$ $\left(\frac{k+2}{k+1}\right)^{k+1} > 2$ $\Rightarrow$ $\left(1+\frac{1}{k+1}\right)^{k+1} > 2$

Using Binomial Expansion :

$\Rightarrow$ $1 + (k+1)\frac{1}{k+1} + {}^{k+1}C_2\left(\frac{1}{k+1}\right)^2 + ...> 2$

$\Rightarrow$ $1 + 1 + {}^{k+1}C_2\left(\frac{1}{k+1}\right)^2 + ...> 2$ which is true

Hence (iii) is true

From (ii) and (iii), we have

$(k+1)! < \frac{(k+1)^{k+1}}{2^k} < \left(\frac{k+2}{2}\right)^{k+1}$

$\Rightarrow$ $(k+1)! < \left(\frac{k+2}{2}\right)^{k+1}$

P(K + 1) is true

Hence P(2) is true and P(k + 1) is true whenever P(k) is true

$\Rightarrow$ By induction, P(n) is true for all n > 1, $n \in N$

Example : 43

Prove that $A_n = \cos n\theta$ if it is given that $A_1 = \cos\theta$, $A_2 = \cos 2\theta$ and for every natural number m > 2, the relation $A_m = 2 A_{m-1} \cos\theta - A_{m-2}$.

Solution

The principle of induction can be extended to the following form :

P(n) is true for all $n \in N$, if

(i) P(1) is true and P(2) is true and

(ii) P(k + 2) is true whenever P(k) and P(k + 1) are true

Let P(n) : $A_n = \cos n\theta$

Hence $A_1 = \cos\theta$, $A_2 = \cos 2\theta$ $\Rightarrow$ P(1) and P(2) are true

Now let us assume that P(k) and P(k + 1) are true

$\Rightarrow$ $A_k = \cos k\theta$ and $A_{k+1} = \cos (k + 1)\theta$

We will now try to show that P(k + 2) is true

Using $A_m = 2 A_{m-1} \cos\theta - A_{m-2}$, (for m > 2)

We have $A_{k+2} = 2A_{k+1} \cos\theta - A_k$ (for k > 0)

$\Rightarrow$ A_{k+2} $= 2 \cos (k + 1)\theta \cos\theta = \cos k\theta$

$= \cos (k + 2)\theta + \cos k\theta - \cos k\theta$

$= \cos (k + 2)\theta$

$\Rightarrow$ P(k + 2) is true

Hence P(1), P(2) are true and P(k + 2) is true whenever P(k), P(k + 1 are true

$\Rightarrow$ By induction, P(n) is true for all $n \in N$

Example : 44

Let $u_1 = 1$, $u_2 = 1$ and $u_{n+2} = u_n + u_{n+1}$ for $n \geq 1$. Use induction to show that $u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

for all $n \geq 1$.

Solution

Let $P(n) : u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

$P(1) : u_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right] = 1$ which is true

$P(2) : u_2 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right] = 1$ which is true

Hence P(1), P(2) are true

Let P(k), P(k + 1) be true

$\Rightarrow$ We have : $u_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$

And $u_{k-1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$

Let us try to prove that P(k + 2) is true

From the given relation : $u_{k+2} = u_k + u_{k+1}$

$\Rightarrow$ $u_{k+2} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right] - \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$

$\Rightarrow \quad u_{k+2} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k\left(1+\frac{1+\sqrt{5}}{2}\right)\right] - \frac{1}{\sqrt{5}}\left[\left(\frac{1-\sqrt{5}}{2}\right)^k\left(1+\frac{1-\sqrt{5}}{2}\right)\right]$

$\Rightarrow \quad u_{k+2} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k\left(\frac{1+\sqrt{5}}{2}\right)^2\right] - \left[\left(\frac{1-\sqrt{5}}{2}\right)^k\left(\frac{1-\sqrt{5}}{2}\right)^2\right]$

$\Rightarrow \quad u_{k+2} = \frac{1}{\sqrt{5}}\left[\left(\frac{1-\sqrt{5}}{2}\right)^{k+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+2}\right]$

$\Rightarrow$ P(k + 2) is true

Hence P(1), P(2) are true and P(k + 2) is true whenever P(k), P(k + 1) are true

$\Rightarrow$ By induction, P(n) is true for all $n \in N$

Example : 45

Use mathematical induction to prove that $\sum_{k=0}^{n} k^2 \ {}^nC_k = n(n+1)\,2^{n-2}$ for $n \ge 1$

Solution

Let P(n) : $\sum_{k=0}^{n} k^2 \ {}^nC_k = n\,(n+1)\,2^{n-2}$

for n = 1 : $\sum_{k=0}^{n} k^2 \ {}^1C_k = 1\,(1+1)\,2^{1-2}$

i.e. 1 = 1 which is true $\Rightarrow$ P(1) is true

Let P(m) be true

$\Rightarrow \quad \sum_{k=0}^{m} k^2 \ {}^mC_k = m\,(m+1)\,2^{m-2}$

consider P(m + 1) : $\sum_{k=0}^{m+1} k^2 \ {}^{m+1}C_k = (m+1)\,(m+2)\,2^{m-1}$

LHS of P(m + 1) : = $\sum_{k=0}^{m+1} k^2 \ {}^{m+1}C_k = \sum_{k=0}^{m+1} k^2 \ ({}^mC_k + {}^mC_{k-1}) = \sum_{k=0}^{m} k^2 \ {}^mC_k + \sum_{k=1}^{m+1} k^2 \ {}^mC_{k-1}$

$= m(m+1)\,2^{m-2} + \sum_{t=0}^{m} (t+1)^2 \ {}^mC_t$ substituting k = t + 1

$= m\,(m+1)\,2^{m-2} + \sum_{t=0}^{m+1} t^2 \ {}^mC_t + 2\sum_{t=0}^{m} t \ {}^mC_t + \sum_{t=0}^{m} {}^mC_t$

using P(k) and $C_1 + 2C_2 + 3C_3 + \ldots\ldots\ldots\ldots {}^nC_n = n2^{n-1}$

$\Rightarrow$ LHS = $m\,(m+1)\,2^{m-2} + m\,(m+1)\,2^{m-2} + 2\,(m2^{m-1}) + 2^m = 2^{m-1}\,[m(m+1) + 2m + 2]$

$= 2^{m-1}\,(m+1)\,(m+2) =$ RHS

$\Rightarrow$ P(m + 1) is true

Hence P(1) is true and P(m + 1) is true whenever P(m) is true

$\Rightarrow$ By induction, P(n) is true for all $n \in N$

Example : 46

Using mathematical induction, prove $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is an integer for all $n \in N$

Solution

Let P(n) : $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is an integer

P(1) : $\frac{1}{5} + \frac{1}{3} = \frac{7}{15}$ = 1 is an integer $\Rightarrow$ P(1) is true

Let us assume that P(k) is true i.e. P(k) : $\frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15}$ is an integer ....(i)

Consider LHS of P(k + 1)

$$\text{LHS of } P(k+1) = \frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{7(k+1)}{15}$$

$$= \frac{k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1}{5} + \frac{k^3 + 3k^2 + 3k + 1}{3} + \frac{7(k+1)}{15}$$

$$= \frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15} + k^4 + 2k^3 + 3k^2 + 2k + \frac{1}{5} = \frac{1}{3} + \frac{7}{15}$$

$= P(k) + k^4 + 2k^3 + 3k^2 + 2k + 1$ [using (i)]

As P(k) and k both are positive integers, we can conclude that P(k + 1) is also an integer

$\Rightarrow$ P(k + 1) is true

Hence by principle of mathematical induction, P(n) si true for all $n \in N$

Example : 47

Using mathematical induction, prove that for any non-negative integers n, m, r and k,

$$\sum_{m=0}^{k} (n-m) \frac{(r+m)!}{m!} = \frac{(r+k+1)!}{k!}\left[\frac{n}{r+1} - \frac{k}{r+2}\right]$$

Solution

In this problem, we will apply mathematical induction on k.

Let P(k) : $\sum_{m=0}^{k} (n-m) \frac{(r+m)!}{m!} = \frac{(r+k+1)!}{k!}\left[\frac{n}{r+1} - \frac{k}{r+2}\right]$

Consider P(0)

$$\text{LHS of } P(0) = \sum_{m=0}^{0} (n-m)\frac{(r+m)!}{m!} = n\,\frac{r!}{0!} = nr!$$

$$\text{RHS of } P(0) = \frac{(r+1)!}{0!}\left[\frac{n}{r+1} - \frac{0}{r+2}\right] = \frac{n(r+1)!}{r+1} = nr!$$

$\Rightarrow$ P(0) is true

Let us assume that P(k) is true for k = p

$$\Rightarrow \quad \sum_{m=0}^{p}(n-m) \frac{(r+m)!}{m!} = \frac{(r+p+1)!}{p!}\left[\frac{n}{r+1} - \frac{p}{r+2}\right] \quad ....(i)$$

Consider LHS of P(p + 1)

$$\text{LHS of } P(p+1) = \sum_{m=0}^{p+1} (n-m) \frac{(r+m!)}{m!} = \sum_{m=0}^{p} (n-m) \frac{(r+m)!}{m!} + (n-p-1)\frac{(r+p+1)!}{(p+1)!}$$

using (i), we get :

LHS of P(p + 1) = $\frac{(r+p+1)!}{p!}\left[\frac{n}{r+1}-\frac{p}{r+2}\right]$ + (n – p – 1) $\frac{(r+p+1)!}{(p+1)!}$

$$= \frac{(r+p+1)!}{(p+1)!}\left[\frac{n(p+1)}{r+1}-\frac{(p+1)}{r+2}+n-(p+1)\right]$$

$$= \frac{(r+p+1)!}{(p+1)!}\left[\left(\frac{n(p+1)}{r+1}+n\right)-\left(\frac{(p+1)}{r+2}+(p+1)\right)\right]$$

$$= \frac{(r+p+1)!}{(p+1)!}\left[\frac{(p+r+2)n}{r+1}-\frac{(p+1)(p+r+2)}{r+2}\right]$$

$$= \frac{(r+p+2)!}{(p+1)!}\left[\frac{n}{r+1}-\frac{(p+1)}{r+2}\right] = \text{RHS of P(p + 1)}$$

$\Rightarrow$ P(p + 1) is true

Hence, by principle of mathematical induction, P(n) is true for all n = 0, 1, 2, 3, ...

Example : 48

If x is not an integral multiple of 2π , use mathematical induction to prove that :

$$\cos x + \cos 2x + \text{.........} + \cos nx = \cos \frac{n+1}{2} x \sin \frac{nx}{2} \text{ cosec } \frac{x}{2}$$

Solution

Let P(n) : $\cos x + \cos 2x + \text{.........} + \cos nx = \cos \frac{n+1}{2} x \sin \frac{nx}{2} \text{ cosec } \frac{x}{2}$

LHS of P(1) = cos x

RHS of P(1) = $\cos \frac{1+1}{2} x \sin \frac{1.x}{2} \text{ cosec } \frac{x}{2}$ = cos x

Let us assume that P(k) is true

i.e. P(k) : $\cos x + \cos 2x + \text{...........} + \cos kx = \cos \frac{k+1}{2} x \sin \frac{kx}{2} \text{ cosec } \frac{x}{2}$

Consider LHS of P(k + 1)

LHS of P(k + 1) = cos x + cos 2x + ... + cos kx + cos (k + 1) x

Using P(k), we get :

LHS of P(k + 1) = $\cos \frac{k+1}{2} x \sin \frac{kx}{2} \text{ cosec } \frac{x}{2} + \cos (k+1) x$

$$= \frac{\cos\frac{k+1}{2}x\sin\frac{kx}{2}-\cos(k+1)x\sin\frac{x}{2}}{\sin\frac{x}{2}} = \frac{2\cos\frac{k+1}{2}x\sin\frac{kx}{2}-2\cos(k+1)x\sin\frac{x}{2}}{2\sin\frac{x}{2}}$$

$$= \frac{\sin\left(\frac{2k+1}{2}\right)x-\sin\frac{kx}{2}+\sin\left(\frac{2k+3}{2}\right)x-\sin\left(\frac{2k+1}{2}\right)x}{2\sin\frac{x}{2}} = \frac{\sin\left(\frac{2k+3}{2}\right)x-\sin\frac{kx}{2}}{2\sin\frac{x}{2}}$$

$$= \frac{2\cos\left(\frac{k+2}{2}\right)x\sin\left(\frac{k+1}{2}\right)x}{2\sin\frac{x}{2}} = \cos\left(\frac{k+2}{2}\right) x \sin\left(\frac{k+1}{2}\right) x \text{ cosec } \frac{x}{2} = \text{RHS of P(k + 1)}$$

$\Rightarrow$ P(k + 1) is true

Hence by principle of mathematical induction, P(n) is true for all $n \in N$

Example : 49

Using mathematical induction, prove that for every integer $n \geq 1$, $3^{2^n} - 1$ is divisible by 2^{n+2} but not divisible by 2^{n+3} .

Solution

Let P(n) : $3^{2^n} - 1$ is divisible by 2^{n+2} , but not divisible by 2^{n+3}.

P(1) : 8 is divisible by 2^3 , but not divisible by 2^4 .

$\Rightarrow$ P(1) : 8 is divisible by 8, but not divisible by 16

$\Rightarrow$ P(1) is true

Let P(k) is true

i.e. $3^{2^k} - 1$ is divisible by 2^{k+2} , but not divisible by 2^{k+3}

$\Rightarrow$ $3^{2^k} - 1 = m\, 2^{k+2}$, where m is odd number so that P(k) is not divisible by 2^{k+3} ....(i)

Consider P(k + 1)

LHS of P (k + 1) = $3^{2^{k+1}} - 1 = \left(3^{2^k}\right)^2 - 1$

Using (i), we get :

LHS of P(k + 1) = $(m2^{k+2} + 1)^2 - 1$

$= m^2\, 2^{2k+4} + 2m.2^{k+2}$

$= 2^{k+3}\, (m^2\, 2^{k+1} + m)$

$= p\, 2^{k+3}$ where p is an odd number because $m^2\, 2^{k+1}$ is even and m is odd.

$\Rightarrow$ P(k + 1) is divisible by 2^{k+3} , but not divisible by 2^{k+4} as p is odd

$\Rightarrow$ P(k + 1) is true

Hence, by mathematical induction, P(n) is true for all $n \in N$

Example : 50

Using mathematical induction, prove that : ${}^mC_0\, {}^nC_k + {}^mC_1\, {}^nC_{k-1} + {}^mC_2\, {}^nC_{k-2} + ... + {}^mC_k\, {}^nC_0 = {}^{m+n}C_k$ for p < q, where m, n and k are possible integers and ${}^pC_q = 0$ for p < q.

Solution

First apply mathematical induction on n

Let P(n) : ${}^mC_0\, {}^nC_k + {}^mC_1\, {}^nC_{k-1} + {}^mC_2\, {}^nC_{k-2} + ... + {}^mC_k\, {}^nC_0 = {}^{m+n}C_k$

Consider P(1)

LHS of P(1) = ${}^mC_{k-1}\, {}^1C_1 + {}^mC_k\, {}^1C_0 = {}^{m+1}C_k$ = RHS of P(1)

$\Rightarrow$ P(1) is true

Assume that P(n) is true for n = s

i.e. P(s) : ${}^mC_0\, {}^sC_k + {}^mC_1\, {}^sC_{k-1} + {}^mC_2\, {}^sC_{k-2} + ... + {}^mC_k\, {}^sC_0 = {}^{m+s}C_k$

Consider LHS of P(s + 1)

LHS of P(s + 1) = ${}^mC_0\, {}^{s+1}C_k + {}^mC_1\, {}^{s+1}C_{k-1} + {}^mC_2\, {}^{s+1}C_{k-2} + ... + {}^mC_k\, {}^{s+1}C_0$

$\Rightarrow$ LHS of P(s + 1) = ${}^mC_0\, ({}^sC_k + {}^sC_{k-1}) + {}^mC_1\, ({}^sC_{k-1} + {}^sC_{k-2}) + ... + {}^mC_k\, {}^{s+1}C_0$

$= [{}^mC_0\, {}^sC_k + {}^mC_1\, {}^sC_{k-1} + ... + {}^mC_k\, {}^sC_0] - [{}^mC_0\, {}^sC_{k-1} + {}^mC_1\, {}^sC_{k-2} + ... + {}^mC_{k-1}\, {}^sC_0]$

= P(s) + P(s)]where k is replaced by k – 1 in the P(s)

$\Rightarrow$ LHS of P(s + 1) = ${}^{m+s}C_k + {}^{m+s}C_{k-1} = {}^{m+s+1}C_k$ = RHS of P(s + 1)

$\Rightarrow$ P(n + 1) is true for all $n \in N$

Similarly we can show that the given statement is true for all $m \in N$.

Example : 51

Let $p \geq 3$ be an integer and α, β be the roots of $x^2 - (p + 1)x + 1 = 0$. Using mathematical induction, show that $\alpha^n + \beta^n$

(i) is an integer and

(ii) is not divisible by p

Solution

It is given that α and β are roots of $x^2 - (p + 1)x + 1 = 0$

$\Rightarrow$ $\alpha + \beta = p + 1$ and $\alpha\beta = 1$ ....(i)

(i) Let P(n) : $\alpha^n + \beta^n$ is an integer

P(1) : $\alpha + \beta = p + 1$ is an integer
As it is given that p is an integer, P(1) is true.
P(2) : $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (p + 1)^2 - 2$ is an integer.
As p is an integer, $(p + 1)^2 - 2$ is also an integer $\Rightarrow$ P(2) is true
Assume that both P(k) and P(k – 1) are true
i.e. $\alpha^k + \beta^k$ and $\alpha^{k-1} + \beta^{k-1}$ both are integers
Consider LHS of P(k + 1) i.e.
LHS of P(k + 1) = $\alpha^{k+1} + \beta^{k-1} = (\alpha - \beta)(\alpha^k + b^k) - \alpha\beta(\alpha^{k-1} + b^{k-1})$
$\Rightarrow$ LHS of P(k + 1) = p P(k) – P(k –1) [using (i)]
$\Rightarrow$ LHS of P(k + 1) = integer because p, P(k – 1) and P(k) all are integer
$\Rightarrow$ P(k + 1) is true. Hence P(n) is true for $n \in N$.

(ii) Let P(n) = $\alpha^n + \beta^n$ is not divisible by p
P(1) : $\alpha + \beta = p + 1$ = a number which is not divisible by p $\Rightarrow$ P(1) is true
P(2) : $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
$= (p + 1)^2 - 2 = p(p + 2) - 1$
= a number which is divisible by p – a number which is not divisible by p
= a number which is not divisible by p $\Rightarrow$ P(2) is true
P(3) : $\alpha^3 + b^3 = (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta) = (p + 1)[(p + 1)^2 - 3] = p[(p + 1)^2 - 3] + p(p + 2) - 2$
$= p[(p + 1)^2 + p - 1] - 2$
= a number which is divisible by p – a number which is not divisible by p
= a number which is not divisible p $\Rightarrow$ P(3) is true
Assume that P(k), P(k – 1) and P(k – 2) all are true
i.e. $\alpha^k + \beta^k$, α^{k-1} and $\alpha^{k+2} + \beta^{k-2}$ all are non-divisible by p.
Consider LHS of P(k + 1) i.e.
LHS of P(k + 1) = $\alpha^{k+1} + \beta^{k+1} = (\alpha + \beta)(\alpha^k + b^k) - \alpha\beta(\alpha^{k-1} + b^{k-1})$
$= p(\alpha^k - b^k) + (\alpha^k + b^k) - (\alpha^{k-1} + b^{k-1})$
$= p\,P(k) + [(p + 1)(\alpha^{k-1} - b^{k-1}) - (\alpha^{k+2} + b^{k-2})] - (\alpha^{k-1} + b^{k-1})$
= p P(k) + p P(k – 1) – P(k – 2)
= p[P(k) + P(k – 1)] – P(k – 2)
= a number which is divisible by p – a number which is not divisible by p
= a number which is not divisible by p
$\Rightarrow$ P(k + 1) is true
Hence, by principle of mathematical induction P(n) is true for all $n \in N$

Example : 52

Use mathematical induction to prove that $\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{(-1)^n}{x^{n+1}}\left(\log x - 1 - \frac{1}{2} - ...... - \frac{1}{n}\right)$ for all $n \in N$ and $x > 0$.

Solution

Let P(n) : $\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{(-1)^n}{x^{n+1}}\left(\log x - 1 - \frac{1}{2} - ...... - \frac{1}{n}\right)$

LHS of P(1) : $\frac{d}{dx}\left(\frac{\log x}{x}\right) = \frac{\frac{1}{x}\,.\,x - \log x}{x^2} = \frac{1 - \log x}{x^2}$

RHS of P(1) = $\frac{(-1)!}{x^2}$ (log x – 1) = $\frac{1 - \log x}{x^2}$

$\Rightarrow$ P(1) is true
Let us assume that P(k) is true i.e.

P(k) : $\frac{d^k}{dx^k}\left(\frac{\log x}{x}\right) = \frac{(-1)^k k!}{x^{k-1}}\left(\log x - 1 - \frac{1}{2} - ...... - \frac{1}{k}\right)$ ....(i)

Consider LHS of P(k + 1) i.e.

LHS of P(k + 1) = $\frac{d^{k+1}}{dx^{k+1}}\left(\frac{\log x}{x}\right) = \frac{d}{dx}\left[\frac{d^k}{dx^k}\left(\frac{\log x}{x}\right)\right]$

$= \frac{d}{dx}$ [LHS of P(k)] = $\frac{d}{dx}$ [RHS of P(k)] [using (1)]

$= \frac{d}{dx}\left[\frac{(-1)^k k!}{x^{k+1}}\left(\log x - 1 - \frac{1}{2} - .... - \frac{1}{k}\right)\right]$

$= \frac{(-1)^k k!(-1)(k+1)}{x^{k+2}}\left(\log x - 1 - \frac{1}{2} - .... - \frac{1}{k}\right) + \frac{(-1)^k k!}{x^{k+1}}\ \frac{1}{x}$

$= \frac{(-1)^{K+1}(K+1)!}{X^{K+2}}\left(\log x - 1 - \frac{1}{2} - ...... - \frac{1}{k+1}\right)$

⇒ P(k + 1) is true

Hence by principle of mathematical induction, P(n) is true for all n ∈ N

Example : 53

Use mathematical induction to prove that $\frac{d^n}{dx^n}$ (x^n log x) = n! $\left(\log x + 1 + \frac{1}{2} + ... + \frac{1}{n}\right)$ for all n ∈ N and x > 0.

Solution

Let P(n) : $\frac{d^n}{dx^n}$ (x^n log x) = n! $\left(\log x + 1 + \frac{1}{2} + ... + \frac{1}{n}\right)$

LHS of P(1) = $\left(\frac{d}{dx}\right)$ (x log x) = log x + $\frac{x}{x}$ = log x + 1

RHS of P(1) = 1! (log x + 1) = log x + 1

⇒ P(1) is true

Let us assume that P(k) is true i.e.

P(k) : $\frac{d^k}{dx^k}$ (x^k log x) = k! $\left(\log x + 1 + \frac{1}{2} + ... + \frac{1}{k}\right)$ ....(i)

Consider LHS of P(k + 1) i.e.

LHS of P(k + 1) = $\frac{d^{k+1}}{dx^{k+1}}$ (x^{k+1} logx)

$= \frac{d^k}{dx^k}\left[\frac{d}{dx}(x^{k+1}\log x)\right]$

$= \frac{d^k}{dx^k}\left[(k+1)x^k \log x + \frac{x^{k+1}}{x}\right]$

= (k + 1) $\frac{d^k}{dx^k}$ [x^k log x] + $\frac{d^k}{dx^k}\left[\frac{x^{k+1}}{x}\right]$

= (k + 1) $\left[k!(\log x + 1 + \frac{1}{2} + ... + \frac{1}{k}\right]$ + k! [using (i)]

= (k + 1)! $\left[k!(\log x + 1 + \frac{1}{2} + ... + \frac{1}{k+1}\right]$

$\Rightarrow$ P(k + 1) is true

Hence by principle of mathematical induction, P(n) is true for all $n \in N$

Example : 54

Use mathematical induction to prove $\int_0^{\pi/2} \frac{\sin^2 nx}{\sin x}\,dx = 1 + \frac{1}{3} + \frac{1}{5} + \dots\dots\dots + \frac{1}{2n-1}$ for all $n \in N$.

Solution

Consider $I_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin x}\,dx = 1 + \frac{1}{3} + \frac{1}{5} + \dots\dots\dots + \frac{1}{2n-1}$

from left hand side, $I_1 = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin x}\,dx = \int_0^{\pi/2} \sin x\,dx = 1$

from right hand side, $I_1 = 1$

$\Rightarrow$ I_1 is true

Assume that I_k is true i.e.

$$I_k = \int_0^{\pi/2} \frac{\sin^2 kx}{\sin x}\,dx = 1 + \frac{1}{3} + \frac{1}{5} + \dots\dots + \frac{1}{2k-1} \qquad \dots\dots\dots\text{(i)}$$

Consider $I_{k-1} - I_k = \int_0^{\pi/2} \frac{\sin^2 (k+1)x}{\sin x}\,dx - \int_0^{\pi/2} \frac{\sin^2 kx}{\sin x}\,dx$

$$\Rightarrow \quad I_{k+1} - I_k = \int_0^{\pi/2} \frac{\sin^2(k+1)x - \sin^2 kx}{\sin x}\,dx = \int_0^{\pi/2} \frac{\sin(2k+1)x \sin x}{\sin x}\,dx$$

$$= \int_0^{\pi/2} \sin(2k+1)x\,dx = -\frac{\cos(2k+1)x}{2k+1}\Bigg]_0^{\pi/2} = \frac{1}{2k+1}$$

$\Rightarrow$ $I_{k+1} = I_k + \frac{1}{2k+1}$ $\qquad \Rightarrow$ $I_{k+1} = I_k + \frac{1}{2k+1}$

$\Rightarrow$ $I_{k+1} = 1 + \frac{1}{3} + \frac{1}{5} + \dots\dots\dots + \frac{1}{2k-1} + \frac{1}{2k+1}$ [using (i)]

$\Rightarrow$ I_{k+1} is true.

Hence by principle of mathematical induction I_n is true for all values of $n \in N$

Example : 55

Let $I_n = \int_0^{\pi} \frac{1-\cos nx}{1-\cos x}\,dx$. Use mathematical induction to prove that $I_n = n\pi$ for all n = 0, 1, 2, 3, ...

Solution

We have to prove $I_n = \int_0^{\pi} \frac{1-\cos nx}{1-\cos x}\,dx = n\pi$

For n = 0

$$I_0 = \int_0^{\pi} \frac{1-\cos\theta}{1-\cos x}\,dx = \int_0^{\pi} 0\,dx = 0.$$

The value of the integral from the RHS = 0 × π = 0

⇒ The given integral is true for n = 0

From n = 1

$$I_1 = \int_0^{\pi} \frac{1-\cos x}{1-\cos x}\,dx = \int_0^{\pi} dx = \pi$$

The value of the integral from the RHS = 1 × π = π

⇒ The given integral is true for n = 1

Assume that the given integral is true for n = k – 1 and n = k i.e.

$$I_{k-1} = \int_0^{\pi} \frac{1-\cos(k-1)x}{1-\cos x}\,dx = (k-1)\,\pi \qquad \text{.........(i)}$$

$$I_k = \int_0^{\pi} \frac{1-\cos kx}{1-\cos x}\,dx = k\pi \qquad \text{.........(ii)}$$

Consider $I_{k+1} - I_k = \int_0^{\pi} \frac{\cos kx - \cos(k+1)x}{1-\cos x}\,dx$

$$\Rightarrow \quad I_{k+1} - I_k = \int_0^{\pi} \frac{2\sin\frac{x}{2}\sin\frac{2k+1}{2}x}{2\sin^2\frac{x}{2}}\,dx = \int_0^{\pi} \frac{\sin\frac{2k+1}{2}x}{\sin\frac{x}{2}}\,dx \qquad \text{..........(iii)}$$

Consider $I_k - I_{k-1} = \int_0^{\pi} \frac{\cos(k-1)x - \cos kx}{1-\cos x}\,dx$

$$\Rightarrow \quad I_k - I_{k-1} = \int_0^{\pi} \frac{2\sin\frac{x}{2}\sin\frac{2k-1}{2}x}{2\sin^2\frac{x}{2}}\,dx = \int_0^{\pi} \frac{\sin\frac{2k-1}{2}x}{\sin\frac{x}{2}}\,dx \qquad \text{..........(iv)}$$

Subtracting (iv) from (iii), we get :

$$I_{k+1} - 2\,I_k + I_{k-1} = \int_0^{\pi} \frac{\sin\frac{2k+1}{2}x - \sin\frac{2k-1}{2}x}{\sin\frac{x}{2}}\,dx$$

$$\Rightarrow \quad I_{k-1} - 2\,I_k + I_{k-1} = \int_0^{\pi} \frac{2\cos kx\sin\frac{x}{2}}{\sin\frac{x}{2}}\,dx = 2\int_0^{x} \cos kx\,dx = 2\left.\frac{\sin kx}{k}\right]_0^{\pi} = 0$$

⇒ $I_{k+1} = 2I_k - I_{k-1} = 2k\pi - (k-1)\pi$ [using (i) and (ii)]

⇒ $I_{k+1} = (k+1)\,\pi$

⇒ The given integral is true for n = k + 1

Hence, by principle of mathematical induction, the given integral is true for all n = 0, 1, 2, 3, ...

Coordinate Geometry

Example : 1

Find the value of t so that the points (1, 1), (2, –1), (3, –2) and (12, t) are concyclic.

Solution

Let A ≡ (1, 1) B ≡ (2, –1) C ≡ (3, –2) D ≡ (12, t)

We will find the equation of the circle passing through A, B and C and then find t so that D lies on that circle. Any circle passing through A, B can be taken as :

$$(x - 1)(x - 2) + (y - 1)(y + 1) + k \begin{vmatrix} x & y & 1 \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 0$$

$\Rightarrow$ $x^2 + y^2 - 3x + 1 + (2x + y - 3) = 0$

C ≡ (3, –2) lies on this circle.

$\Rightarrow$ $9 + 4 - 9 + 1 + k(6 - 2 - 3) = 0$

$\Rightarrow$ $k = -5$

$\Rightarrow$ circle through A, B and C is :

$x^2 + y^2 - 3x + 1 - 5(2x + y - 3) = 0$

$x^2 + y^2 - 12x - 5y + 16 = 0$

D ≡ (12, t) will lie on this circle if :

$\Rightarrow$ $144 + t^2 - 156 - 5t + 16 = 0$

$\Rightarrow$ $t^2 - 5t + 4 = 0$

$\Rightarrow$ $y = 1, 4$

$\Rightarrow$ for t = 1, 4 the points are concyclic

Example : 2

Find the equation of a circle touching the line x + 2y = 1 at the point (3, –1) and passing through the point (2, 1).

Solution

The equation of any circle touching x + 2y – 1 = 0 at the point (3, –1) can be taken as :

$(x - 3)^2 + (y - 1)^2 + k(x - 2y - 1) = 0$ (using result 5 from family of circles)

As the circle passes through (2, –1) :

$(2 - 3)^2 + (1 + 1)^2 + k(2 + 2 - 1) = 0$

$\Rightarrow$ $k = -5/3$

$\Rightarrow$ the required circle is : $3(x^2 + y^2) - 23x - 4y + 35 = 0$

Notes :

1. Let A ≡ (3, –1) and B ≡ (2, 1)

Let L_1 be the line through A perpendicular to x + 2y = 1. Let L_2 be the right bisector of AB. The centre of circle is the point of intersection of L_1 and L_2 . The equation of the circle can be found by this method also.

2. Let (h, k) be the centre of the circle.

The centre (h, k) can be found from these equations.

$$\Rightarrow \quad \frac{|h + 2k - 1|}{\sqrt{5}} = \sqrt{(h - 3)^2 + (k + 1)^2} = \sqrt{(h - 2)^2 + (k - 1)^2}$$

The centre (h, k) can be found from these equations

Example : 3

Find the equation of a circle which touches the Y-axis at (0, 4) and cuts an intercept of length 6 units on X-axis.

Solution

The equation of circle touching x = 0 at (0, 4) can be taken as :

$(x - 0)^2 + (y - 4)^2 + k(x) = 0$

$x^2 + y^2 + kx - 8y + 16 = 0$

The circle cuts X-axis at points $(x_1, 0)$ and $(x_2, 0)$ given by :

$x^2 + kx + 16 = 0$

X-intercept = difference of roots of this quadratic :

$6 = |x_2 - x_1|$

$\Rightarrow$ $36 = (x_2 + x_1)^2 - 4x_2 x_1$

$\Rightarrow$ $36 = k_2 - 4\ (16)$

$\Rightarrow$ $k = \pm 10$

Hence the required circle is : $x^2 + y^2 \pm 10x - 8y + 16 = 0$

Note :

1. If a circle of radius r touches the X-axis at (1, 0), the centre of the circle is (a, ± r)

2. If a circle of radius r touches the Y-axis at (0, b), the centre of the circle is (± r, b).

Example : 4

Find the equation of the circle passing through the points (4, 3) and (3, 2) and touching the line $3x - y - 17 = 0$

Solution

Using result 4 from the family of circles, any circle passing through $A \equiv (4, 3)$ and $B \equiv (3, 2)$ can be taken as :

$$(x-4)(x-3) + (y-3)(y-2) + k \begin{vmatrix} x & y & 1 \\ 4 & 3 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 0$$

$x^2 + y^2 - 7x - 5y + 18 + k(x - y - 1) = 0$

This circle touches $3x - y - 17 = 0$

centre $\equiv \left(\frac{7-k}{2}, \frac{k+5}{2}\right)$ and radius $= \sqrt{\frac{(7-k)^2}{4} + \frac{(k+5)^2}{4} - (18-k)}$

For tangency, distance of centre from line $3x - y - 17 = 0$ is radius

$$\Rightarrow \quad \frac{\left|3\left(\frac{7-k}{2}\right) - \left(\frac{k+5}{2}\right) - 17\right|}{\sqrt{9+1}} = \sqrt{\frac{(7-k)^2}{4} + \frac{(k+5)^2}{4} - 18 + k}$$

$$\Rightarrow \quad \left(\frac{-4k-18}{\sqrt{10}}\right)^2 = (7-k)^2 + (k+5)^2 - 72 + 4k$$

$\Rightarrow$ $4(4k^2 + 81 + 36k) = 10(2k^2 + 2)$

$\Rightarrow$ $k^2 - 36k - 76 = 0$ $\Rightarrow$ $k = -2, 38$

$\Rightarrow$ there are two circles through A and B and touching $3x - y - 17 = 0$. The equation are :

$x^2 + y^2 - 7x - 5y + 18 - 2(x - y - 1) = 0$ and

$x^2 + y^2 - 7x - 5y + 18 + 38(x - y - 1) = 0$

$\Rightarrow$ $x^2 + y^2 - 9x - 3y + 20 = 0$ and

$x^2 + y^2 + 31x - 43y - 20 = 0$

Notes :

1. Let $C \equiv (h, k)$ be the centre of required circle and $M \equiv (7/2, 5/2)$ be the mid point of AB.

C lies on right bisector of AB

$\Rightarrow$ slope (CM) = slope (AB) = – 1

$$\Rightarrow \quad \left(\frac{k-5/2}{h-7/2}\right) \times (1) = -1$$

Also CA = distance of centre from $(3x - y - 17 = 0)$

$$\Rightarrow \quad \sqrt{(h-4)^2 (k-3)^2} = \frac{|3h-k-17|}{\sqrt{10}}$$

We can get h, k from these two equations.

Example : 5

Find the points on the circle $x^2 + y^2 = 4$ whose distance from the line $4x + 3y = 12$ is 4/5 units

Solution

Let A, B be the points on $x^2 + y^2 = 4$ lying at a distance 4/5 from $4x + 3y = 12$

$\Rightarrow$ AB will be parallel to $4x + 3y = 12$

Let the equation of AB be : $4x + 3y = c$

distance between the two lines is : $\dfrac{|c-12|}{\sqrt{9+16}} = \dfrac{4}{5}$

$\Rightarrow$ c = 16, 8

$\Rightarrow$ the equation of AB is : $5x + 3y = 8$ and $4x + 3y = 16$

The points A, B can be found by solving for points of intersection of x2 + y2 = 4 with AB.

AB $\equiv$ $(4x + 3y - 8 = 0)$

$$\Rightarrow \quad x^2 + \left(\frac{8-4x}{3}\right)^2 = 4$$

$\Rightarrow$ $25x^2 - 64x + 28 = 0$

$\Rightarrow$ x = 2, 14/25

$\Rightarrow$ y = 0, 448/25

AB $\equiv$ $(4x + 3y - 16 = 0)$

$$\Rightarrow \quad x^2 + \left(\frac{16-4x}{3}\right)^2 = 4$$

$\Rightarrow$ $25x^2 - 128x + 220 = 0$

$\Rightarrow$ $D < 0$ $\Rightarrow$ no real roots

Hence there are two points on circle at distance 4/5 from line.

A $\equiv$ (2, 0) and B $\equiv$ (14/25, 48/25)

Alternate Method :

Let P $\equiv$ (2 cos,2 sin) be the point on the circle $x^2 + y^2 = 4$ distant 4/5 from given line.

The distance from line = 4/5.

$$\Rightarrow \quad \frac{|4(2\cos\theta) + 3(2\sin\theta) - 12|}{5} = \frac{4}{5}$$

Solve for θ to get the point P.

Example : 6

Find the equation of circle passing through (–2, 3) and touching both the axes.

Solution

As the circle touches both the axes and lies in the IInd quadrant, its centre is :

C $\equiv$ (–r, r), where r is the radius

Distance of centre from (–2, 3) = radius

$$\Rightarrow \quad \sqrt{(r-2)^2 + (3-r)^2} = r$$

$\Rightarrow$ $r = 5 \pm 2\sqrt{3}$

$\Rightarrow$ the circles are : $(x + r)^2 + (y - r)^2 = r^2$

$\Rightarrow$ $x^2 + y^2 + 2(5 \pm 2\sqrt{3})\, x - 2\,(5 \pm 2\sqrt{3})\, y + (5 \pm 2\sqrt{3})^2 = 0$

Example : 7

Tangents PA and PB are drawn from the point P(h, k) to the circle $x^2 + y^2 = a^2$. Find the equation of circumcircle of ΔPAB and the area of ΔPAB

Solution

AB is the chord of contact for point P.

Equation of AB is : $hx + ky = a^2$

The circumcircle of ΔPAB passes through the intersection of circle

$x^2 + y^2 - a^2 = 0$ and the line $hx + ky - a^2 = 0$

Using S + k L = 0, we can write the equation of the circle as :
$(x^2 + y^2 - a^2) + k(hx + ky - a^2) = 0$ where k is parameter
As this circle passes through P(h, k) ;
$\Rightarrow \quad h^2 + k^2 - a^2 + k(h^2 + k^2 - a^2) = 0$
$\Rightarrow \quad k = -1$
The circle is $x^2 + y^2 - hx - ky = 0$
Area of $\Delta PAB = 1/2\ (PM) \times (AB)$ (PM is perpendicular to AB)

PM = distance of P from AB = $\dfrac{|h^2 + k^2 - a^2|}{\sqrt{h^2 + k^2}}$

PA = length of tangent from P = $\sqrt{h^2 + k^2 - a^2}$

Area = $\dfrac{1}{2}$ PM $\left[2\sqrt{PA^2 - PM^2}\right]$ = PM $\sqrt{PA^2 - PM^2}$

Area = $\dfrac{|h^2 + k^2 - a^2|}{\sqrt{h^2 + k^2}} \; \dfrac{a\left(\sqrt{h^2 + k^2 - a^2}\right)}{\sqrt{h^2 + k^2}}$

Area = $\dfrac{a(h^2 + k^2 - a^2)^{3/2}}{h^2 + k^2}$

Note that $h^2 + k^2 - a^2 > 0 \quad \because \quad$ (h, k) lies outside the circle

Example : 8

Examine if the two circles $x^2 + y^2 - 8y - 4 = 0$ and $x^2 + y^2 - 2x - 4y = 0$ touch each other. Find the point of contact if they touch.

Solution

For $\quad x^2 + y^2 - 2x - 4y = 0 \quad$ centre $C_1 \equiv (1, 2)$
and $\quad x^2 + y^2 - 8y - 4 = 0 \quad$ centre $C_2 \equiv (0, 4)$

using $r = \sqrt{g^2 + f^2 - c}$: $\quad r_1 = \sqrt{5} \quad$ and $r_2 = 2\sqrt{5}$

Now $\quad C_1 C_2 = \sqrt{(0-1)^2 + (4-2)^2} = \sqrt{5}$

$\Rightarrow \quad r_2 - r_1 = 2\sqrt{5} - \sqrt{5} = \sqrt{5}$
$\Rightarrow \quad C_1 C_2 = r_2 - r_1$
$\Rightarrow \quad$ the circle touch internally
For point of contact :

Let P(x, y) be the point of contact. P divides $C_1 C_2$ externally in the ratio of $\sqrt{5} : 2\sqrt{5} \equiv 1 : 2$
using section formula, we get :

$x = \dfrac{1(0) - 2(1)}{1 - 2} = 2$

$y = \dfrac{1(4) - 2(2)}{1 - 2} = 0$

$\Rightarrow \quad P(x, y) \equiv (2, 0)$ is the point of contact

Example : 9

Find the equation of two tangents drawn to the circle $x^2 + y^2 - 2x + 4y = 0$ from the point (0, 1)

Solution

Let m be the slope of the tangent. For two tangents there will be two values of m which are required
As the tangent passes though (0, 1), its equation will be :
$y - 1 = m(x - 0) \quad \Rightarrow \quad mx - y + 1 = 0$

Now the centre of circle $(x^2 + y^2 - 2x + 4y = 0) \equiv (1, -2)$ and $r = \sqrt{5}$
So using the condition of tangency : distance of centre (1, –2) from line = radius (r)

$$\frac{|m(1)-(-2)+1|}{\sqrt{m^2+1}} = \sqrt{5}$$

⇒ $(3 + m)^2 = 5(1 + m^2)$ ⇒ $m = 2, -1/2$

⇒ equations of tangents are :

$2x - y + 1 = 0$ (slope = 2) and $x + 2y - 2 = 0$ (slope = $-1/2$)

Example : 10

Find the equations of circles with radius 15 and touching the circle $x^2 + y^2 = 100$ at the point (6, –8).

Solution

Case – 1 :

If the require circle touches $x^2 + y^2 = 100$ at (6, –8) externally, then P(6, –8) divides OA in the ratio 2 : 3 internally.

Let centre of the circle be (h, k). Now using section formula :

⇒ $$\frac{2k+3(0)}{2+3} = 6$$

⇒ $$\frac{2k+3(0)}{2+3} = -8$$

⇒ k = 15 and k = – 20

⇒ $(x - 15)^2 + (y + 20)^2 = 225$ is the required circle.

Case – 2 :

If the required circle touches $x^2 + y^2 = 100$ at (6, –8) internally, then P(6, –8) divides OA in the ratio 2 : 3 externally. Let centre of the circle be (h, k). Now using section formula :

⇒ $$\frac{2h-3(0)}{2-3} = 6$$

⇒ $$\frac{2k-3(0)}{2-3} = -8$$

⇒ h = – 3 and k = 4

⇒ $(x + 3)^2 + (y - 4)^2 = 225$ is the required circle.

Example : 11

For what values of m, will the line y = mx does not intersect the circle $x^2 + y^2 + 20x + 20y + 20 = 0$?

Solution

If the line y = mx does not intersect the circle, the perpendicular distance of the line from the centre of the circle must be greater than its radius.

Centre of circle ≡ (–10, –10) ; radius $r = 6\sqrt{5}$

distance of line mx – y = 0 from (–10, –10) = $\dfrac{|m(-10)-(-10)|}{\sqrt{m^2+1}}$

⇒ $$\frac{|10-10m|}{\sqrt{m^2+1}} > 6\sqrt{5}$$

⇒ $(2m + 1)(m + 2) < 0$

⇒ $-2 < m < -1/2$

Example : 12

Find the equation of circle passing through (–4, 3) and touching the lines x + y = 2 and x – y = 2.

Solution

Let (h, k) be the centre of the circle. The distance of the centre from the given line and the given point must be equal to radius

⇒ $$\frac{|h+k-2|}{\sqrt{2}} = \frac{|h-k-2|}{\sqrt{2}} = \sqrt{(h+4)^2+(k-3)^2}$$

Consider $\dfrac{|h+k-2|}{\sqrt{2}} = \dfrac{|h-k-2|}{\sqrt{2}}$

$\Rightarrow$ h + k – 2 = ± (h – k – 2)

Case 1 : (k = 0)

$$\frac{|h-2|}{\sqrt{2}} = \sqrt{(h+4)^2+9}$$

$(h-2)^2 = 2(h+4)^2 + 18 \quad \Rightarrow \quad h^2 + 20h + 46 = 0$

$\Rightarrow \quad h = -10 \pm 3\sqrt{6}$

radius = $\dfrac{|h+k-2|}{\sqrt{2}} = \left|\dfrac{-12 \pm 3\sqrt{6}}{\sqrt{2}}\right|$

$\Rightarrow \quad$ circle is : $(x + 10 \mp 3\sqrt{6})^2 + (y-0)^2 = \dfrac{(-12 \pm 3\sqrt{6})^2}{2}$

$\Rightarrow \quad x^2 + y^2 + 2(10 \pm 3\sqrt{6})x + (10 \pm 3\sqrt{6})^2 - \dfrac{(-12 \pm 3\sqrt{6})^2}{2} = 0$

$\Rightarrow \quad x^2 + y^2 + 2(10 \pm 3\sqrt{6})x + 55 \pm 24\sqrt{6} = 0$

Case – 2 : (h = 2)

$$\frac{|k|}{\sqrt{2}} = \sqrt{36+(k-3)^2}$$

$\Rightarrow \quad k^2 = 72 + 2(k-3)^2 \quad \Rightarrow \quad k^2 - 12k + 90 = 0$

The equation has no real roots. Hence no circle is possible for h = 2

Hence only two circles are possible (k = 0)

$$x^2 + y^2 + 2(10 \pm 3\sqrt{6})x + 55 \pm 24\sqrt{6} = 0$$

Example : 13

The centre of circle S lies on the line 2x – 2y + 9 = 0 and S cuts at right angles the circle $x^2 + y^2 = 4$. Show that S passes through two fixed points and find their coordinates.

Solution

Let the circle S be : $\quad x^2 + y^2 + 2gx + 2fy + c = 0$

centre lies on 2x – 2y + 9 = 0

$\Rightarrow \quad -2g + 2f + 9 = 0 \quad$ ....(i)

S cuts $x^2 + y^2 - 4 = 0$ orthogonally,

$\Rightarrow \quad 2g(0) + 2f(0) = c - 4$

$\Rightarrow \quad c = 4 \quad$ ....(ii)

Using (i) and (ii) the equation of S becomes :

$x^2 + y^2 + (2f + 9)x + 2fy + 4 = 0$

$\Rightarrow \quad (x^2 + y^2 + 9x + 4) + f(2x + 2y) = 0$

We can compare this equation with the equation of the family of circle though the point of intersection of a circle and a line (S + fL = 0, where f is a parameter).

Hence the circle S always passes through two fixed points A and B which are the points of intersection of $x^2 + y^2 + 9x + 4 = 0$ and $2x + 2y = 0$

Solving these equations, we get :

$x^2 + x^2 + 9x + 4 = 0$

$\Rightarrow \quad x = -4, -1/2 \quad \Rightarrow \quad y = 4, 1/2$

$\Rightarrow \quad A \equiv (-4, 4) \quad$ and $\quad B \equiv (-1/2, 1/2)$

Example : 14

A tangent is drawn to each of the circle $x^2 + y^2 = a^2$, $x^2 + y^2 = b^2$. Show that if the two tangents are perpendicular to each other, the locus of their point of intersection is a circle concentric with the given circles.

Solution

Let $P \equiv (x_1, y_1)$ be the point of intersection of the tangents PA and PB where A, B are points of contact with the two circles respectively.

As PA perpendicular to PB, the corresponding radii OA and OB are also perpendicular.

Let $\angle AOX = \theta$

$\Rightarrow \quad \angle BOX = \theta + 90^o$

Using the parametric form of the circles we can take :

$A \equiv (a \cos \theta , a \sin \theta)$

$B \equiv [b \cos (\theta + 90^o), b \sin (\theta + 90^o)]$

$B \equiv (-b \sin \theta, b \cos \theta)$

The equation of PA is : $x (a \cos \theta) + y (a \sin \theta) = a^2$

$\Rightarrow \quad x \cos \theta + y \sin \theta = a$

The equation of PB is :

$x(-b \sin \theta) + y (b \cos \theta) = b^2$

$\Rightarrow \quad y \cos \theta - x \sin \theta = b$

$\Rightarrow \quad P \equiv (x_1, y_1)$ lies on PA and PB both

$\Rightarrow \quad x_1 \cos \theta + y_1 \sin \theta = a$ and $y_1 \cos \theta - x_1 \sin \theta = b$

As θ is changing quantity (different for different positions of P), we will eliminate.

Squaring and adding, we get :

$x_1^2 + y_1^2 = a^2 + b^2$

$\Rightarrow$ the locus of P is $x^2 + y^2 = a^2 + b^2$ which is concentric with the given circles.

Example : 15

Secants are drawn from origin to the circle $(x - h)^2 + (y - k)^2 = r^2$. Find the locus of the mid-point of the portion of the secants intercepted inside the circle.

Solution

Let $C \equiv (h, k)$ be the centre of the given circle and $P \equiv (x_1 , y_1)$ be the mid-point of the portion AB of the secant OAB.

$\Rightarrow \quad CP \perp AB$

$\Rightarrow$ slope (OP) × slope (CP) = – 1

$$\Rightarrow \quad \left(\frac{y_1 - 0}{x_1 - 0}\right) \times \left(\frac{y_1 - k}{x_1 - h}\right) = -1$$

$\Rightarrow \quad x_1^2 + y_1^2 - hx_1 - ky_1 = 0$

$\Rightarrow$ the locus of the point P is : $x^2 + y^2 - hx - ky = 0$

Example : 16

The circle $x^2 + y^2 - 4x - 4y + 4 = 0$ is inscribed in a triangle which has two of its sides along the coordinate axes. The locus of the circumcentre of the triangle is $x + y - xy + k (x^2 + y^2)^{1/2} = 0$. Find value of k.

Solution

The given circle is $(x - 2)^2 + (y - 2)^2 = 4$

$\Rightarrow$ centre = (2, 2) and radius = 2

Let OAB be the triangle in which the circle is inscribed. As ΔOAB is right angled, the circumcentre is mid-point of AB.

Let $P \equiv (x_1, y_1)$ be the circumcentre.

$\Rightarrow \quad A \equiv (2x_1, 0)$ and $B \equiv (0, 2y_1)$

$$\Rightarrow \quad \text{the equation of AB is : } \frac{x}{2x_1} + \frac{y}{2y_1} = 1$$

As ΔAOB touches the circle, distance of C from AB = radius

$$\Rightarrow \qquad \frac{\left|\frac{2}{2x_1}+\frac{2}{2y_1}-1\right|}{\sqrt{\frac{1}{4x_1^2}+\frac{1}{4y_1^2}}} = 2 \qquad ....(i)$$

As the centre (2, 2) lies on the origin side of the line $\frac{x}{2x_1} + \frac{y}{2y_1} - 1 = 0$

the expression $\frac{2}{2x_1} + \frac{2}{2y_1} - 1$ has the same sign as the constant term (–1) in the equation

$\Rightarrow \qquad \frac{2}{2x_1} + \frac{2}{2y_1} - 1$ is negative

$\Rightarrow$ equation (i) is : $\quad -\left(\frac{2}{2x_1}+\frac{2}{2y_1}-1\right) = 2\sqrt{\frac{1}{4x_1^2}+\frac{1}{4y_1^2}}$

$\Rightarrow \qquad -(x_1 + y_1 - x_1y_1) = \sqrt{x_1^2+y_1^2}$

$\Rightarrow$ the locus is : $x + y - xy + \sqrt{x^2+y^2} = 0$

$\Rightarrow \qquad k = 1$

Alternate Solution

We know, $\quad r = \Delta/S$ where r is inradius, Δ is the area triangle and S is the semi-perimeter

$$\Rightarrow \qquad 2 = \frac{\frac{1}{2}(2x_1)(2y_1)}{\frac{2x_1+2y_1+\sqrt{4x_1^2+4y_1^2}}{2}}$$

$$\Rightarrow \qquad 2 = \frac{\frac{1}{2}(2x_1)(2y_1)}{\frac{2x_1+2y_1+\sqrt{4x_1^2+4y_1^2}}{2}}$$

$$\Rightarrow \qquad 2 = \frac{2x_1y_1}{x_1+y_1+\sqrt{x_1^2+y_1^2}}$$

$\Rightarrow \qquad x_1 + y_1 - x_1y_1 + \sqrt{x_1^2+y_1^2} = 0$

$\Rightarrow$ the locus is : $x + y - xy = \sqrt{x^2+y^2} = 0 \quad \Rightarrow \quad k = 1$

Example : 17

A and B are the points of intersection of the circles $x^2 + y^2 + 2ax - c^2 = 0$ and $x^2 + y^2 + 2bx - c^2 = 0$. A line through A meets one circle at P. Another line parallel to AP but passing through B cuts the other circle at Q. Find the locus of the mid-point of PQ.

Solution

Let us solve for the point of intersection A and B

$x^2 + y^2 + 2ax - c^2 = 0$ and $x^2 + y^2 + 2bx - c^2 = 0$

$\Rightarrow \quad x = 0$ and $y = \pm c$

$\Rightarrow \quad A \equiv (0, c)$ and $B \equiv (0, -c)$

Let the equation of AP be : $y = mx + c$, where m is changing quantity and c is fixed quantity (Y-intercept)

$\Rightarrow$ the equation BQ is : $y = mx - c$ (AP || BQ)

Coordinates of P, Q :

Solve y = mx + c and $x^2 + y^2 + 2ax - c^2 = 0$

$\Rightarrow$ $x^2 (mx + c)^2 + 2ax + c^2 = 0$

$\Rightarrow$ $x = -\dfrac{2(a+mc)}{1+m^2}$ and $x = 0$

$\Rightarrow$ $y = -\dfrac{2m(a+mc)}{1+m^2} + c$ and $y = c$

$\Rightarrow$ $P = \left[\dfrac{2(a+mc)}{1+m^2}, -\dfrac{2m(a+mc)}{1+m^2}+c\right]$

Similarly the coordinates Q are :

$\Rightarrow$ $Q \equiv \left[-\dfrac{2(b-mc)}{1+m^2}, -\dfrac{2m(b-mc)}{1+m^2}-c\right]$

mid-point of PQ is :

$$\left[-\frac{(a+b)}{1+m^2}, -\frac{m(a+b)}{1+m^2}\right] \equiv (x_1, y_1)$$

$\Rightarrow$ $x_1 = -\dfrac{(a+b)}{1+m^2}$; $y_1 = -\dfrac{m(a+b)}{1+m^2}$

Elimiate m to get the locus of the midpoint

$x_1^2 + y_1^2 = -(a + b)\, x_1$

$\Rightarrow$ $x^2 + y^2 + (a + b)x = 0$ is the locus

Example : 18

Find the equation of the circumcircle of the triangle having x + y = 6, 2x + y = 4 and x + 2y = 5 as its sides.

Solution

Consider the following equation :

$(x + y - 6)(2x + y - 4) + \lambda (2x + y - 4)(x + 2y - 5) + \mu (x + 2y - 5)(x + y - 6) = 0$ ....(i)

Equation (i) represents equation of curve passing through the intersection of the three lines taken two at a time (i.e. passes through the vertices of the triangle). For this curve to represent a circle,

Coefficient of x^2 = Coefficient of y^2 and Coefficient of xy = 0

$\Rightarrow$ $2 + 2\lambda + \mu = 1 + 2\lambda + 2\mu$ ....(ii)

and $3 + 5\lambda + 3\mu = 0$ ....(iii)

Solving (ii) and (iii), we get $\lambda = -6/5$ and $\mu = 1$

Putting values of λ and μ in (i), we get :

$(x + y - 6)(2x + y - 4) - 6/5\,(2x + y - 4)(x + 2y - 5) + 1\,(x + 2y - 5)(x + y - 6) = 0$

$\Rightarrow$ $x^2 + y^2 - 17x - 19y + 50 = 0$

Hence equation of circumcircle of the triangle is : $x^2 + y^2 - 17x - 19y + 50 = 0$

Example : 19

Find the equation of the circle passing through the origin and through the points of contact of tangents from the origin to the circle $x^2 + y^2 - 11x + 13y + 17 = 0$

Solution

Let $S = x^2 + y^2 - 11x + 13y + 17 = 0$

Equation of the chord of contact of circle S with respect to the point (0, 0) is

$L \equiv -11x + 13y + 34 = 0$

Equation of family of circles passing through the intersection of circle S and chord of contact L is

$S + kL = 0$

$\Rightarrow$ $x^2 + y^2 - 11x + 13y + 17 + k(-11x + 13y + 34) = 0$ ....(i)

Since required circle passes through the origin, find the member of this family that passes through the origin

i.e. Put (0, 0) and find corresponding value of k.

$\Rightarrow$ $0^2 + 0^2 - 11 \times 0 + 13 \times 0 + 17 + k(-11 \times 0 + 13 \times 0 + 34) = 0$

$\Rightarrow$ $k = -1/2$

Put k = –1/2 in (i) to get equation of the required circle i.e. $2x^2 + 2y^2 - 11x + 13y = 0$

Alternate Solution

Let centre of the circle S be C. As points of contact, origin and C form a cyclic quadrilateral, OC must be the diameter of the required circle.

$C \equiv (11/2, -13/2)$ and $O \equiv (0, 0)$

Apply diametric form to get the equation of the required circle,

i.e. $(x - 11/2)(x - 0) + (y + 13/2)(y - 0) = 0$

$\Rightarrow \quad 2x^2 + 2y^2 - 11x + 13y = 0$

Hence required circle is : $2x^2 + 2y^2 - 11x + 13y = 0$

Example : 20

If $\left(m_i, \frac{1}{m_i}\right)$, $m_i > 0$ for i = 1, 2, 3, 4 are four distinct points on a circle. Show that $m_1m_2m_3m_4 = 1$.

Solution

Let equation of circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

As $\left(m_i, \frac{1}{m_i}\right)$ lies on the circle, it should satisfy the equation of the circle

i.e. $m_1^2 + \frac{1}{m_i^2} + 2gm_1 + 2f\frac{1}{m_i} + c = 0$

$\Rightarrow \quad m_i^4 + 2gm_i^3 + cm_i^2 + 2fm_i + 1 = 0$

This is equation of degree four in m whose roots are m_1, m_2, m_3, and m_4.

Product of the roots = $m_1m_2m_3m_4 = \frac{\text{coefficient of } x^0}{\text{coefficient of } x^4} = \frac{1}{1} = 1$

Hence $m_1m_2m_3m_4 = 1$

Example : 21 $\left(\frac{ma}{1+m^2}, \frac{ma}{1+m^2}\right)$

y = mx is a chord of the circle of radius a and whose diameter is along the axis of x. Find the equation of the circle whose diameter is this chord and hence find the locus of its centre for all values of m.

Solution

The circle whose chord is y = mx and centre lies on x -axis will touch y axis at origin

The equation of such circle is given by :

$(x - a)^2 + y^2 = a^2 \quad \Rightarrow \quad x^2 + y^2 - 2ax = 0$ ....(i)

Further, family of circles passing through the intersection of circle (i) and the line y = mx is :

$x^2 + y^2 - 2ax + k(y - mx) = 0 \quad \Rightarrow \quad x^2 + y^2 - x(2a + km) + ky = 0$ ....(i)

centre of the circle is $\equiv (a + km/2, -k/2)$

We require that member of this family whose diameter is y = mx

$\Rightarrow$ centre of the required circle lies on y = mx.

$\Rightarrow \quad -k/2 = am + km^2/2 \quad \Rightarrow \quad k = -2ma/(1 + m^2)$

Put the value of k in (i) to get the equation of the required circle,

$$x^2 + y^2 - x\left(2a - \frac{2am^2}{1+m^2}\right) - \frac{2am}{1+m^2}y = 0$$

$\Rightarrow \quad (1 + m^2) - (x^2 + y^2) - 2a(x + my) = 0$

(ii) Let the coordinates of the point whose locus is required be (x_1, y_1)

$\Rightarrow \quad (x_1, y_1)$ is the centre of the circle (ii)

$\Rightarrow \quad (x_1, y_1) \equiv$

$\Rightarrow \quad x_1 = \frac{a}{1+m^2}$ ....(iii) and $y_1 =$ ....(iv)

On squaring and adding (iii) and (iv), we get :

$x_1^2 + y_1^2 =$ $\Rightarrow$ $1 + m^2 =$

Substitute the value of $(1 + m^2)$ in (iii) to get : $x_1^2 + y_1^2 = ax_1$

$\Rightarrow$ required locus is : $x^2 + y^2 = ax$.

Example : 22

Find the equation of a circle having the lines $x^2 + 2xy + 3x + 6y = 0$ as its normals and having size just sufficient to contain the circle $x (x – 4) + y (y – 3) = 0$

Solution

On factorising the equation of the pair of straight lines $x^2 + 2xy + 3x + 6y = 0$, we get :

$(x + 2y) (x + 3) = 0$

$\Rightarrow$ Two normals are $x = –2y$ ....(i) and $x = – 3$ ....(ii)

The point of intersection of normals (i) and (ii) is centre of the required circle as centre lies on all normal lines.

Solving (i) and (ii), we get :

centre $\equiv C_1 \equiv (–3, 3/2)$

Given circle is $C_2 \equiv$` $x (x – 4) + y(y – 3) = 0$ $\Rightarrow$ $x^2 + y^2 – 4x – 3y = 0$

$\Rightarrow$ centre $\equiv C_2 \equiv (2, 3/2)$ and radius = r = 5/2

If the required circle just contains the given circle, the given circle should touch the required circle internally from inside.

$\Rightarrow$ radius of the required circle = $|C_1 – C_2| + r$

$\Rightarrow$ radius of the required circle = 5 + 5/2 = 15/2

Hence, equation of required circle is $(x + 3)^2 + (y – 3/2)^2 = 225/4$

Example : 23

A variable circle passes through the point (a, b) and touches the x-axis. Show that the locus of the other end of the diameter through A is $(x – a)^2 = 4by$

Solution

Let the equation of the variable circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Let $B \equiv (x_1, y_1)$ be the other end of the diameter whose locus is required

centre of the circle $\equiv (–g, –f) \equiv$ mid point of the diameter AB $\equiv \left(\frac{x_1 + a}{2}, \frac{y_1 + b}{2}\right)$

$\Rightarrow$ $–2g = x_1 + a$ ....(i) and $–2f = y_1 + b$ ....(ii)

As circle touches x axis, we can write : | f | = radius of the circle

$\Rightarrow$ $| f |^2 = g^2 + f^2 – c$ $\Rightarrow$ $g^2 = c$

Substituting the value of g from (i), we get : $c = (x_1 + a^2)/4$ ....(iii)

Since point $B \equiv (x_1, y_1)$ lies on circle, we can have :

$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

On substituting the values of g, f and c from (i), (ii) and (iii), we get :

$x_1^2 + y_1^2 – (x_1 + a) x_1 – (y_1 + b) y_1 + (x_1 + a)^2 /4 = 0$

$\Rightarrow$ $(x_1 – a)^2 = 4by_1$

Hence, required locus is $(x – a)^2 = 4by$

Alternate Solution

Let $B \equiv (x_1, y_1)$ be the other end of the diameter whose locus is required

centre of the circle $\equiv (–g, –f) \equiv$ mid point of the diameter AB $\equiv \left(\frac{x_1 + a}{2}, \frac{y_1 + b}{2}\right)$

length of the diameter of the circle = $[(x_1 – a)^2 + (y_1 – b)^2]^{1/2}$

$\Rightarrow$ radius = r = $1/2 [(x_1 – a)^2 + (y_1 – b)^2]^{1/2}$

As circle touches x-axis, | f | = r $\Rightarrow$ $| f |^2 = r^2$

$\Rightarrow$ $(y_1 + b)^2 = (x_1 – a)^2 + (y_1 – b)^2$

$\Rightarrow$ $(x_1 – a)^2 = 2by_1$

Hence, required locus is $(x – a)^2 = 4by$

Example : 24

A circle is drawn so that it touches the y-axis cuts off a constant length 2a, on the axis of x. Show that the equation of the locus of its centre is $x^2 - y^2 = a^2$.

Solution

Let (x_1, y_1) be the centre of the circle.

As circle touches y-axis, radius of the circle = x_1.

So equation of circle is : $(x - x_1)^2 + (y - y_1)^2 = x_1^2$

$\Rightarrow \quad x^2 + y^2 - 2x_1x - 2y_1y + y_1^2 = 0$

Intercept made by the circle on x-axis = $2\,(g^2 - c)^{1/2} = 2a$ (given)

$\Rightarrow \quad g^2 - c = a^2 \quad \Rightarrow \quad x_1^2 - y_1^2 = a^2$

Hence required locus is $x^2 - y^2 = a^2$

Example : 25

A circle is cut by a family of circles all of which pass through two given points $A \equiv (x_1, y_1)$ and $B(x_2, y_2)$. prove that the chords of intersection of the fixed circle with any circle of the family passes through a fixed point.

Solution

Let $S_0 \equiv 0$ be the equation of the fixed circle.

Equation of family of circles passing through two given points A and B is :

$$S_2 \equiv (x - x_1)(x - x_2) + (y - y_1)(y - y_2) + kL_1 = 0$$

where L_1 is equation of line passing through A and B

$\Rightarrow \quad S_2 \equiv S_1 + kL_1$ ....(i)

where $S_1 \equiv (x - x_1)(x - x_2) + (y - y_1)(y - y_2)$

The common chord of intersecting of circles $S_0 = 0$ and $S_2 = 0$ is given by :

$$L \equiv S_2 - S_0 = 0$$

Using (i), we get

$$L \equiv S_2 - S_1 - kL_1 = 0$$

$\Rightarrow \quad L \equiv L_2 - kL_1$ where $L_2 \equiv S_2 - S_1$ is the equation fo common chord of S_1 and S_2 .

On observation we can see that L represents a family of straight lines passing the intersection of L_2 and L_1.

Hence all common chords (represented by L) pass through a fixed point

Example : 26

The circle $x^2 + y^2 = 1$ cuts the x-axis at P and Q. Another circle with centre at Q and variable radius intercepts the first circle at R above x-axis and the line segment PQ at S. Find the maximum area of the triangle QSR

Solution

Equation of circle I is $x^2 + y^2 = 1$. It cuts x-axis at point P (1, 0) and Q(–1, 0).

Let the radius of the variable circle be r. Centre of the variable circle is Q(–1, 0)

$\Rightarrow \quad$ Equation of variable circle is $(x + 1)^2 + y^2 = r^2$ ....(ii)

Solving circle I and variable circle we get coordinates of R as $\left(\dfrac{r^2 - 2}{2}, \dfrac{r}{2}\sqrt{4 - r^2}\right)$

Area of the triangle QSR = 1/2 × QS × RL = $\dfrac{1}{2}\, r\, \dfrac{r}{2}\, \sqrt{4 - r^2}$

To maximise the area of the triangle, maximise its square i.e.

Let $\quad A(r) = \dfrac{1}{16}\, r^4 (4 - r^2) = \dfrac{4r^4 - r^6}{16}$

$\Rightarrow \quad A'(r) = \dfrac{16r^3 - 6r^5}{16}$

For A(r) to be maximum or minimum, equate $A'(r) = 0$

$\Rightarrow \quad r = \sqrt{\dfrac{8}{3}}$

See yourself that $A''\left(\sqrt{\dfrac{8}{3}}\right) < 0$

$\Rightarrow$ Area is maximum for $r = \sqrt{\dfrac{8}{3}}$

Maximum Area of the triangle QRS = $\dfrac{1}{2} \cdot \dfrac{1}{2}\ \dfrac{8}{3} \cdot \sqrt{\dfrac{4}{3}} = \dfrac{4}{3\sqrt{3}}$ sq. units.

Example : 27

Two circles each of radius 5 units touch each other at (1, 2). If the equation of their common tangent is $4x + 3y = 10$, find the equation of the circles.

Solution

Equation of common tangent is $4x + 3y = 10$. The two circles touch each other at (1, 2).

Equation of family of circles touching a given line $4x + 3y = 10$ at a given point (1, 2) is :

$$(x - 1)^2 + (y - 2)^2 + k\,(4x + 3y - 10) = 0$$

$\Rightarrow$ $x^2 + y^2 + (4k - 2)\,x + (3k - 4)\,y + 5\ \ 10k = 0$ ......(i)

$\Rightarrow$ centre $\equiv \left(1 - 2k, \dfrac{4 - 3k}{2}\right)$ and radius $= g^2 + f^2 - c = (2k - 1)^2 + \left(\dfrac{3k - 4}{2}\right)^2 - (5 - 10k)$

As the radius of the required circle is 5, we get : $(2k - 1)^2 + \left(\dfrac{3k - 4}{2}\right)^2 - (5 - 10k) = 5$

$\Rightarrow$ $k^2 = 20/25$ $\Rightarrow$ $k = \pm \dfrac{2}{\sqrt{5}}$

Put the values of k in (i) to get the equations of required circles.

The required circles are : $\sqrt{5}\ (x^2 + y^2) + (8 - 2\sqrt{5}\,)\,x + (6 - 4\sqrt{5}\,)\,y + 5\sqrt{5}\ - 20 = 0$

and $\sqrt{5}\ (x^2 + y^2) + (8 + 2\sqrt{5}\,)\,x - (6 + 4\sqrt{5}\,)\,y + 5\sqrt{5}\ + 20 = 0$

Example : 28

The line $Ax + By + C = 0$ cuts the circle $x^2 + y^2 + ax + by + c = 0$ in P and Q. The line $A'x + B'y + c' = 0$ cuts the circle $x^2 + y^2 + a'x + b'y + c' = 0$ in R and S. If P, Q, R and S are concyclic then show that

$$\begin{vmatrix} a - a' & b - b' & c - c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$$

Solution

Let the given circles be $S_1 \equiv x^2 + y^2 + ax + by + c = 0$ and $S_2 \equiv x^2 + y^2 + a'x + b'y + c' = 0$. Assume that the points P, Q, R and S lie on circle $S_3 = 0$

The line PQ $\equiv Ax + By + C = 0$ intersects both S_1 and S_3.

$\Rightarrow$ Line PQ is radical axis of S_1 and S_3

The line RS $\equiv A'x + B'y + c' = 0$ intersects both S_2 and S_3

$\Rightarrow$ Line RS is radical axis of S_2 and S_3 .

Also radical axis of $S_1 = 0$ and $S_2 = 0$ is given by : $S_1 - S_2 = 0$

or $(a - a')x + (b - b')y + c - c' = 0$ ....(i)

The lines PQ, RS and line (i) are concurrent lines because radical axis of three circles taken in pair are concurrent. Using the result of three concurrent lines, we get :

$$\begin{vmatrix} a - a' & b - b' & c - c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$$

Example : 29

If two curves whose equations are : $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and

$a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0$ intersect in four concyclic points, prove that $\frac{a-b}{h} = \frac{a'-b'}{h'}$.

Solution

The equation of family of curves passing through the points of intersection of two curves is :

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c + k\,(a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c') = 0$$

It above equation represents a circle, then coefficient of x^2 = coefficient of y^2 and coefficient of xy = 0

$\Rightarrow \quad a + ka' = b + kb' \quad$ ....(i)

and $\quad 2(h + kh') = 0 \quad \Rightarrow \quad k = -h/h'$

On substituting the value of k in (i), we get :

$$\frac{a-b}{h} = \frac{a'-b'}{h'}$$

Example : 30

Find all the common tangents to the circles $x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$.

Solution

The centre and radius of first circle are : $C_1 \equiv (1, 3)$ and $r_1 = 1$

The centre and radius of second circle are : $C_1 \equiv (-3, 1)$ and $r_2 = 3$

Direct common tangents

Let P be the point of intersection of two direct common tangents.

Using the result that divides C_1C_2 externally in the ratio of radii i.e. 1 : 3

the coordinates of point P are $\quad P \equiv \left(\frac{1(-3) - 3\,.\,1}{1-3}, \frac{1.1 - 3(3)}{1-3}\right) \equiv (3, 4)$

Let m be the slope of direct common tangent.

So equation of direct common tangent is : $y - 4 = m\,(x - 3)$ ....(i)

Since direct common tangent touches circles, apply condition of tangency with first circle

i.e. $\quad \frac{|-1+2m|}{\sqrt{1+m^2}} = 1 \quad \Rightarrow \quad 1 = 4m^2 - 4m = 1 + m^2$

$\Rightarrow \quad 3m^2 + 4m = 0 \quad \Rightarrow \quad m\,(3m + 4) = 0$

$\Rightarrow \quad m = 0$ and $m = 4/3$

On substituting the values of m in (i), we get the equations of two direct common tangents

i.e. $\quad y = 4$ and $4x - 3y = 0$

Hence equations of direct common tangents are : $y = 4$ and $4x - 3y = 0$

<u>Transverse common tangents</u>

Let Q be the point of intersection fo two transverse (indirect) common tangents.

Using the result that P divides C_1C_2 internally in the ratio radii i.e. 1 : 3

the coordinates of point P are $P \equiv \left(\frac{1(-3) + 3.1}{1+3}, \frac{1.1 + 3(3)}{1+3}\right) \equiv \left(0, \frac{5}{2}\right)$

Let m be the slope of direct common tangent.

So equation of direct common tangent is : $y - 5/2 = mx$ ....(i)

Since direct common tangent touches circles, apply condition of tangency with first circle

i.e. $\quad \frac{|m - 1/2|}{\sqrt{1+m^2}} = 1 \quad \Rightarrow \quad 1 + 4m^2 - 4m = 4 + 4m^2$

$\Rightarrow \quad 0m^2 + 4m + 3 = 0$

As coefficient of m^2 is 0, one root must be ∞ and other is $m = -3/4$

$\Rightarrow \quad m = \infty$ and $m = -3/4$

On substituting the values of m in (i), we get the equations of two direct common tangents

i.e. $\quad x = 0$ and $3x + 4y = 10$

Hence equations of direct common tangents are : $x = 0$ and $3x + 4y = 10$.

Example : 31

Find the intervals of values of a for which the line y + x = 0 bisects two chords drawn from a point $\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$ to the circle $2x^2 + 2y^2 - (1 + \sqrt{2}a)\,x - (1 - \sqrt{2}a)\,y = 0$.

Solution

Let $(m, n) \equiv \left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$

$\Rightarrow$ Equation fo circle reduces to $x^2 + y^2 - mx - ny = 0$.

Let P (t, –t) be a point on the line y + x = 0.

Equation fo chord passing through (t, –t) as mid-point is :

$$xt - yt + \frac{-m}{2}(x + t) + \frac{-n}{2}(y - t) = t^2 + t^2 - mt + nt \qquad \text{...........(i)}$$

Since chord (i) also passes through (m, n), it should satisfy the equation of chord

i.e. $mt - nt + \frac{-m}{2}(m + t) + \frac{-n}{2}(n - t) = t^2 + t^2 - mt + nt$

$\Rightarrow$ $4t^2 + m^2 + n^2 = 3t\,(m - n)$

On substituting the values of m and n, we get $\Rightarrow$ $4t^2 - 3\sqrt{2}at + (1 + 2a^2)/2 = 0$ ....(ii)

Now if there exists two chords passing through (m, n) and are bisected by the line y + x = 0, then equation of (ii) should have two real and distinct roots.

$\Rightarrow$ $D > 0$ $\Rightarrow$ $18a^2 - 16\,(1 + 2a^2)/2 > 0$

$\Rightarrow$ $a^2 - 4 > 0$ $\Rightarrow$ $(a + 2)\,(a - 2) > 0$

$\Rightarrow$ $a \in (-\infty, -2) \cup (2, \infty)$

Hence values of a are $a \in (-\infty, -2) \cup (2, \infty)$.

Complex Numbers

Example : 1

Express the following complex numbers in the trigonometric forms and hence calculate their principal arguments. Show the complex numbers on the Argand plane

(i) $z_1 = -\sqrt{3} + i$ (ii) $z_2 = -1 - \sqrt{3}$ (iii) $z_3 = 1 - i$

Solution

(i) $z_1 = -\sqrt{3} + i$ $(|z| = 2)$

$\Rightarrow$ $z_1 = 2\left(-\frac{\sqrt{2}}{3} + \frac{1}{2}i\right)$ $\left(\text{as} \cos\theta = -\frac{\sqrt{3}}{2}, \sin\theta = \frac{1}{2}\right)$

$\Rightarrow$ $z_1 = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$ $\Rightarrow$ the argument = $\frac{5\pi}{6}$

(ii) $z_3 = -1 - \sqrt{3}\,i$ $(|z| = 2)$

$\Rightarrow$ $z_2 = 2\left(-\frac{1}{2} - \frac{\sqrt{3}\,i}{2}\right)$ $\left(\cos\theta = -\frac{1}{2}, \sin\theta = -\frac{\sqrt{3}}{2}\right)$

$\Rightarrow$ $z_2 = 2\left[\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right]$

$\Rightarrow$ argument = $\frac{-2\pi}{3}$

(iii) $z_3 = 1 - i$ $(|z| = \sqrt{2})$

$\Rightarrow$ $z_3 = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$ $\left(\cos\theta = -\frac{1}{\sqrt{2}}, \sin\theta = -\frac{1}{\sqrt{2}}\right)$

$\Rightarrow$ $z_3 = \sqrt{2}\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right]$

$\Rightarrow$ argument = $\frac{-\pi}{4}$

Example : 2

If $z_1 = r_1 (\cos\alpha + i\sin\alpha)$ and $z_2 = r_2 (\cos\beta + i\sin\beta)$, show that :

(i) $|z_1 z_2| = r_1 r_2$ (ii) $\arg(z_1 z_2) = \alpha + \beta$

(iii) $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}$ (iv) $\arg\left(\frac{z_1}{z_2}\right) = \alpha - \beta$

Solution

For (i) and (ii) :

$z_1 z_2 = r_1 r_2 (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$= r_1 r_2 (\cos\alpha\cos\beta - \sin\alpha\sin\beta + i\sin\alpha\cos\beta + i\cos\alpha\sin\beta)$

$= r_1 r_2 [\cos(\alpha + \beta) + i\sin(\alpha + \beta)]$

comparing with $z = |z| (\cos\theta + i\sin\theta)$, we get :

$|z_1 z_2| = r_1 r_2$ and $\arg(z_1 z_2) = \alpha + \beta$

For (iii) and (iv) :

$$\frac{z_1}{z_2} = \frac{r_1(\cos\alpha + i\sin\alpha)}{r_2(\cos\beta + i\sin\beta)}$$

$$= \frac{r_1}{r_2}(\cos\alpha + i\sin\alpha)(\cos\beta - i\sin\beta)$$

$$= \frac{r_1}{r_2}\ [\cos\alpha\cos\beta + \sin\alpha\sin\beta + i\sin\alpha\cos\beta - i\cos\alpha\sin\beta]$$

$$= \frac{r_1}{r_2}\ [\cos(\alpha-\beta) + i\sin(\alpha+\beta)]$$

$$\Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} \quad \text{and} \quad \arg\left(\frac{z_1}{z_2}\right) = \alpha - \beta$$

Example : 3

Show that $|z - 2i| = 2\sqrt{2}$, if $\arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{4}$

Solution

Let $z = x + yi$ $\quad x, y \in R$

$$\Rightarrow \quad \arg\left(\frac{x-2+yi}{x+2+yi}\right) = \frac{\pi}{4}$$

$$\Rightarrow \quad \arg\left[\frac{(x-2+yi)(x+2-yi)}{(x+2)^2+y^2}\right] = \frac{\pi}{4}$$

$$\Rightarrow \quad \arg\left[\frac{(x^2-4+y^2)+4yi}{(x+2^2)+y^2}\right] = \frac{\pi}{4}$$

$$\Rightarrow \quad \frac{4y}{x^2-4+y^2} = \tan\frac{\pi}{4}$$

$\Rightarrow \quad x^2 + y^2 - 4y - 4 = 0$

$\Rightarrow \quad x^2 + (y-2)^2 = 8$

$\Rightarrow \quad |x + (y-2)\,i| = 2\sqrt{2}$

$\Rightarrow \quad |z - 2i| = 2\sqrt{2}$

Example : 4

If $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$, then show that :

(i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

(iii) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = \sin 2\alpha + \sin 2\beta + \sin 3\gamma = 0$

Solution

For (i) and (ii) :

Let $z_1 = \cos\alpha + i\sin\alpha$;

$z_2 = \cos\beta + i\sin\beta$;

$z_3 = \cos\gamma + i\sin\gamma$

$z_1 + z_2 + z_3 = \Sigma\cos\alpha + i\,\Sigma\sin\alpha = 0$

for $3\alpha, 3\beta, 3\gamma$ we have to consider z_1^3 , z_2^3 , z_3^3

$z_1^3 + z_2^3 + z_3^3 = (\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^2 + (\cos\gamma + i\sin\gamma)^3$

$= (\cos 3\alpha + i\sin 3\alpha) + (\cos 3\beta + i\sin 3\beta) + (\cos 3\gamma + i\sin 3\gamma)$

$= (\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i\,(\sin 3\alpha + \sin 3\beta + \sin 3\gamma)$ ....(i)

Now $z_1^3 + z_2^3 + z_3^3 = 3z_1 z_2 z_3$ because $z_1 + z_2 + z_3 = 0$

$\Rightarrow \quad z_1^3 + z_2^3 + z_3^3 = 3\,(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$

$z_1^3 + z_2^3 + z_3^3 = 3\,[\cos(\alpha + \beta + \gamma) + i\sin(a + b + g)]$ ....(ii)

Equating the RHS of (i) and (ii), we get :

$$\Sigma\cos 3\alpha + i\,\Sigma\sin 3\alpha = 3\cos(\alpha + \beta + \gamma) + 3i\sin(\alpha + \beta + \gamma)$$

Equating real and imaginary parts,

$\Sigma\cos 3\alpha = 3\cos(\alpha + \beta + \gamma)$ and $\Sigma\sin 3\alpha = 3\sin(\alpha + \beta + \gamma)$

For (iii) :

Consider $z_1^2 + z_2^2 + z_3^3$

$$z_1^2 + z_2^2 + z_3^2 = (z_1 + z_2 + z_3)^2 - 2(z_1 z_2 + z_2 z_3 + z_3 z_1)$$

$$= 0 - 2z_1 z_2 z_3 \left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right)$$

$$= 2z_1 z_2 z_3 \left[\frac{1}{\cos\alpha + i\sin\alpha} + \frac{1}{\cos\beta + i\sin\beta} + \frac{1}{\cos\gamma + i\sin\gamma}\right]$$

$$= -2z_1 z_2 z_3 [\cos\alpha - i\sin\alpha + \cos\beta - i\sin\beta + \cos\gamma - i\sin\gamma]$$

$$= -2z_1 z_2 z_3 [\Sigma\cos\alpha - i\,\Sigma\sin\alpha]$$

$$= -2_1 z_2 z_3 [0 - i(0)] = 0$$

$\Rightarrow$ $(\cos\alpha + i\sin\alpha)^2 + (\cos\beta + i\sin\beta)^2 + (\cos\gamma + i\sin)^2 = 0$

$\Rightarrow$ $(\cos\alpha + i\sin 2\alpha) + (\cos 2\beta + i\sin\beta)^2 + (\cos 2\gamma + i\sin 2\gamma) = 0$

$\Rightarrow$ $\Sigma\cos 2\alpha = 0$ and $\Sigma\sin 2\alpha = 0$

Example : 5

Express $\sin 5\theta$ in terms of $\sin\theta$ and hence show that $\sin 36^o$ is a root of the equation $16x^4 + 20x^2 + 5 = 0$.

Solution

Expand $(\cos\theta + i\sin\theta)^5$ using binomial theorem.

$(\cos\theta + i\sin\theta)^5 = {}^5C_0\ \cos^5\theta + 5C_1 \cos 4\theta\,(i\sin\theta) + \ldots\ldots\ldots + {}^5C_5\ I\ 5\sin^5\theta$

using DeMoiver's theorem on L.H.S. :

$(\cos 5\theta + i\sin 5\theta) = (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + i\,5[\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta]$

Equating imaginary parts :

$\sin 5\theta = \sin\theta\,[5\cos^4\theta - 10\cos^2\theta\sin^2\theta\sin^2\theta + \sin^4\theta]$

$\sin 5\theta = \sin\theta\,[5(1 + \sin^4\theta - 2\sin^2\theta) - 10(1 - \sin^2\theta)\sin^2\theta] + \sin^4\theta$

$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

for $\theta = 36^o$, $\sin 5\theta = \sin 180^o = 0$

$\Rightarrow$ $16\sin^5 36^o - 20\sin^3 36^o + 5\sin 36^o = 0$

$\Rightarrow$ $\sin 36^o$ is a root of $16x^5 - 20x^3 + 5x = 0$

i.e. $16x^4 - 20x^2 + 5 = 0$

Example : 6

If $(1 + x)^n = P_0 + P_1x + P_2x^2 + \ldots\ldots.. + P_nx^n$, the show that

(a) $P_0 - P_2 + P_4 + \ldots\ldots.. = 2^{n/2}\cos(n\pi)/4$

(b) $P_1 - P_3 + P_5 + \ldots\ldots.. = 2^{n/2}\sin(n\pi)/4$

Solution

Consider the identity

$(1 + x)^n = P_0 + P_1x + P_2x^2 + P_3x^3 + \ldots\ldots\ldots + P_nx^n$.

Put x = i on both the sides

$(1 + i)^n = P_0 + P_1i + P_2\,i^2 + P_3i^3 + \ldots\ldots + P_n\,i^n$

$$\left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n = (P_0 - P_2 + P_4 + \ldots..) + i\,(P_1 - P_3 + P_5 + \ldots\ldots)$$

$$2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) = (P_0 - P_2 + P_4 + \ldots\ldots) + i\,(P_1 - P_3 + P_5 + \ldots..)$$

equate the real and imaginary parts.

$$P_0 - P_2 + P_4 - P_0 + \ldots\ldots. = 2^{n/2}\cos\frac{n\pi}{4}$$

$$P_1 - P_3 + P_5 - \ldots\ldots.. = 2^{n/2}\sin\frac{n\pi}{4}$$

Example : 7

If a, b, c and d are the roots of the equation $x^4 + P_1x^3 + P_2 x^2 + P_3x + P_4 = 0$, then show that :

$(1 + a^2) (1 + b^2) (1 + c^2) (1 + d^2) = (1 – P_2 + P_4)^2 + (P_3 – P_1)^2$

Solution

As a, b, c and d are the roots of the given equation :

$\Rightarrow$ (x – a), (x – b), (x – c) and (x – d) are the factors of LHS

$\Rightarrow$ $x^4 + P_1 x^3 + P_2 x^2 + P_3x + P_4 = (x – a) (x – b) (x – c) (x – d)$ is an identity ....(i)

Put x = i on both sides :

$i^4 + P_1i^3 + P_2 i^2 + P_3 i + P_4 = (i – a) (i – b) (i – c) (i – d)$

$(1 – P_2 + P_4) + i (P_3 – P_1) = (i – a) (i – b) (i – c) (i – d)$ ....(ii)

Put x = – i in (i) :

$i^4 – P_1i^3 + P_2i^2 – P_3 i + P_4 = (–i – a) (i – b) (–i – c) (–i – d)$

$(1 – P_2 + P_4) – i (P_3 – P_1) = (–i – a) (–i – b) (–i – c) (–i – d)$ ....(iii)

multiply (ii) and (iii) to get

$(1 – P_2 + P_4)^2 + (P_3 – P_1)^2 = (1 + a^2) (1 + b^2) (1 + c^2) (1 + d^2)$

Example : 8

Show that $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2 \text{ Re } (z1\ \bar{z}_2)$.

Solution

$|z_1 \pm z_2|^2 = (z_1 \pm z_2)\ (\bar{z}_1 \pm \bar{z}_2)$

$= z_1\ \bar{z}_2 + z_2\ \bar{z}_2 \pm (z_1\ \bar{z}_2 + \bar{z}_1\ z_2)$

$= |z|^2 + |z_2|^2 \pm (z_1\ \bar{z}_2 + \bar{z}_1\ \bar{z}_2)$

$= |z_1|^2 + |z_2|^2 \pm 2 \text{ Re } (z_1\ \bar{z}_2)$ because z + z = 2 Re (z)

Example : 9

If 1, ω, ω^2 are cube roots of unity. Show that :

$(1 – \omega + \omega^2) (1 – \omega^2 + \omega^4) (1 – \omega^4 + \omega^8)$ .... 2n factors = 2^{2n}

Solution

LHS = $(1 – \omega + \omega^2) (1 – \omega^2 + \omega^4) (1 – \omega^4 + \omega^8)$ .... 2n factors

using $\omega^4 = \omega^{16} = ... = \omega$ and $\omega^8 = \omega^{32} = ... = \omega^2$

L.H.S. = $(1 – \omega + \omega^2) (1 – \omega^2 + \omega) (1 – \omega + \omega^2) (1 – \omega^2 + \omega)$ .... 2n factors.

L.H.S. = $[(1 – \omega + \omega^2) (1 – \omega^2 + \omega)]^n = [(–2\omega) (–2\omega^2)]^n$

L.H.S. = 2^{2n} = R.H.S.

Example : 10

Prove that the area of the triangle whose vertices are the points z_1, z_2 , z_3 on the argand diagram is :

$$\sum\left[\frac{(z_2 - z_3)\,|\,z_1\,|^2}{4\,iz_1}\right]$$

Solution

Let the vertices of the triangle be

A (x_1, y_1) : $z_1 = x_1 + iy_1$

B (x_2, y_2) : $z_2 = x_{2i} + iy_2$

C (x_3, y_3) : $z_3 = x_3 + iy_3$

Area of triangle ABC is :

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

We have to express the area in terms of z_1, z_2 and z_3.

Operating $C_1 \to C_1 + iC_2$ (properties of Determinants)

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1+iy_1 & y_1 & 1 \\ x_2+iy_2 & y_2 & 1 \\ x_3+iy_3 & y_3 & 1 \end{vmatrix}$$

$$\Delta = \frac{1}{2}\begin{vmatrix} z_1 & y_1 & 1 \\ z_2 & y_2 & 1 \\ z_3 & y_3 & 1 \end{vmatrix}$$

$$\Delta = \frac{1}{4i}\begin{vmatrix} z_1 & z_1-\bar{z}_1 & 1 \\ z_2 & z_2-\bar{z}_2 & 1 \\ z_3 & z_3-\bar{z}_3 & 1 \end{vmatrix}$$

Operating $C_2 \rightarrow C_2 - C_1$ (properties of Determinants)

$$\Delta = \frac{1}{4i}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & z_3 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \frac{1}{4i}\,[\,\bar{z}_1\ (z_2 - z_3) + \bar{z}_2\ (z_1 - z_3) - \bar{z}_3\ (z_1 - z_2)]$$

$$\Rightarrow \quad \Delta = \frac{1}{4i}\,[\,\bar{z}_1\ (z_2 - z_3) + \bar{z}_2\ (z_3 - z_1) - \bar{z}_3\ (z_1 - z_2)]$$

$$\Rightarrow \quad \Delta = \frac{1}{4i}\sum \bar{z}_1(z_2 - z_3)$$

$$\Rightarrow \quad \Delta = \frac{1}{4i}\sum\left[\frac{|\,\bar{z}_1|^2\ (z_2 - z_3)}{z_1}\right]$$

Example : 11

Show that the sum of nth roots of unity is zero.

Solution

Let $S = 1 + e^{i2\pi/n} + e^{i4\pi/n} + \ldots.. + e^{i2\pi(n-1)/n}$

the series on the RHS is a GP

$$\Rightarrow \quad S = \frac{1\left(1-e^{i\frac{2\pi}{n}n}\right)}{1-e^{i\frac{2\pi}{n}}} \qquad \Rightarrow \quad S = \frac{1-e^{i2\pi}}{1-e^{i\frac{2\pi}{n}}}$$

$$\Rightarrow \quad S = \frac{1-1}{1-e^{i\frac{2\pi}{n}}} = 0$$

Example : 12

Find the value of : $\sum_{r=1}^{r=6}\left[\sin\frac{2\pi r}{7} - i\cos\frac{2\pi r}{7}\right]$

Solution

Let $\quad S = \sum_{r=1}^{r=6}\left[\sin\frac{2\pi r}{7} - i\cos\frac{2\pi r}{7}\right] = -\,i\sum_{r=1}^{r=6}\left[\cos\frac{2\pi r}{7} + i\sin\frac{2\pi r}{7}\right]$

$$= -i\sum_{r=1}^{r=6} e^{i\frac{2\pi r}{7}} = -i\left[\sum_{r=0}^{r=6} e^{i\frac{2\pi r}{7}} - 1\right]$$

= – i (sum of 7th roots of unity – 1)
= – i(0 – 1) = i

Example : 13

Find the sixth roots of z = i

Solution

$$z = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$z^{1/6} = 1^{1/6}\left(\cos\frac{\pi/2+2k\pi}{6} + i\sin\frac{\pi/2+2k\pi}{6}\right) \qquad \text{where } k = 0, 1, 2, 3, 4, 5$$

⇒ The sixth roots are :

k = 0 ⇒ $z_n = \left(\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

k = 1 ⇒ $z_1 = \cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}$

k = 2 ⇒ $z_2 = \cos\frac{9\pi}{12} + i\sin\frac{9\pi}{12}$

k = 3 ⇒ $z_3 = \cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12} = \cos\frac{11\pi}{12} - i\sin\frac{11\pi}{12}$

k = 4 ⇒ $z_4 = \cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12} = -\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}$

k = 5 ⇒ $z_5 = \cos\frac{21\pi}{12} + i\sin\frac{21\pi}{12} = \cos\frac{3\pi}{12} - i\sin\frac{3\pi}{12}$

Example : 14

Prove that $(x + y)^n - x^n - y^n$ is divisible by $xy(x + y)(x^2 + y^2 + xy)$ if n is odd but no a multiple of 3.

Solution

Let $f(x) = (x + y)^n - x^n - y^n$
$f(0) = (0 + y)^n - (0)^n - y^n = 0$
⇒ (x – 0) is a factor of f(x)
⇒ x is a factor of f(x)
By symmetry y is also a factor f(x)
$f(-y) = (-y + y)^n - (-y)^n - y^n = 0$ (because n is odd)
⇒ (x + y) is also factor of f(x).
Now consider f (ωy)
$f(\omega y) = (\omega y + y)^n - (w\, y)^n - y^n$
$= y^n(-\omega^2)^n - \omega^n y^n - y^n$
$= y^n[-\omega^{2n} - \omega^n - 1]$ (because n is odd)
$= -y^n[\omega^{2n} + \omega^n + 1]$
n is not a multiple of 3.
⇒ n = 3k + 1 or n = 3k + 2 where k is an integer
⇒ $[\omega^{2n} + w^n + 1] = 0$ (for both cases)
⇒ f(ω y) = 0
⇒ (x – ωy) is also a factor of f(x)
Similarly we can show that $f(\omega^2 y) = 0$
⇒ $(x - w^2 y)$ is also a factor of f(x)
Combining all the factors :

we get : $xy(x + y)(x - \omega^2 y)(x - \omega^2 y)$ is a factor of f(x)

now $(x - \omega y)(x - \omega^2 y) = x^2 + xy + y^2$

$\Rightarrow$ f(x) is divisible by $x\,y\,(x + y)(x - \omega\, y)(x - \omega^2 y)$

Example : 15

Interpret the following equations geometrically on the Argand plane :

(i) $|z - 2 - 3i| = 4$ (ii) $|z - 1| + |z + 1| = 4$

(ii) $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$ (iv) $\frac{\pi}{6} < \arg(z) < \frac{\pi}{3}$

Solution

To interpret the equations geometrically, we will convert them to Cartesian form in terms of x and y coordinates by substituting z = x + iy

(i) $|x + iy - 2 - 3i| = 4$

$\Rightarrow$ $(x - 2)^2 + (y - 3)^2 = 4^2$

$\Rightarrow$ the equation represents a circle centred at (2, 3) of radius 4 units

(ii) `$|x + iy - 1| = |x + iy + 1| = 4$

$\Rightarrow$ $\sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2} = 4$

simplify to get : $\frac{x^2}{4} + \frac{y^2}{3} = 1$

$\Rightarrow$ the equation represents an ellipse centred at (0, 0)

(iii) $\text{Arg}\left(\frac{x+iy-1}{x+iy+1}\right) = \frac{\pi}{4}$

$\Rightarrow$ $\text{Arg}(x + iy - 1) - \text{Arg}(x + iy + 1) = \frac{\pi}{4}$

$\Rightarrow$ $\frac{\frac{y}{x-1} - \frac{y}{x+1}}{1 + \frac{y^2}{x^2-1}} = \tan\frac{\pi}{4}$ $\Rightarrow$ $\frac{2y}{x^2 + y^2 - 1} = 1$

$\Rightarrow$ $x^2 + y^2 - 2y - 1 = 0$

$\Rightarrow$ the equation represents a circle centred at z = 0 + i and of radius = $\sqrt{2}$.

(iv) $\frac{\pi}{6} < \tan^{-1}\left(\frac{y}{x}\right) < \frac{\pi}{3}$

$\Rightarrow$ $\frac{1}{\sqrt{3}}\,x < y < \sqrt{3}\,x$

$\Rightarrow$ this inequation represents the region between the lines :
$y = \sqrt{3}\,x$ and $y = (1/\sqrt{3})\,x$ in Q_1

Example : 16

Find the complex number having least positive argument and satisfying $|z - 5i| \le 3$

Solution

We will analyses the problem geometrically.

All complex numbers (z) satisfying $|z - 5i| \le 3$ lies on or inside the circle of radius 3 centred at $z_0 = 5i$.

The complex number having least positive argument in this region is at the point of contact of a tangent drawn from origin to the circle.

From triangle OAC

$OA = \sqrt{5^2 - 3^2} = 4$

and $\quad 0_{min} = \sin^{-1}\left(\frac{OA}{OC}\right) = \sin^{-1}\left(\frac{4}{5}\right)$

the complex number at A has modulus 4 and argument $\sin^{-1} 4/5$

$\Rightarrow \quad z_A = 4 (\cos\theta + i \sin\theta) = 4\left(\frac{3}{5} + i\frac{4}{5}\right)$

$\Rightarrow \quad z_A = \frac{12}{5} + i\,\frac{16}{5}$

Example : 17

Show that the area of the triangle on the Argand plane formed by the complex numbers z, i z and (z + i z) is $(1/2)\ z^2$.

Solution

$iz = ze^{i\pi/2}$

$\Rightarrow$ iz is the vector obtained by rotating z in anti-clockwise direction through 90

As $|\ iz\ | = |\ i\ |\ |\ z\ |$, the triangle is an isosceles right angled triangle.

Area = 1/2 = base × height = $1/2\ |\ z\ |\ |\ iz\ |$

Example : 18

If $|\ z\ |^2 = 5$, find the area of the triangle formed by the complex numbers z, ω z and z = ω z as its sides.

Solution

$\omega z = ze\ i2\pi/3$ and $|\omega z| = |\ z\ |$

$\Rightarrow$ ωz is the vector obtained by rotating vector z anti-clockwise through an angle of 120

As seen from the figure, the triangle formed is equilateral because angle between equal sides is 60º

$\Rightarrow$ Area = $\sqrt{3}/4\ (\text{side})^2 = \sqrt{3}/4\ |\ z\ |^2 = \sqrt{3}$ sq. units.

Note that the third side is

$z + \omega z = (1 + \omega) z = -\omega^2 z = e^{i\pi}\ e^{-12\pi/3}\ z = z\ e^{i\pi/3}$

$\Rightarrow$ this vector is obtained by rotating the vector z anticlockwise through 60º. This can be verified from the figure

Example : 19

Show that z_1, z_2, z_3 represent the vertices of an equilateral triangle if and only if :

$z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = 0$

Solution

The problem has two parts :

(i) If the triangle is equilateral then prove the condition

(ii) If the condition is given then prove the triangle is equilateral.

Part (i)

If the triangle ABC is equilateral, the vector BC

can be obtained by rotating AB anti-clockwise through 120º

$\Rightarrow \quad (z_3 - z_2) = (z_2 - z_1)\ e^{i2\pi/3}$

$\Rightarrow \quad z_3 - z_2 = (z_2 - z_1)\ \omega$

$\Rightarrow \quad z_1\omega - z_2\ \omega - z_2 + z_3 = 0$

$\Rightarrow \quad z_1 - z_2\ \omega^3 - z_2\ \omega^2 + z_3\ \omega^2 = 0$

$\Rightarrow \quad z_1 - (1 + \omega^2)\ z^2 + \omega^2\ z_3 = 0$

$\Rightarrow \quad z_1 + \omega z_2 + \omega^2\ z_3 = 0$

Taking LHS :

$z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = (z_1 + \omega\ z_2 + \omega^2 z_3)\ (z_1 + \omega^3 z_2 + \omega z_3) = 0$ (using the above proved result)

Part (ii)

Give that :

$z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = 0$

$\Rightarrow \quad (z_1 + \omega z_2 + \omega^2 z_3)\ (z_1 + w^2 z_2 + \omega z_3) = 0$

$\Rightarrow \quad (z_1 + \omega z_2 + \omega^2 z_3 = 0 \quad \text{OR} \quad (z_1 + \omega^2\ z_2 + \omega z_3) = 0$

Case (1) :

$(z_1 + w\ z_2 + \omega^2 z_3) = 0$

$\Rightarrow \quad z_1 + \omega\ z_2 + (-1 - \omega)\ z_3 = 0$

$\Rightarrow$ $(z_1 - z_3) = \omega (z_3 - z_2)$

$\Rightarrow$ $(z_1 - z_2)$ is obtained by rotating the vector $(z_3 - z_2)$ anti-clockwise through 120º

$\Rightarrow$ $|z_1 - z_3| = |z_3 - z_2|$ and the angle inside the triangle is 60º

$\Rightarrow$ triangle ABC is equilateral

Case (2) :

$(z_1 + \omega^2 z_2 + \omega z_3) = 0$

$\Rightarrow$ $z_1 + \omega z_3 + (-1 - \omega) z_2 = 0$

$\Rightarrow$ $(z_1 - z_2) = \omega (z_2 - z_3)$

$\Rightarrow$ $|z_1 - z_2|$ is obtained by rotating the vector $(z_3 - z_3)$ anti-clockwise through 120º

$\Rightarrow$ $|z_1 - z_2| = |z_2 - z_3|$ and the angle inside the triangle is 60º

$\Rightarrow$ triangle ABC is equilateral

Example : 20

Let the complex numbers z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle. Prove that : $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.

Solution

For an equilateral triangle with vertices z_1, z_2 and z_3 :

$z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = 0$ ....(i)

As circumcentre coincides with centroid, z_0 is centroid also.

$\Rightarrow$ $z_0 = (z_1 + z_2 + z_3)/3$

$\Rightarrow$ $9z_0^2 = z_1^2 + z_2^2 + z_3^2 + 2 (z_1z_2 + z_2z_3 + z_3z_1)$

using (i), we have

$\Rightarrow$ $9 z_0^2 = z_1^2 + z_2^2 + 2 (z_1^2 + z_2^2 + z_3^2)$

$\Rightarrow$ $9z_0^2 = 3 (z_1^2 + z_2^2 + z_3^2)$

$\Rightarrow$ $3 z_0^2 = z_1^2 + z_2^2 + z_3^2$

Example : 21

If $z_1^2 + z_2^2 - 2z_1z_2 \cos \theta = 0$, then the origin, z_1, z_2 from vertices of an isosceles triangle with vertical angle θ .

Solution

$z_1^2 + z_2^2 - 2z_1z_2 \cos \theta = 0$

$\Rightarrow$ $z_1^2 - (2 z_2 \cos \theta) z_1 + z_2^2 = 0$

Solving as a quadratic in z_1, we get :

$$z_1 = \frac{2z_2 \cos\theta \pm z_2\left(\sqrt{4\cos^2\theta - 4}\right)}{2}$$

$\Rightarrow$ $z_1 = z_2 (\cos \theta \pm i \sin \theta)$

$\Rightarrow$ $z_1 = z_2 e^{\pm i\theta}$

$\Rightarrow$ $z_1 = z_2 e^{i\theta}$ or $z_2 = z_1 e^{i\theta}$

$\Rightarrow$ z_1 is obtained by rotating z_2 anticlockwise through θ or z_2 is obtained by rotating z_1 anti-clockwise through θ.

In both the cases, $|z_1| = |z_2|$ and the angle between z_1 and z_2 is θ

Hence origin, z_1 and z_2 from an isosceles triangle with vertex at origin and vertical angle as θ

Example : 22

Find the locus of the point z which satisfies :

(i) $2 < | z | \le 3$ (ii) $| z | = |z - i| = |z - 1|$

(iii) $|z - 2| < |z - 6|$ (iv) $\text{Arg}\left(\frac{z-1-i}{z-2}\right) = \frac{\pi}{2}$

Solution

Important Note : $(z - z_0)$ represents an arrow going from a fixed point z_0 to a moving point z.

(i) $2 < | z | \le 3$

$| z |$ is the length of vector from origin to the moving point z.

$| z | > 2$ $\Rightarrow$ z is outside the circle $x^2 + y^2 = 4$

$| z | \le 3$ $\Rightarrow$ z is on or inside the circle $x^2 + y^2 = 9$

$\Rightarrow$ locus is the region between two circles as shown

(ii) $|z - 0| = |z - i| = |z - 1|$

distance of moving point from origin

= distance from i

= distance from 1 + 0i

$\Rightarrow$ the moving point is equidistant from vertices $z_1 = 0$, $z_2 = i$ and $z_3 = 1 + 0i$ of a triangle.

Hence it is at the circumcentre of this triangle

(iii) $|z - 2| < |z - 6|$

$\Rightarrow$ distance of z from $z_1 = 2$ is less than its distance from $z_2 = 6$

$\Rightarrow$ z lies to the left of the right bisector of segment joining z_1 and z_2

Alternatively : $|z + iy - 2| < |x + iy - 6|$

$$\Rightarrow \sqrt{(x-2)^2 + y^2} < \sqrt{(x-6)^2 + y^2}$$

$\Rightarrow$ $(x - 2)^2 - (x - 6)^2 < 0$

$\Rightarrow$ $2x - 8 < 0$ $\Rightarrow$ $x < 4$

$\Rightarrow$ $\text{Re}(z) < 4$

Hence z lies in the region to the left of the line x = 4

(iv) $\text{Arg}\left(\frac{z - z_1}{z - z_2}\right)$ is the angle between vectors joining the fixed points z_1 and z_2 to the moving point z.

$\text{Arg}\left(\frac{z - z_1}{z - z_2}\right) = \pi/3$ $\quad z_1 = 1 + i,\ z_2 = 2$

$\Rightarrow$ the point z moves such that the angle subtended at z by segment joining z_1 and z_3 is $\pi/3$

$\Rightarrow$ the locus is an arc of a circle. The equation of the locus can be found by taking z = x + iy.

$$\text{Arg}\left(\frac{x + iy - 1 - i}{x + iy - 2}\right) \frac{\pi}{3}$$

$$\Rightarrow \tan^{-1}\left(\frac{y-1}{x-1}\right) - \tan^{-1}\left(\frac{y}{x-2}\right) = \frac{\pi}{3}$$

$$\Rightarrow \frac{\frac{y-1}{x-1} - \frac{y}{x-2}}{1 + \frac{(y-1)\,y}{(x-1)\,(x-2)}} = \sqrt{3}$$

$$\Rightarrow \frac{-x - y + 2}{x^2 - 3x + y^2 - y + 2} = \sqrt{3}$$

$\Rightarrow$ $\sqrt{3}\,(x^2 + y^2) - 3\sqrt{3} - 1)\,x - (\sqrt{3} - 1)\,y + 2\sqrt{3} - 2 = 0$

Locus of z is the arc of this circle lying to the non-origin side of line joining $z_1 = 1 + i$ and $z_2 = 2$.

Example : 23

If $|z| \le$, $|w| \le 1$, show that : $|z - w|^2 \le (|z| - |w|)^2 + (\text{Arg } z - \arg w)^2$

Solution

Let O be the origin and points W and Z are represented by complex numbers z and w on the Argand plane.

Apply cosine rule in ΔOWZ i.e.

$|w - z|^2 = |z|^2 + |w|^2 - 2\,|z|\,|w| \cos\theta$

$= |z|^2 + |w|^2 - 2\,|z|\,|w| \left(1 - 2\sin^2\frac{\theta}{2}\right)$

$= (|z| - |w|)^2 + 4\,|z|\,|w| \sin^2\theta/2.$

As $|z| \le$ and $|w| \le 1$, make RHS greater than LHS by replacing $|z| = 1$, $|w| = 1$

$|w - z|^2 \le (|z| - |w|)^2 + 4\sin^2\theta/2$

On RHS, replace $\sin\theta/2$ $\quad(\because\ \theta > \sin\theta$ for $\theta > 0)$

⇒ $|w - z|^2 \le (|z| - |w|)^2 + 4\ \theta/2 \times \theta/2$

⇒ $|w - z|^2 \le (|z| - |w|)^2 + \theta^2$

⇒ $|w - z|^2 \le (|z| - |w|)^2 + (\text{Arg}\ (z) - \text{Arg}\ (w))^2$

hence proved

Example : 24

If $iz^3 + z^2 - z + i = 0$, then show that $|z| = 1$.

Solution

Consider : $iz^3 + z^2 - z + i = 0$

By inspection, we can see that $z = i$ satisfies the above equation.

⇒ $z - i$ is a factor of the LHS

Factoring LHS, we get : $(z - i)\ (iz^2 - 1) = 0$

⇒ $z = i$ and $z^2 = 1/i = -i$

Case – 1

$z = i \Rightarrow |z| = 1$

Case – II

$z^2 = -i$

Take modulus of both sides,

$|z|^2 = |-i| = 1 \Rightarrow |z| = 1$

Hence, in both cases $|z| = 1$

Example : 25

If z_1 and z_2 are two complex numbers such that $\left|\dfrac{z_1 - z_2}{z_1 + z_2}\right| = 1$, Prove that $\dfrac{iz_1}{z_2} = k$, where k is a real number. Find the angle between the lines from the origin to the points $z_1 + z_2$ and $z_1 - z_2$ in terms of k.

Solution

Consider $\left|\dfrac{z_1 - z_2}{z_1 + z_2}\right| = 1$

Divide N and D on LHS by z_2 to get :

$$\Rightarrow \quad \frac{\left|\dfrac{z_1}{z_2} - 1\right|}{\left|\dfrac{z_1}{z_2} + 1\right|} = 1 \quad \Rightarrow \quad \left|\frac{z_1}{z_2} - 1\right| = \left|\frac{z_1}{z_2} + 1\right|$$

On squaring, $\left|\dfrac{z_1}{z_2}\right|^2 + 1 - 2\ \text{Re}\left(\dfrac{z_1}{z_2}\right) = \left|\dfrac{z_1}{z_2}\right|^2 + 1 + 2\ \text{Re}\left(\dfrac{z_1}{z_2}\right)$

⇒ $4\ \text{Re}\left(\dfrac{z_1}{z_2}\right) = 0 \Rightarrow \dfrac{z_1}{z_2}$ is purely imaginary number.

⇒ $\dfrac{z_1}{z_2}$ can be written as : $i\dfrac{z_1}{z_2} = k$ where k is real number ....(i)

(ii) If θ is the angle between $z_1 - z_2$ and $z_1 + z_2$, then $\theta = \text{Arg}\ \dfrac{z_1 + z_2}{z_1 - z_2}$

$$\Rightarrow \quad \theta = \text{Arg}\left[\frac{\dfrac{z_1}{z_2} + 1}{\dfrac{z_1}{z_2} - 1}\right]$$

Using (i), we get

$$\theta = \text{Arg}\left[\frac{-ik+1}{-ik-1}\right] = \text{Arg}\left[\frac{-1+ik}{1+ik}\right] = \text{Arg}\left[\frac{k^2-1+2ik}{1+k^2}\right]$$

$\Rightarrow \quad \theta = \tan^{-1}\dfrac{2k}{k^2-1}$

Example : 26

For any $z_1\ z_2 \in$ C, show that $|z_1 + z_2|^2 + |z_1 + z_2|^2 = 2\,|z_1|^2 + 2|z_2|^2$

Solution

Consider $\quad$ LHS $= |z_1 + z_2|^2 + |z_1 - z_2|^2$

$\Rightarrow \quad$ LHS $= (z_1 + z_2) \qquad + (z_1 - z_2)\ \overline{(z_1 - z_2)}$

$= (z_1 + z_2)\ (\overline{z}_1 + \overline{z}_2) + (z_1 - z_2)\ (\overline{z}_1 - \overline{z}_2)$

$= (|z_1|^2 + |z_2|^2 + z_1\ \overline{z}_2 + z_2\ \overline{z}_1) + (|z_1|^2 + |z_2|^2 - z_1\ \overline{z}_2 - z_2\ \overline{z}_1)$

$= 2|z_1|^2 + 2\,|z_2|^2$

Example : 27

If $\quad S_1 = {}^nC_0 + {}^nC_3 + {}^nC_6 + \ldots\ldots\ldots\ldots$

$S_2 = {}^nC_1 + {}^nC_2 + {}^nC_7 + \ldots\ldots\ldots\ldots$

$S_3 = {}^nC_2 + {}^nC_5 + {}^nC_8 + \ldots\ldots\ldots\ldots$

each series being continued as far as possible, show that the values of S_1 , S_2 and S_3 are 1/3 $(2^n + 2\cos r\pi/3)$ where r = n_1 n – 2, n + 2 respectively and n $\in$ N.

Solution

Consider the identity :

$(1 + x)^n = C_0 + C_1 x + C_2x^2 + C_3 x^3 + \ldots\ldots\ldots + C_nx^n$

Put x = 1, x = ω and x = ω^2 in above identity to get :

$2^n = C_0 + C_1 + C_2 + C_3 + \ldots\ldots\ldots C_n \qquad \overline{(z_1 + z_2)}$ ....(i)

$(1 + \omega)^n = C_0 + C_1\omega + C_2\ \omega^2 + C_3\ \omega^3 + \ldots\ldots\ldots + C_n\ \omega^n$ ....(ii)

$(1 + \omega^2)^n = C_0 + C_1\omega^2 + C_2\ \omega + C_3 + \ldots\ldots\ldots\ldots + C_n\ \omega^{2n}$ ....(iii)

Find S_1

Add (i), (ii) and (iii) to get :

$3C_0 + C_1\ (1 + \omega + \omega^2) + C_2\ (1 + \omega^2 + \omega) + 3C_3 + \ldots\ldots = 2^n + (1 + \omega)^n + (1 + w^2)^n$

$\Rightarrow \quad {}^3C_0 + 3C_3 + 3C_6 + \ldots\ldots\ldots\ldots = 2^n + \left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^n + \left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^n$

$\Rightarrow \quad 3S_1 = 2^n + \left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}i\right)^n + \left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}i\right)^n$

$\Rightarrow \quad S_1 = \dfrac{2^n + 2\cos\frac{n\pi}{3}}{3} \qquad$ (using demoivre's Law)

Find S_2

Multiply (ii) with ω^2, (iii) with ω and add to (i) to get :

$C_0\ (1 + \omega^2 + \omega) + 3C_1 + C_2\ (1 + \omega + \omega^2) + C_3\ (1 + \omega^2 + \omega) + \ldots\ldots\ldots = 2^n + w^2\ (1 + \omega)^n + \omega\ (1 + \omega^2)^n$

$3C_1 + 3C_4 + 3C_7 + \ldots\ldots\ldots = 2^n + \left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)\left(\cos\frac{n\pi}{3}-i\sin\frac{n\pi}{3}\right) + \left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\left(\cos\frac{n\pi}{3}-i\sin\frac{n\pi}{3}\right)$

$\Rightarrow \quad 3S_2 = 2^n + \cos\dfrac{(n-2)\pi}{3} + i\sin\dfrac{(n-2)\pi}{3} + \cos\dfrac{(n-2)\pi}{3} - i\sin\dfrac{(n-2)\pi}{3} = 2^n + 2\cos\dfrac{(n-2)\pi}{3}$

$\Rightarrow \quad S_2 = \dfrac{2^n + 2\cos\dfrac{(n-2)\pi}{3}}{3}$

Find S_3

Multiply (ii) be ω , (iii) with ω^2 and add to (i) to get

$3(C_2 + C_5 + C_8 + \text{..........}) = 2^n + 2\cos\dfrac{(n+2)\pi}{3}$

$\Rightarrow \quad S_3 = \dfrac{2^n + 2\cos\dfrac{(n+2)\pi}{3}}{3}$

Example : 28

Prove that the complex number z_1, z_2 and the origin form an isosceles triangle with vertical angle $2\pi/3$. If $x^2_1 + z^2_2 + z_1 z_2 = 0$

Solution

Let A and B are the points represented by z_1 and z_2 respectively on the Argand plane

Consider $z^2_1 + z^2_2 + z_1z_2 = 0$

On factoring LHS, we get :

$(z_2 - \omega z_1)(z_2 - \omega^2 z_1) = 0$

$\Rightarrow \quad z_2 = \omega_1 z_1 \quad$ or $\quad z_2 = \omega^2 z_1$

consider $z_2 = \omega z_1$ ....(i)

Take modulus of both sides

$|z_2| = |\omega z_1|$

$\Rightarrow \quad |z_2| = |\omega|\,|z_1| = |z_1| \quad (\because |\omega| = 1)$

$\Rightarrow \quad OA = OB \quad \Rightarrow \quad \Delta OAB$ is isosceles.

Take argument on both sides,

$\text{Arg}(z_2) = \text{Arg}(\omega z_1) = \text{Arg}(\omega) + \text{Arg}(z_1)$

$\Rightarrow \quad \text{Arg}(z_2) - \text{Arg}(z_1) = 2\pi/3 \quad (\because \text{Arg}(\omega) = 2\pi/3)$

$\Rightarrow \quad \angle AOB = 2\pi/3$. Hence vertical angle $= \angle AOB = 2\pi/3$.

Note : As $z_2 = \omega z_1 \quad \Rightarrow \quad z_2 = z_1 e^{i2\pi/3}$,we can directly conclude that z_2 is obtained by rotating z_1 through $2\pi/3$ in anti-clockwise direction

$\Rightarrow \quad \angle AOB = 2\pi/3$ and $OA = OB$

Consider $z_2 = \omega^2 z_1$

Similarly show that ΔAOB is isosceles with vertical angle $2\pi/3$

Example : 29

For every real number $c \geq 0$, find all complex numbers z which satisfy the equation : $|z|^2 - 2iz + 2c(1 + i) = 0$.

Solution

Let z = x = iy

$\Rightarrow \quad (x^2 + y^2 + 2y + 2c) - i(2x - 2c) = 0$

Comparing the real and imaginary parts, we get :

$\Rightarrow \quad x^2 + y^2 + 2y + 2c = 0$ ....(i)

and $\quad x = c$ ....(ii)

Solving (i) and (ii), we get

$\Rightarrow \quad y^2 + 2y + c^2 + 2c = 0$

$\Rightarrow \quad y = \dfrac{-2 \pm \sqrt{4 - 4(c^2 + 2c)}}{2} = -1 \pm \sqrt{1 - c^2 - 2c}$

as y is real, $1 - c^2 - 2c \geq 0$

$\Rightarrow \quad -\sqrt{2} - 1 \leq c \leq \sqrt{2} - 1$

$\Rightarrow \quad c \leq \sqrt{2} - 1 \quad (\because c \geq 0)$

$\Rightarrow$ the solution is

$z = x + iy = c + i\left(-1 \pm \sqrt{1 - c^2 - 2c}\right)$ for $0 \le c \le \sqrt{2} - 1$

$z = x + iy \equiv$ no solution for $c > \sqrt{2} - 1$

Example : 30

Let $\bar{b}z + b\bar{z} = c$, $b \ne 0$, be a line in the complex plane, where $\bar{b}$ si the complex conjugate of b. Ig a point z_1 is the reflection of a point z_2 through the line, then show that $c = \bar{z}_1 b + z_2 \bar{b}$.

Solution

Since z_1 is image of z_2 in line $bz + b\bar{z} = c$.

therefore mid-point of z_1 and z_2 should lie on the line i.e.

$\frac{z_1 + z_2}{2}$ lies on $\bar{b}z + b\bar{z} = c$

$$\Rightarrow \quad \bar{b}\left(\frac{z_1 + z_2}{2}\right) + b\,\frac{\bar{z}_1 + \bar{z}_2}{2} = c$$

$$\Rightarrow \quad \frac{\bar{b}z_1 + b\bar{z}_2}{2} + \frac{\bar{b}z_2 + b\bar{z}_1}{2} = c$$

Let z_b and z_c be two points on the given line.

As $z_1 - z_2$ is perpendicular to $z_b - z_c$, we can take : $\frac{z_c - z_b}{|z_c - z_b|}\, e^{i\pi/2} = \frac{z_1 - z_2}{|z_1 - z_2|}$ ....(ii)

$$\Rightarrow \quad \frac{z_1 - z_2}{z_c - z_b} = -\frac{\bar{z}_1 - \bar{z}_2}{\bar{z}_c - \bar{z}_b} \qquad \Rightarrow \qquad \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} = -\frac{z_c - z_b}{\bar{z}_c - \bar{z}_b}$$

As z_b and z_c also lie on line, we get :

$\bar{b}z_b + b\bar{z}_b = c$ and $\bar{b}z_c + b\bar{z}_c = c$

On subtracting, $\bar{b}(z_c - z_b) + b\,(\bar{z}_c - \bar{z}_b) = 0$

$$\Rightarrow \quad \frac{z_c - z_b}{\bar{z}_c - \bar{z}_b} = -\frac{b}{\bar{b}} \qquad \text{...........(iii)}$$

combining (ii) and (iii),

$$(z_1 - z_2)\,\bar{b} = b\,(\bar{z}_1 - \bar{z}_2)$$

$$\Rightarrow \quad \bar{b}z_1 + b\bar{z}_2 = b\bar{z}_1 + \bar{b}z_2 \qquad \text{...........(iv)}$$

combining (i) and (iv) we get :

$$\frac{\bar{b}z_2 + b\bar{z}_1}{2} + \frac{\bar{b}z_2 + b\bar{z}_1}{2} = c$$

$$\Rightarrow \quad \bar{b}z_2 + b\bar{z}_1 = c$$

Hence proved

Coordinate Geometry (Conic Section)

Example : 1

What does the equation $x^2 - 5xy + 4y^2 = 0$ represent?

Solution

$x^2 - 5xy + 4y^2 = 0$

$\Rightarrow$ $x^2 - 4xy - xy + 4y^2 = 0$

$\Rightarrow$ $(x - 4y)(x - y) = 0$

$\Rightarrow$ the equation represent two straight lines through origin whose equation are $x - 4y = 0$ and $x - y = 0$

Example : 2

Find the area formed by the triangle whose sides are $y^2 - 9xy + 18x^2 = 0$ and $y = 9$

Solution

$y^2 - 9xy + 18x^2 = 0$

$\Rightarrow$ $(y - 3x)(y - 6x) = 0$

$\Rightarrow$ the sides of the triangle are $y - 3x = 0$ and $y - 6x = 0$ and $y - 9 = 0$

$\Rightarrow$ By solving these simultaneously, we get the vertices as

$A \equiv (0, 0)$ $B \equiv (3/2, 9)$ $C \equiv (3, 9)$

$$\text{Area} = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{3}{2} & 9 & 1 \\ 3 & 9 & 1 \end{vmatrix} = \frac{27}{4} \text{ sq. units.}$$

Example : 3

Find the angle between the lines $x^2 + 4y^2 - 7xy = 0$

Solution

Using the result given in section 1.3, we get :

$$\text{Angle between the lines} = \theta = \tan^{-1}\frac{2\sqrt{h^2 - ab}}{a + b} = \tan^{-1}\left[\frac{2\sqrt{\left(\frac{-7}{2}\right)^2 - 1(4)}}{1 + 4}\right] \tan^{-1}\left[\frac{\sqrt{33}}{5}\right]$$

Example : 4

Find the equation of pair of lines through origin which form an equilateral triangle with the lines $Ax + By + C = 0$. Also find the area of this equilateral triangle.

Solution

Let PQ be the side of the equilateral triangle lying on the line $Ax + By + C = 0$

Let m be the slope of line through origin and making an angle of 60º with $Ax + By + C = 0$

$\Rightarrow$ m is the slopes of OP or OQ

$\Rightarrow$ As the triangle is equilateral, $Ax + By + C = 0$ line makes an angle of 60º with OP and OQ

i.e. $$\tan 60^\circ = \left|\frac{m(-A/B)}{1 + m\left(\frac{-A}{B}\right)}\right| \quad \Rightarrow \quad 3 = \left(\frac{mB + A}{B - mA}\right)^2 \qquad ....(i)$$

This quadratic will give two values of m which are slopes of OP and OQ.

As OP and OQ pass through origin, their equations can be taken as : $y = mx$ ....(ii)

Since we have to find the equation of OP and OQ, we will not find values of m but we will eliminate m between (i) and (ii) to directly get the equation of the pair of lines : OP and OQ

$$\Rightarrow \quad 3 = \left(\frac{By/x + A}{B - yA/x}\right)^2 \quad \Rightarrow \quad 3 = \left(\frac{By + Ax}{Bx - yA}\right)^2$$

$\Rightarrow$ $3(B^2x^2 + y^2A^2 - 2ABxy) = (B^2y^2 + A^2x^2 + 2ABxy)$

⇒ $(A^2 - 3B^2)x^2 + 8ABxy + (B^2 - 3A^2)y^2 = 0$ is the pair of lines through origin makes an equilateral triangle (OPQ) with $Ax + By + C = 0$

Area of equilateral ΔOPQ = $\frac{\sqrt{3}}{4}$ (side)^2 = $\frac{\sqrt{3}}{4}\left(\frac{P}{\sin 60}\right)^2$ where P = altitude.

⇒ area = $\frac{\sqrt{3}}{4} \times \frac{4}{3} P^2 = \frac{1}{\sqrt{3}} P^2 = \frac{1}{\sqrt{3}}\left[\frac{|C|}{\sqrt{A^2+B^2}}\right]^2 = \frac{C^2}{\sqrt{3}(A^2+B^2)}$

Example : 5

If a pair of lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ is such that each pair bisects the angle between the other pair, prove that $pq = -1$

Solution

The pair of bisectors for $x^2 - 2pxy - y^2 = 0$ is : $\frac{x^2 - y^2}{1-(-1)} = \frac{xy}{-p}$

⇒ $x^2 - y^2 = \frac{2xy}{-p}$

⇒ $x^2 + \frac{2}{p}xy - y^2 = 0$

As $x^2 + \frac{2}{p}xy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ coincide, we have

$\frac{1}{1} = \frac{2/p}{-2q} = \frac{-1}{-1}$

⇒ $\frac{2}{p} = -2q$ ⇒ $pq = -1$

Example : 6

Prove that the angle between one of the lines given by $ax^2 + 2hxy + by^2 = 0$ and one of the lines $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$ is equal to the angle between the other two lines of the system.

Solution

Let $L_1 L_2$ be one pair and P_1P_2 be the other pair.
If the angle between L_1P_1 is equal to the angle between L_2P_2 ,
the pair of bisectors of L_1L_2 is same as that of P_1P_2

⇒ Pair of bisectors of P_1P_2 is $\frac{x^2 - y^2}{(a+\lambda)-(b+\lambda)} = \frac{xy}{h}$

⇒ $\frac{x^2 - y^2}{x - b} = \frac{xy}{h}$

Which is same as the bisector pair of L_1L_2
Hence the statement is proved.

Example : 7

Show that the orthocentre of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $\ell x + my = 1$ is given by $\frac{x}{\ell} = \frac{y}{m} = \frac{a+b}{am^2 - 2h\ell m + b\ell^2}$.

Solution

Let the triangle be OBC where O is origin and BC is the line $\ell x + my = 1$.
⇒ The equation of pair of lines OB and OC is $ax^2 + 2hxy + by^2 = 0$.

The equation of the altitude from O to BC is :

$$y - 0 = m/\ell\ (x - 0)$$

$\Rightarrow \quad mx - \ell y = 0$ ....(i)

Let equation of OB be $y - m_1x = 0$ and that of OC be $y - m_2x = 0$

$$\Rightarrow \quad B \equiv \left[\frac{1}{\ell + mm_1}, \frac{m_1}{\ell + mm_1}\right]$$

Slope of altitude from B to OC is $-1/m_2$

$\Rightarrow$ equation of altitude from B is :

$$y - \frac{m_1}{\ell + mm_1} = \frac{-1}{m_2}\left[x - \frac{1}{\ell + mm_1}\right]$$

$\Rightarrow \quad (l + mm_1)\ x + m_2\ (\ell + mm_1)\ y - (1 + m_1m_2) = 0$ ....(ii)

Solving (i) and (ii), we get orthocentre

$$\frac{x}{-\ell(1 + m_1m_2)} = \frac{y}{-m(1 + m_1m_2)} = \frac{1}{-\ell(\ell + mm_1) - m(\ell + mm_1)m_2}$$

using values of m_1m_2 and $m_1 + m_2$, we get :

$$\Rightarrow \quad \frac{x}{\ell} = \frac{y}{m} = \frac{-(1 + a/b)}{-\ell^2 - m^2m_1m_2 - \ell m(m_1 + m_2)} = \frac{a + b}{b\ell^2 + am^2 - 2h\ell m}$$

Example : 8

Prove that the equation $6x^2 - xy - 12y^2 - 8x + 29y - 14 = 0$ represent a pair of lines. Find the equations of each line.

Solution

Using the result given in section 2.1, we get

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 6 & -1/2 & -4 \\ -1/2 & -6 & \frac{29}{2} \\ -4 & \frac{29}{2} & -14 \end{vmatrix} = 0$$

Hence the given equation represents a pair of lines.

To find the equation of each line, we have to factories the LHS. We first factories the second degree term.

The second degree terms in the expression are :

$$6x^2 - xy - 12y^2 = 6x^2 - 9xy + 8xy - 12y^2 = (3x + 4y)\ (2x - 3y).$$

Let the two factors be $3x + 4y + C_1$ and $2x - 3y + C_2$.

$\Rightarrow \quad 6x^2 - xy - 12y^2 - 8x + 29y - 14 = (3x + 4y + C_1)\ (2x - 3y + C_2)$

Comparing the coefficients of x and y, we get :

$-8 = 3C_2 + 2C_1$ and $29 = 4C_2 - 3C_1$

Solving for C_1 and C_2 , we get :

$C_2 = 2$ and $C_1 = -7$

$\Rightarrow$ the lines are $3x + 4y - 7 = 0$ and $2x - 3y + 2 = 0$

Example : 9

Find the equation of the lines joining the origin to the points of intersection of the line $4x - 3y = 10$ with the circle $x^2 + y^2 + 3x - 6y - 20 = 0$ and show that they are perpendicular.

Solution

To find equation of pair of lines joining origin to the points of intersection of given circle and line, we will make the equation of circle homogeneous by using : $1 = \frac{4x - 3y}{10}$

$\Rightarrow$ the pair of lines is : $x^2 + y^2 + (3x - 6y)\left(\frac{4x-3y}{10}\right) - 20\left(\frac{4x-3y}{10}\right)^2 = 0$

$\Rightarrow$ $10x^2 + 15xy - 10y^2 = 0$

Coefficient x^2 + coefficient of $y^2 = 10 - 10 = 0$

$\Rightarrow$ The lines of the pair are perpendicular.

This question can also be asked as :

["Show that the chord $4x - 3y = 10$ of the circle $x^2 + y^2$ $3x - 6y - 20 = 0$ subtends a right angle at origin."]

Example : 10

A variable chord of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ always subtends a right angle at origin. Find the locus of the foot of the perpendicular drawn from origin to this chord.

Solution

Let the variable chord be $\ell x + my = 1$ where ℓ, m are changing quantities (i.e. parameters that change with the moving chord)

Let $P(x_1, y_1)$ be the foot of the perpendicular from origin to the chord.

If AB is the chord, then the equation of pair OA and OB is :

$x^2 + y^2 + (2gx + 2fy)(\ell x + my) + c(\ell x + my)^2 = 0$

$\Rightarrow$ $x^2(1 + 2g\ell + c\ell^2) + y^2(1 + 2fm + cm^2) + (2gm + 2f\ell) + 2c\ell m)\,xy = 0$

As OA is perpendicular OB,

coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow$ $(1 + 2g\ell + c\ell^2) + (1 + 2fm + cm^2) = 0$

As P lies on AB, $\ell x_1 + my_1 = 1$

As OP $\perp$ AB $\left(\frac{y_1}{x_1}\right)\left(\frac{-\ell}{m}\right) = -1$

We have to eliminate ℓ, m using (i), (ii) and (iii)

From (ii) and (iii), we get $m = \frac{y_1}{x_1^2 + y_1^2}$ and $\ell = \frac{x_1}{x_1^2 + y_1^2}$

Now from (i), we get :

$$1 + \frac{2gx_1}{x_1^2 + y_1^2} + \frac{cx_1^2}{(x_1^2 + y_1^2)^2} + \frac{2fy_1}{x_1^2 + y_1^2} + 1 + \frac{cy_1^2}{(x_1^2 + y_1^2)^2} = 0$$

$\Rightarrow$ $2(x_1^2 + y_1^2) + 2gx_1 + 2fy_1 + c = 0$

$\Rightarrow$ the locus of P is : $2(x^2 + y^2) + 2gx + 2fy + c = 0$

Example : 11

Show that the locus of a point, such that two of the three normals drawn from it to the parabola $y^2 = 4ax$ are perpendicular is $y^2 = a(x - 3a)$.

Solution

Let $P \equiv (x_1, y_1)$ be the point from where normals AP, BP, CP are drawn to $y^2 = 4ax$.

Let $y = mx - 2am - 2m^3$ be one of these normals

P lies on it $\Rightarrow$ $y_1 = mx_1 - am - am^3$.

Slopes m_1, m_2, m_3 of AP, BP, CP are roots of the cubic

$y_1 = mx_1 - 2am - am^2$

$\Rightarrow$ $am^3 + (2a - x_1)m + y_1 = 0$

$\Rightarrow$ $m_1 + m_2 + m_3 = 0$

$\Rightarrow$ $m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a - x_1}{a}$

$\Rightarrow$ $m_1m_2m_3 = -\frac{y_1}{a}$

As two of the three normals are perpendicular, we take $m_1m_2 = -1$ (i.e. we assume AP perpendicular BP)

To get the locus, we have to eliminate m_1, m_2, m_3 .

$m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a - x_1}{a}$

$\Rightarrow \quad -1 + m_3(-m_3) = \frac{2a - x_1}{a}$

$\Rightarrow \quad -1 - \left(\frac{+y_1}{a}\right)^2 = \frac{2a - x_1}{a}$ [using $m_1m_2m_3 = -y_1/a$ and $m_1m_2 = -1$]

$\Rightarrow \quad a^2 + y_1^2 = -2a^2 + ax_1$

$\Rightarrow \quad y_1^2 = a(x_1 - 3a)$

$\Rightarrow \quad y^2 = a(x - 3a)$ is the required locus.

Example : 12

Suppose that the normals drawn at three different points on the parabola $y^2 = 4x$ pass through the point (h, k). Show that h > 2

Solution

Let the normal(s) be $y = mx - 2am - 2m^3$. they pass through (h, k).

$\Rightarrow \quad k = mh - 2am - am^3$.

The three roots m_1, m_2, m_3 of this cubic are the slope of the three normals. Taking a = 1, we get :

$m^3 + (2 - h) m + k = 0$

$\Rightarrow \quad m_1 + m_2 + m_3 = 0$

$\Rightarrow \quad m_1m_2 + m_2m_3 \ \ m_3m_1 = 2 - h$

$\Rightarrow \quad m_1m_2m_3 = -k$

As m_1, m_2, m_3 are real, $m_1^2 + m_2^2 + m_3^2 > 0$ (and not all are zero)

$\Rightarrow \quad (m_1 + m_2 + m_3)^2 - 2(m_1m_2 + m_2m_3 + m_3m_1) > 0$

$\Rightarrow \quad 0 - 2(2 - h) > 0$

$\Rightarrow \quad h > 2.$

Example : 13

If the normals to the parabola $y^2 = 4ax$ at three points P, Q and R meet at A and S be the focus, prove that SP . SQ . SR = $a(SA)^2$.

Solution

Since the slopes of normals are not involved but the coordinates of P, Q, R are important, we take the normal as :

$tx + y = 2at = at^3$

Let A ≡ (h, k)

$\Rightarrow \quad t_1, t_2, t_3$ are roots of the $th + k = 2at^3$ i.e. $at^3 + (2a - h)t - k = 0$

$\Rightarrow \quad t_1 + t_2 + t_3 = 0$

$\Rightarrow \quad t_1t_2 + t_2t_3 + t_3t_1 = \frac{2a - h}{a}$

$\Rightarrow \quad t_1t_2t_3 = k/a$

Remainder that distance of point P(t) from focus and from directrix is $SP = a(1 + t^2)$

$\Rightarrow \quad SP = a(1 + t_1^2)$, $SQ = a(1 + t^2)$, $SR = a(1 + t_3)^2$

SP, SQ, SR = $a^3 (t_1^2 + t_2^2 + t_3^2) + (t_1^2 t_2^2 + t_2^2 t_3^2 + t_3^2 t_1^2) + (t_1^2 + t_2^2 + t_3^2) + 1$

we can see that : $t_1^2 + t_2^2 + t_3^2 = (t_1 + t_2 + t_3)^2 - 2\Sigma t_1t_2 = 0 - 2\ \frac{(2a - h)}{a}$

and also $\Sigma t_1^2 t_2^2 = (\Sigma t_1t_2)^2 - 2\Sigma (t_1t_2)(t_2t_3)$ [using : $\Sigma a^2 = (\Sigma a)^2 - 2\Sigma ab$]

$= \frac{(2a - h)^2}{a^2} - 2t_1t_2t_3(0) = \frac{(2a - h)^2}{a^2}$

$\Rightarrow \quad$ SP, SQ, SR = $a^3 \left\{\frac{k^2}{a^2} + \frac{(2a - h)^2}{a^2} + \frac{2h - 4a}{a} + 1\right\} = a\{(h - a)^2 + k^2\} = aSA^2$

Example : 14

Show that the tangent and the normal at a point P on the parabola $y^2 = 4ax$ are the bisectors of the angle between the focal radius SP and the perpendicular from P on the directrix.

Solution

Let $P \equiv (at^2, 2at)$, $S \equiv (a, 0)$

Equation of SP is : $y - 0 = \dfrac{2at - 0}{at^2 - a}(x - a)$

$\Rightarrow \quad 2tx + (1 - t^2) y + (-2at) = 0$ ....(i)

Equation of PM is : y – 2sat = 0 ....(ii)

Angle bisectors of (i) and (ii) are :

$$\frac{y - 2at}{\sqrt{0+1}} = \pm \frac{2tx + (1-t^2)y - 2at}{\sqrt{4t^2 + (1-t^2)^2}}$$

$$\Rightarrow \quad y - 2at = \pm \frac{2tx + (1-t^2)y - 2at}{1+t^2}$$

$\Rightarrow \quad ty = x + at^2$ and $tx + y = 2at + at^3$

$\Rightarrow$ tangent and normal at P are bisectors of SP and PM.

Alternate Method :

Let the tangent at P meet X-axis in Q.

As MP is parallel to X-axis, $\angle MPQ = \angle PQS$

Now we can find SP and SQ.

$SP = \sqrt{(1 - at^2)^2 + (0 - 2at)^2} = a(1 + t^2)$

Equation of PQ is $ty = x + at^2$

$\Rightarrow \quad Q \equiv (-at^2, 0)$

$\Rightarrow \quad SQ = \sqrt{(a + at^2) + 0} = a(1 + t^2)$

$\Rightarrow$ SP = SQ

$\Rightarrow \quad \angle SPQ = \angle SQP = \angle MPQ$

Hence PQ bisects $\angle SPM$

It obviously follows that normal bisects exterior angle.

Example : 15

In the parabola $y^2 = 4ax$, the tangent at the point P, whose abscissa is equal to the latus rectum meets the axis in T and the normal at P cuts the parabola again in Q. Prove that PT : PQ = 4 : 5

Solution

Latus rectum = $x_p = 4a$

Let $P \equiv (at^2, 2at)$

$\Rightarrow \quad at^2 = 4a \quad \Rightarrow \quad t = \pm 2$

We can do the problem by taking only one of the values.

Let t = 2

$\Rightarrow \quad P \equiv (4a, 4a)$

$\Rightarrow$ tangent at P is 2y = x + 4a

T lies on X-axis, $\Rightarrow \quad T \equiv (-4a, 0)$

$\Rightarrow \quad PT = \sqrt{(8a)^2 + (4a)^2} = 4a\sqrt{5}$

Let us nor find PQ.

If normal at P(t) cuts parabola again at $Q(t_1)$, then $t_1 = -t - 2/t$

$\Rightarrow \quad t_1 = -2 - 2/2 = -3$

$\Rightarrow \quad Q \equiv (9a, -6a)$

$\Rightarrow \quad PQ = \sqrt{25a^2 + 100a^2} = 5a\sqrt{5}$

$\Rightarrow$ PT : PQ = 4 : 5

Example : 16

A variable chord PQ of $y^2 = 4ax$ subtends a right angle at vertex. Prove that the locus of the point of intersection of normals at P, Q is $y^2 = 16a (x – 6x)$.

Solution

Let the coordinates of P and Q be $(at_1^2 , 2at_1)$ and $(at_2^2 , 2at_2)$ respectively.

As OP and OQ are perpendicular, we can have :

$$\left(\frac{2at_1 - 0}{at_1^2 - 0}\right)\left(\frac{2at_2 - 0}{at_2^2 - 0}\right) = -1$$

$\Rightarrow \quad t_1t_2 = -4$ ....(i)

Let the point of intersection of normals drawn at P and Q be $\equiv (x_1, y_1)$

Using the result given in section 1.4, we get :

$x_1 = 2a + a(t_1^2 + t_2^2 + t_1t_2)$ and ....(i)

$y_1 = -a\, t_1t_2 (t_1 + t_2)$

Eliminating t_1 and t_2 from (i), (ii) and (iii), we get :

$y_1^2 = 16a (x_1 - 6a)$

The required locus is $y^2 = 16a (x - 6a)$

Example : 17

The normal at a point P to the parabola $y^2 = 4ax$ meets the X-axis in G. Show that P and G are equidistant from focus.

Solution

Let the coordinates of the point P be $(at^2, 2at)$

$\Rightarrow$ The equation of normal at P is : $tx + y = 2at + at^3$

The point of intersection of the normal with X-axis is $G \equiv (2a + at^2, 0)$.

$SP = a(1 + t^2)$ and $SG = \sqrt{(a + at^2)^2 + O^2} = a(1 + t^2)$.

$\Rightarrow$ SP = SG

Hence P and G are equidistant from focus.

Example : 18

Tangents to the parabola $y^2 = 4ax$ drawn at points whose abscise are in the ratio $\mu^2 : 1$. Prove that the locus of their point of intersection is $y^2 = [\mu 1/2 + \mu^{-1/2}]\, ax$.

Solution

Let the coordinates of the two points on which the tangents are drawn at $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$.

As the abscissas are in the ratio $\mu^2 : 1$, we get :

$$\frac{at_1^2}{at_2^2} = \mu^2$$

$\Rightarrow \quad t_1 = \mu t_2$ ....(i)

Let the point of intersecting of two tangents be $M \equiv (x_1, y_1)$.

Using the result given in section 1.2, we get :

$M \equiv (x_1, y_1) \equiv [at_1\, t_2, a(t_1 + t_2)]$

$\Rightarrow \quad x_1 = at_1\, t_2$ ....(ii)

and $\quad y_1 = a(t_1 + t_2)$ ....(iii)

Eliminate t_1 and t_2 from equations (i), (ii) and (iii) to get :

$y_1^2 = [\mu^{1/2} + \mu^{-1/2}]^2\, ax_1$

$\Rightarrow$ The required locus of M is : $y^2 = [\mu^{1/2} + \mu^{-1/2}]^2\, ax$.

Example : 19

Find the equation of common tangent to the circle $x^2 + y^2 = 8$ and parabola $y^2 = 16x$.

Solution

Let $ty = x + at^2$ (where a = 4) be a tangent to parabola which also touches circle.

$\Rightarrow \quad ty = x + 4t^2$ and $x^2 + y^2 = 8$ have only one common solution.

$\Rightarrow \quad (ty - 4t^2)^2 + y^2 = 8$ has equal roots as a quadratic in y.

$\Rightarrow$ $(1 + t^2)\, y^2 - 8t^3y + 16t^4 - 8 = 0$ has equal roots.

$\Rightarrow$ $64t^6 = 64t^6 + 64t^4 - 32 - 32t^2$

$\Rightarrow$ $t^2 + 1 - 2t^4 = 0 \quad \Rightarrow \quad t^2 = 1, -1/2$

$\Rightarrow$ $t = \pm 1$

$\Rightarrow$ the common tangents are $y = x + 4$ and $y = - x - 4$.

Example : 20

Through the vertex O of the parabola $y^2 = 4ax$, a perpendicular is drawn to any tangent meeting it at P and the parabola at Q. Show that OP. OQ = constant.

Solution

Let $ty = x + at^2$ be the equation of the tangent

OP = perpendicular distance of tangent from origin

$$\Rightarrow \quad OP = -\frac{at^2}{\sqrt{1+r^2}}$$

Equation of OP is $y - 0 = - t(x - 0) \quad \Rightarrow \quad y = - tx$

Solving $y = - tx$ and $y^2 = 4ax$, we get

$$Q \equiv \left(\frac{4a}{t^2}, \frac{-4a}{t}\right)$$

$$\Rightarrow \quad OQ^2 = \frac{16a^2}{t^4} + \frac{16a^2}{t^2}$$

$\Rightarrow$ $OP \,.\, OQ = 4a^2$

Example : 21

Prove that the circle drawn on any focal chord as diameter touches the directrix.

Solution

Let $P(t_1)$ and $Q(t_2)$ be the ends of a focal chord.

Using the result given in section 1.3, we get : $t_1t_2 = - 1$

Equation of circle with PQ as diameter is :

$(x - at_1^2)(x - at_2^2) + (y - 2at_1)(y - 2at_2) = 0$ (using diametric form of equation of circle)

For the directrix to touch the above circle, equation of circle and directrix must have a unique solution i.e.

Solving $x = - a$ and circle simultaneously, we get

$a^2 (1 + t^2)(1 + t_2^2) + y^2 - 2ay (t_1 + t_2) + 4a^2 t_1t_2 = 0$

This quadratic in y has discriminant = $D = B^2 - 4AC$

$\Rightarrow$ $D = 4a^2 (t + t_2)^2 - 4a^2 [(1 + t_1^2)(1 + t_1^2) + 4t_1t_2] = 0$ (using $t_1t_2 = - 1$)

$\Rightarrow$ circle touches $x = - a$

$\Rightarrow$ circle touches the directrix.

Example : 22

Find the eccentricity, foci, latus rectum and directories of the ellipse $2x^2 + 3y^2 = 6$

Solution

The equation of the ellipse can be written as : $\dfrac{x^2}{3} + \dfrac{y^2}{2} = 1$

On comparing the above equation of ellipse with the standard equation of ellipse, we get

$a = \sqrt{3}$ and $b = \sqrt{2}$

We known that : $b^2 = a^2 (1 - e^2)$

$\Rightarrow$ $2 = 3 (1 - e^2) \quad \Rightarrow \quad e = 1/\sqrt{3}$

Using the standard results, foci are (ae, 0) and (–ae, 0)

$\Rightarrow$ foci are (1, 0) and (–1, 0)

Latus rectum = $2b^2/a = 4\sqrt{3}$

Directrices are $x = \pm a/e \quad \Rightarrow \quad x = \pm 3$

Example : 23

If the normal at a point P(θ) to the ellipse $\frac{x^2}{14} + \frac{y^2}{5} = 1$ intersect it again at Q(2θ), show that $\cos\theta = -2/3$.

Solution

The equation of normal at P(θ) : $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

As Q ≡ (a cos 2θ, b sin 2θ) lies on it, we can have :

$$\frac{a}{\cos\theta}(a\cos 2\theta) - \frac{b}{\sin\theta}(b\sin 2\theta) = a^2 - b^2$$

$$\Rightarrow \quad a^2\,\frac{(2\cos^2\theta - 1)}{\cos\theta} - 2b^2\cos\theta = a^2 - b^2$$

Put $a^2 = 14$, $b^2 = 5$ in the above equation to get :

$14(2\cos^2\theta - 1) - 10\cos^2\theta = 9\cos\theta$

$\Rightarrow \quad 18\cos^2\theta - 9\cos\theta - 14 = 0$

$\Rightarrow \quad (6\cos\theta - 7)(3\cos\theta + 2) = 0$

$\Rightarrow \quad \cos\theta = 7/6$ (reject) or $\cos\theta = -2/3$

Hence $\cos\theta = -2/3$

Example : 24

If the normal at end of latus rectum passes through the opposite end of minor axis, find eccentricity.

Solution

The equation of the normal at L ≡ (ae, b^2/a) is given by :

$$\frac{a^2x}{ae} - \frac{b^2y}{b^2/a} = a^2 - b^2$$

$$\Rightarrow \quad \frac{x}{e} - y = \frac{a^2 - b^2}{a}$$

According to the question, B′ (0, –b) lies on the above normal.

$\Rightarrow \quad 0/e + b = (a^2 - b^2)/a$

$\Rightarrow \quad a^2 - b^2 - ab = 0$

Using $b^2 = a^2(1 - e^2)$, we get :

$a^2e^2 - ab = 0$

$\Rightarrow \quad b = ae^2$

$\Rightarrow \quad a^2e^4 = a^2(1 - e^2)$ [using : $b^2 = a^2(1 - e^2)$]

$\Rightarrow \quad e^4 = 1 - e^2$

$$\Rightarrow \quad e^2 - \frac{\sqrt{5} - 1}{2}$$

Example : 25

Show that the locus of the foot of the perpendicular drawn from the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ on any tangent is $(x^2 + y^2) = a^2x^2 + b^2y^2$.

Solution

Let the tangent be $y = mx + \sqrt{a^2m^2 + b^2}$

Drawn CM is perpendicular to tangent and let M ≡ (x_1, y_1)

M lies on tangent, $\Rightarrow \quad y_1 = mx_1 + \sqrt{a^2m^2 + b^2}$ ....(i)

Slope (CM) = $-1/m$

$\Rightarrow \quad \frac{y_1}{x_1} = -\frac{1}{m}$

$\Rightarrow \quad m = m = -\frac{x_1}{y_1}$ ....(ii)

Replace the value of m from (ii) into (i) to get :

$(x_1^2 + y_1^2)^2 = a^2 x_1^2 + b^2 y_1^2$

Hence the required locus is : $(x^2 + y^2)^2 = a^2 x^2 + b^2 y^2$

Example : 26

The tangent at a point P on ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cuts the directrix in F. Show that PF subtends a right angle at the corresponding focus.

Solution

Let $P \equiv (x_1, y_1)$ and $S \equiv (ae, 0)$

The equation of tangent at P is : $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

To find F, we put x = a/e in the equation of the tangent

$\Rightarrow \quad \frac{ax_1}{a^2e} + \frac{yy_1}{b^2} = 1$

$\Rightarrow \quad y = \frac{(ae - x_1)b^2}{aey_1}$

$\Rightarrow \quad F \equiv \left[\frac{a}{e}, \frac{(ae - x_1)b^2}{aey_1}\right]$

$\Rightarrow \quad \text{slope (SF)} = \frac{(ae - x_1)b^2}{aey_1} \; \frac{1}{\frac{a}{e} - ae}$ ....(i)

slope (SP) = $\frac{y_1 - 0}{x_1 - ae}$ ....(ii)

From (i) and (ii),

slope of (SF) × slope (SP) = – 1

SF and SP are perpendicular

Hence PF subtends a right angle at the focus.

Example : 27

Show that the normal of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at any point P bisects the angle between focal radii SP and S′P.

Solution

Let PM be the normal and $P \equiv (x_1, y_1)$

$\Rightarrow \quad$ equation of normal PM is $\frac{xa^2}{x_1} - \frac{yb^2}{y_1} = a^2 - b^2$

We will try to show that : $\frac{S'P}{SP} = \frac{MS'}{MS}$

M is the point of intersection of normal PM with X-axis

⇒ Put y = 0 is normal PM to get $M \equiv \left[\frac{(a^2 - b^2)\,x_1}{a^2}, 0\right]$ = $[e^2x_1, 0]$

⇒ $MS = ae - e^2x_1$ and $MS' = ae = e^2x_1$

⇒ $\frac{MS'}{MS} = \frac{e(a+ex_1)}{e(a-ex_1)} = \frac{a+ex_1}{a-ex_1} = \frac{SP'}{SP}$ (using result given in section 1.1)

Example : 28

A tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches at the point P on it in the first quadrant and meets the axes in A and B respectively. If P divides AB is 3 : 1, find the equation of tangent.

Solution

Let the coordinates of the point $P \equiv (a \cos \theta , b \sin \theta)$

⇒ the equation of the tangent at P is : $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ ....(i)

⇒ The coordinates of the points A and B are :

$A \equiv \left(\frac{a}{\cos\theta}, 0\right)$ and $B \equiv \left(0, \frac{b}{\sin\theta}\right)$

By section formula, the coordinates of P are $\left(\frac{a}{4\cos\theta}, \frac{3b}{3\sin\theta}\right) \equiv (a \cos \theta , b \sin \theta)$

⇒ $\frac{a}{4\cos\theta} = a \cos \theta$ and $\frac{3b}{4\sin\theta} = b \sin \theta$

⇒ $\cos \theta = \pm \frac{1}{2}$ and $\sin \theta = \pm \frac{\sqrt{3}}{2}$

⇒ $\theta = 60^o$

For equation of tangent, replace the value of θ in (i)

⇒ The equation of tangent is : $\frac{x}{a} + \frac{\sqrt{3}y}{b} = 2$

Example : 29

If the normal at point P of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with centre C meets major and minor axes at G and g respectively, and if CF be perpendicular to normal, prove that PF . PG = b^2 and PF . Pg = a^2.

Solution

If Pm is tangent to the ellipse at point P, then CMPF is a rectangle.

⇒ CM = PF ....(i)

Let the coordinates of point P be $(a \cos \theta, b \sin \theta)$

The equation of normal at P is : $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

The point of intersection of the normal at P with X-axis is $G \equiv \left[\frac{(a^2 - b^2)\cos\theta}{a}, 0\right]$

The point of intersection of the normal at P with Y-axis is $g \equiv \left[0, \frac{(b^2 - a^2)\sin\theta}{b}\right]$

$\Rightarrow$ $PG^2 = \frac{b^2}{a^2}$ $[b^2 \cos^2\theta + a^2 \sin^2\theta]$ ....(ii)

and $Pg^2 = \frac{a^2}{b^2}$ $[b^2 \cos^2\theta + a^2 \sin^2\theta]$ ....(iii)

From (i),

$\Rightarrow$ PF = MC = distance of centre of the ellipse from the tangent at P

$$= \frac{1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}} = \frac{ab}{\sqrt{b^2\cos^2 + a^2\sin^2\theta}} \quad \text{...........(iv)}$$

Multiplying (iii) and (iv), we get :

$PF^2 \,.\, PG^2 = b^4$

Multiplying (iii) and (iv), we get :

$PF^2 \,.\, Pg^2 = a^4$

Hence proved

Example : 30

Any tangent to an ellipse is cut by the tangents at the ends of the major axis in T and T′. Prove that circle on TT′ as diameter passes through foci.

Solution

Consider a point P on the ellipse whose coordinates are $(a \cos \theta , b \sin \theta)$

The equation of tangent drawn at P is : $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ ....(i)

The two tangents drawn at the ends of the major axis are x = a and x = – a.

Solving tangent (i) and x = a we get T = $\left[a, \frac{b(1-\cos\theta)}{\sin\theta}\right] \equiv \left[a, b\tan\frac{\theta}{2}\right]$

Solving tangent (i) and x = – a, we get T′ $\equiv \left[-a, \frac{b(1+\cos\theta)}{\sin\theta}\right] = \left[-a, \cot\frac{\theta}{2}\right]$

Circle on TT′ as diameter is $x^2 - a^2 + (y - b \tan \theta/2)(y - b \cot \theta/2) = 0$

(using diametric form of equation of circle)

Put $x = \pm ae$, $y = 0$ in LHS to get :

$a^2e^2 - a^2 + b^2 = 0$ = RHS

Hence foci lie on this circle.

Example : 31

A normal inclined at 45º to the X-axis is drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It cuts major and minor axes in P and Q. If C is centre of ellipse, show that are $(\Delta CPQ) = \frac{(a^2 - b^2)^2}{a(a^2 + b^2)}$.

Solution

Consider a point M on the ellipse whose coordinates are $(a \cos \theta, b \sin \theta)$

The equation of normal drawn at M is : $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

As the normal makes an angle 45º with X-axis, slope of normal = tan 45º

$\Rightarrow$ $\tan 45^\circ = \frac{a\sin\theta}{b\cos\theta}$ $\Rightarrow$ $\tan\theta = \frac{b}{a}$

$\Rightarrow \quad \sin\theta = \dfrac{b}{\sqrt{a^2+b^2}}$ and $\cos\theta = \dfrac{a}{\sqrt{a^2+b^2}}$ ....(i)

The point of intersecting of the normal with X-axis is $P \equiv \left[\dfrac{a^2-b^2}{a}\cos\theta, 0\right]$

$\Rightarrow \quad CP = \left|\dfrac{a^2-b^2}{a}\cos\theta\right|$ ....(ii)

The point of intersection of the normal with Y-axis is $Q \equiv \left[0, \dfrac{b^2-a^2}{b}\sin\theta\right]$

$\Rightarrow \quad CQ = \left|\dfrac{b^2-a^2}{b}\right|$ ....(iii)

Ar $(\Delta CPQ) = \dfrac{1}{2}\, PC \times CQ$

Using (ii) and (iii), Ar $(\Delta CPQ) = \dfrac{1}{2}\left|\dfrac{(a^2-b^2)^2}{ab}\sin\theta\cos\theta\right|$

Using (i), Ar $(\Delta CPQ) = \dfrac{1}{2}\,\dfrac{(a^2-b^2)^2}{a^2+b^2}$

Example : 32

If P, Q are points on $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, whose centre is C such that CP is perpendicular to CQ, show that

$\dfrac{1}{CP^2} + \dfrac{1}{CQ^2} = \dfrac{1}{a^2} - \dfrac{1}{b^2}$ given that $(a < b)$

Solution

Let y = mx be the equation of CP. Solving y = mx and $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, we get coordinates of P.

$\Rightarrow \quad \dfrac{x^2}{a^2} - \dfrac{m^2x^2}{b^2} = 1 \quad \Rightarrow \quad x^2 = \dfrac{a^2b^2}{b^2-a^2m^2},\ y^2 = \dfrac{a^2b^2m^2}{b^2-a^2m^2}$

$\Rightarrow \quad CP^2 = x^2 + y^2 = \dfrac{a^2b^2(1+m^2)}{b^2-a^2m^2}$

Similarly, be replacing m by – 1/m, we get coordinates of Q because equation of CQ is $y = \dfrac{-1}{m}x$.

$\Rightarrow \quad CQ^2 = \dfrac{a^2b^2\left(1+\dfrac{1}{m^2}\right)}{b^2-\dfrac{a^2}{m^2}} = \dfrac{a^2b^2(m^2+1)}{b^2m^2-a^2}$

$\Rightarrow \quad \dfrac{1}{CP^2} + \dfrac{1}{CQ^2} = \dfrac{b^2-a^2m^2+b^2m^2-a^2}{a^2b^2(1+m^2)} = \dfrac{b^2-a^2}{a^2b^2} = \dfrac{1}{a^2} - \dfrac{1}{b^2}$

Example : 33

Find the locus of the foot of the perpendicular drawn from focus S of hyperbola $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to any tangent.

Solution

Let the tangent be $y = mx + \sqrt{a^2m^2 - b^2}$

Let m (x_1, y_1) be the foot of perpendicular SM drawn to the tangent from focus S (ae, 0).

Slope (SM) × slope (PM) = – 1

$$\Rightarrow \quad \left(\frac{y_1 - 0}{x_1 - ae}\right) m = -1$$

$x_1 + my_1 = ae$ ....(i)

As M lies on tangent, we also have $y_1 = m_1x + \sqrt{a^2m^2 - b^2}$

$$\Rightarrow \quad -mx_1 + y_1 = \sqrt{a^2m^2 - b^2} \quad ....(ii)$$

We can now eliminate m from (i) and (ii).

Substituting value of m from (i) in (ii) leads to a lot of simplification and hence we avoid this step.

By squaring and adding (i) and (ii), we get

$x_1^2 (1 + m^2) + y_1^2 (1 + m^2) = a^2e^2 + a^2m^2 - b^2$

$\Rightarrow \quad (x_1^2 + y_1^2) (1 + m^2) = a^2 (1 + m^2)$

$\Rightarrow \quad x_1^2 + y_1^2 = a^2$

$\Rightarrow$ Required locus is : $x^2 + y^2 = a^2$ (Note that M lies on the auxiliary circle)

Example : 34

Prove that the portion of the tangent to the hyperbola intercepted between the asymptotes is bisected at the point of contact and the area of the triangle formed by the tangent and asymptotes is constant.

Solution

Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the hyperbola and let the point of contact be P $(a \sec\theta, b \tan\theta)$

Let the tangent meets the asymptotes $y = \frac{bx}{a}$ and $y = -\frac{bx}{b}$ in points M, N respectively.

Solving the equation of tangent and asymptotes, we can find M and N

Solve : $\frac{x\cos\theta}{a} - \frac{y\tan\theta}{b} = 1$ and $y = \frac{bx}{a}$ to get :

$$x = \frac{a}{\sec\theta - \tan\theta}, \quad y = \frac{b}{\sec\theta - \tan\theta}$$

$$\Rightarrow \quad M \equiv \left[\frac{a}{\sec\theta - \tan\theta}, \frac{b}{\sec\theta - \tan\theta}\right],$$

Similarly solving $y = -\frac{bx}{a}$ and $\frac{x}{a}\sec\theta - \frac{y}{a}\tan\theta = 1$, we get :

$$N \equiv \left[\frac{a}{\sec\theta + \tan\theta}, \frac{-b}{\sec\theta + \tan\theta}\right]$$

$$\text{Mid point of MN} \equiv \left[\frac{a\sec\theta}{\sec^2\theta - \tan^2\theta}, \frac{b\tan\theta}{\sec^2\theta - \tan^2\theta}\right] \equiv (a \sec\theta, b \tan\theta)$$

Hence P bisects MN.

Area of ΔCNM = $\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ x_N & y_N & 1 \\ x_M & y_M & 1 \end{vmatrix} = \frac{1}{2}\{x_N y_M - x_M y_N\} = \frac{1}{2}$ (ab + ab) = ab

hence area does not depend on 'θ' or we can say that area is constant.

Example : 35

Show that the locus of the mid-point of normal chords of the rectangular hyperbola $x^2 - y^2 = a^2$ is $(y^2 - x^2)^3 = 4a^2x^2y^2$.

Solution

Let the mid point of a chord be $P(x_1, y_1)$

$\Rightarrow$ Equation of chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ whose mid-point is (x_1, y_1) is :

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2 - y_1^2}{a^2}$$

As the hyperbola is rectangular hyperbola, a = b

$\Rightarrow$ Equation of the chord is : $xx_1 - yy_1 = x_1^2 - y_1^2$ ....(i)

Normal chord is a chord which is normal to hyperbola at one of its ends.

$\Rightarrow$ Equation of normal chord at (a sec θ, b tan θ) is : $\frac{ax}{\sec\theta} - \frac{by_1}{\tan\theta} = a^2 + b^2$

but here $a^2 = b^2$,

$\Rightarrow$ normal chord is : x cos θ – y cot θ = 2a ....(ii)

We now compare the two equations of same chord i.e. compare (i) and (ii) to get :

$$\Rightarrow \quad \frac{x_1}{\cos\theta} = \frac{y_1}{\cot\theta} = \frac{x_1^2 - y_1^2}{2a}$$

$$\Rightarrow \quad \sec\theta = \frac{x_1^2 - y_1^2}{2ax_1} \quad \text{and} \quad \cot\theta = \frac{2ay_1}{x_1^2 - y_1^2}$$

Eliminating θ using $\sec^2\theta - \tan^2\theta = 1$, we get :

$$\left(\frac{x_1^2 - y_1^2}{2ax_1}\right)^2 - \left(\frac{x_1^2 - y_1^2}{2ay_1}\right)^2 = 1$$

$\Rightarrow$ $(y_1^2 - x_1^2)^3 = 4a^2 x_1^2 y_1^2$

$\Rightarrow$ $(y^2 - x^2)^3 = 4a^2x^2y^2$ is the locus.

Calculus

Example : 1

Evaluate (i) $\int_1^3 x^2\,dx$ (ii) $\int_0^{\pi/2} \sin x\,dx$

Solution

(i) $\int_1^3 x^2\,dx = \left| \frac{x^3}{3} \right|_1^3 = \frac{1}{3}(3^3 - 1^3) = \frac{26}{3}$

(ii) $\int_0^{\pi/2} \sin x\,dx = \left| -\cos x \right|_0^{\pi/2} = (\cos \pi/2 - \cos 0) = 1$

Example : 2

$\int_0^{\pi/2} \sin^3 x \cos x\,dx$

Solution

Let $I = \int_0^{\pi/2} \sin^3 x \cos x\,dx$

Let $\sin x = t \Rightarrow \cos x\,dx = dt$

For $x = \frac{\pi}{2}$, $t = 1$ and for $x = 0$, $t = 0$

$\Rightarrow I = \int_0^1 t^3\,dt = \left| \frac{t^4}{4} \right|_0^1 = \frac{1}{4}$

Note : Whenever we use substitution in a definite integral, we have to change the limits corresponding to the change in the variable of the integration
In the example we have applied New-ton-Leibnitz formula to calculate the definite integral. New-Leibnitz formula is applicable here since $\sin^3 x \cos x$ (integrate) is a continuous function in the interval $[0, \pi/2]$

Example : 3

Evaluate : $\int_{-1}^{2} |x|\,dx$

Solution

$\int_{-1}^{2} |x|\,dx = \int_{-1}^{0} |x|\,dx + \int_{0}^{2} |x|\,dx$ (using property – 1)

$= \int_{-1}^{0} -x\,dx + \int_{0}^{2} x\,dx$ ($\because |x| = -x$ for $x < 0$ and $|x| = x$ for $x \geq 0$)

$= -\left| \frac{x^2}{2} \right|_{-1}^{0} + \left| \frac{x^2}{2} \right|_0^2 = -\left(0 - \frac{1}{2}\right) + \left(\frac{4}{2} - 0\right) = \frac{5}{2}$

Example : 4

Evaluate : $\int_{-4}^{3} |x^2 - 4|\, dx$

Solution

$$\int_{-4}^{3} |x^2-4|\,dx = \int_{-4}^{-2} |x^2-4|\,dx + \int_{-2}^{+2} |x^2-4|\,dx + \int_{-2}^{3} |x^2-4|\,dx$$

$$= \int_{-4}^{-2} (x^2-4)\,dx + \int_{-2}^{2} (4-x^2)\,dx + \int_{2}^{3} (x^2-4)\,dx$$

($\because$ $|x^2 - 4| = 4 - x^2$ in $[-2, 2]$ and $|x^2 - 4| = x^2 - 4$ in other intervals]

$$= \left|\frac{x^3}{3} - 4x\right|_{-4}^{-2} + \left|4x - \frac{x^3}{3}\right|_{-2}^{2} + \left|\frac{x^3}{3} - 4x\right|_{2}^{3}$$

$$= \left(-\frac{8}{3}+8\right) - \left(\frac{64}{3}+16\right) + \left(8-\frac{8}{3}\right) + \left(\frac{27}{3}-12\right) - \left(\frac{8}{3}-8\right) = \frac{71}{3}$$

Example : 5

Evaluate : $\int_{0}^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$

Solution

Let $I = \int_{0}^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$ ....(i)

Using property – 4, we have :

$$I = \int_{0}^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\pi/2 - x)} + \sqrt{\cos(\pi/2 - x)}}$$

$$I = \int_{0}^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx \qquad \text{...........(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_{0}^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx + \int_{0}^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$$

$$\Rightarrow \quad 2I = \int_{0}^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$$

$$\Rightarrow \quad 2I = \int_{0}^{\pi/2} dx = \frac{\pi}{2}$$

$$\Rightarrow \quad I = \frac{\pi}{4}$$

Example : 6

If f(a – x) = f(x), then show that $\int_0^a x\, f(x)\, dx = \frac{a}{2}\int_0^a f(x)\, dx$

Solution

Let $I = \int_0^a x\, f(x)\, dx$

$\Rightarrow \quad I = \int_0^a (a-x)\, f\,(a-x)\, dx$ (using property – 4)

$\Rightarrow \quad I = \int_0^a (a-x)\, f\,(x)\, dx$ [using f(x) = f(a – x)]

$\Rightarrow \quad I = \int_0^a a\, f(x)\, dx - \int_0^a x\, f(x)\, dx$

$\Rightarrow \quad I = a\int_0^a f(x)\, dx - I$

$\Rightarrow \quad 2I = a\int_0^a f(x)\, dx$

$\Rightarrow \quad I = \frac{a}{2}\int_0^a f(x)\, dx = \text{RHS}$

Example : 7

Evaluate : $\int_0^{\pi} \frac{x}{1+\cos^2 x}\, dx$

Solution

Let $I = \int_0^{\pi} \frac{x}{1+\cos^2 x}\, dx$ ....(i)

$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)}{1+\cos^2(\pi - x)}\, dx$ (using property – 4) ....(ii)

Adding (i) and (ii), we get :

$\Rightarrow \quad 2I = \int_0^{\pi} \frac{\pi}{1+\cos^2 x}\, dx$

$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi} \frac{dx}{1+\cos^2 x} = \frac{2\pi}{2}\int_0^{\pi/2} \frac{dx}{1+\cos^2 x}$ (using property – 6)

Divide N and D by $\cos^2 x$ to get :

$$I = \int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x + 1}\, dx$$

Put tan x = t ⇒ $\sec^2 x\, dx = dt$

For x = π/2, $t \to \infty$ and for x = 0, t = 0

$$\Rightarrow \quad I = \pi \int_0^{\infty} \frac{dt}{2 + t^2}$$

$$\Rightarrow \quad I = \left| \frac{1}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}} \right|_0^{\infty} = \frac{\pi}{\sqrt{2}} \times \frac{\pi}{2} = \frac{\pi^2}{2\sqrt{2}}$$

Example : 8

Evaluate : $\int_0^{\pi/2} \ell og \sin x\, dx$

Solution

Let $I = \int_0^{\pi/2} \ell og \sin x\, dx$ ....(i)

$$\Rightarrow \quad I = \int_0^{\pi/2} \ell og \sin\left(\frac{\pi}{2} - x\right) dx$$ (using property – 4)

$$\Rightarrow \quad I = \int_0^{\pi/2} \ell og \cos x\, dx$$ ....(ii)

Adding (i) and (ii) we get :

$$2I = \int_0^{\pi/2} \ell og (\sin x \cos x)\, dx = \int_0^{\pi/2} \ell og \left(\frac{\sin 2x}{2}\right) dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \ell og \sin 2x\, dx - \int_0^{\pi/2} \log 2\, dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \ell og \sin 2x\, dx - \frac{\pi}{2}\, \ell og 2$$ ....(iii)

Let $I_1 = \int_0^{\pi/2} \ell og \sin 2x\, dx$

Put t = 2x ⇒ dt = 2dx

For $x = \frac{\pi}{2}$, t = π and for x = 0, t = 0

$$\Rightarrow \quad I_1 = \frac{1}{2} \int_0^{\pi} \log \sin t\, dt = \frac{2}{2} \int_0^{\pi/2} \log \sin t\, dt$$ (using property – 6)

$$\Rightarrow \quad I_1 = \int_0^{\pi/2} \log \sin x\, dx$$ (using property – 3)

$\Rightarrow \quad I_1 = I$

Substituting in (iii), we get : $2I = I - \pi/2 \log 2$

$\Rightarrow \quad I = -\pi/2 \log 2$ (learn this result so that you can directly apply it in other difficult problems)

Example : 9

Show that : $\int_0^{\pi/2} f(\sin 2x) \sin x\, dx = \int_0^{\pi/2} f(\cos x\, dx = \sqrt{2} \int_0^{\pi/4} f(\cos 2x) \cos x\, dx$

Solution

Let $I = \int_0^{\pi/2} f(\sin 2x) \sin x\, dx$ ....(i)

$\Rightarrow \quad I = \int_0^{\pi/2} f[\sin 2(\pi/2 - x)] \sin(\pi/2 - x)\, dx$ (using property – 4)

$\Rightarrow \quad I = \int_0^{\pi/2} f[\sin(\pi - 2x)] \cos x\, dx$

$\Rightarrow \quad I = \int_0^{\pi/2} f(\sin 2x) \cos x\, dx$ ....(ii)

Hence the first part is proved

$I = \int_0^{\pi/2} f(\sin 2x) \sin x\, dx$

$= \int_0^{\pi/4} f(\sin 2x) \sin x\, dx + \int_0^{\pi/4} f[\sin 2(\pi/2 - x)] \sin(\pi/2 - x)\, dx$ (using property – 5)

$= \int_0^{\pi/4} f(\sin 2x) \sin x\, dx + \int_0^{\pi/4} f(\sin 2x) \cos x\, dx$

$= \int_0^{\pi/4} f(\sin 2x)(\sin x + \cos x)\, dx$

$= \int_0^{\pi/4} f(\sin 2x)(\sin x + \cos x)\, dx$

$= \int_0^{\pi/4} f[\sin 2(\pi/4 - x)][\sin(\pi/4 - x) + \cos(\pi/4 - x)\, x]\, dx$ (using property – 4)

$= \int_0^{\pi/4} f(\cos 2x)\left[\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right] dx$

$= \sqrt{2} \int_0^{\pi/4} f(\cos 2x) \cos x\, dx$

Hence the second part is also proved

Example : 10

Evaluate : $\int_{2}^{3} x\sqrt{5-x}\ dx$

Solution

Let $I = \int_{2}^{3} x\sqrt{5-x}\ dx$

$\Rightarrow \quad I = \int_{2}^{3} (2+3-x)\sqrt{5-(2+3-x)}\ dx$ (using property – 7)

$\Rightarrow \quad I = \int_{2}^{3} (5-x)\sqrt{x}\ dx$

$\Rightarrow \quad I = \int_{2}^{3} 5\sqrt{x}\ dx - \int_{2}^{3} x\sqrt{x}\ dx$

$\Rightarrow \quad I = 5\left|\frac{2}{3}x\sqrt{x}\right|_{2}^{3} - \frac{2}{5}\left|x^2\sqrt{x}\right|_{2}^{3}$

$\Rightarrow \quad I = \frac{10}{3}\left(3\sqrt{3}-2\sqrt{2}\right) - \frac{2}{5}\left(9\sqrt{3}-4\sqrt{2}\right)$

Example : 11

Evaluate : $\int_{a}^{b} \frac{f(x)}{f(x)+f(a+b+x)}\ dx$

Solution

Let $I = \int_{a}^{b} \frac{f(x)}{f(x)+f(a+b-x)}\ dx$ ....(i)

$\Rightarrow \quad I = \int_{a}^{b} \frac{f(a+b-x)}{f(a+b-x)+f[a+b-(a+b-x)]\ dx}$

$\Rightarrow \quad I = \int_{a}^{b} \frac{f(a+b-x)}{f(a+b-x)+f(x)}\ dx$ ....(ii)

Adding (i) and (ii), we get

$\Rightarrow \quad 2I = \int_{a}^{b} \frac{f(x)+f(a+b-x)}{f(x)+f(a+b-x)}\ dx$

$\Rightarrow \quad 2I = \int_{a}^{b} dx = b-a$

$\Rightarrow \quad I = \frac{b-a}{2}$

Example : 12

Evaluate : $\int_{-1}^{+1} \log\left(\frac{2-x}{2+x}\right) \sin^2 x \, dx$

Solution

Let $f(x) = \log\left(\frac{2-x}{2+x}\right) \sin^2 x \, dx$

$\Rightarrow \quad f(-x) = \log\left(\frac{2+x}{2-x}\right) \sin^2 (-x)$

$\Rightarrow \quad f(-x) = \log\left(\frac{2-x}{2+x}\right)^{-1} \sin^2 x = -\log\left(\frac{2-x}{2+x}\right) \sin^2 x = -f(x)$

$\Rightarrow$ f(x) is an odd function

Hence $\int_{-1}^{+1} f(x)\, dx = 0$ (using property – 8)

Example : 13

Evaluate : $\int_{0}^{\pi/2} \sqrt{1-\sin 2x}\, dx$

Solution

Let $I = \int_{0}^{\pi/2} \sqrt{1-\sin 2x}\, dx$

$\Rightarrow \quad I = \int_{0}^{\pi/2} \sqrt{(\sin x - \cos x)^2}\, dx$

$\Rightarrow \quad I = \int_{0}^{\pi/2} |\sin x - \cos x|\, dx$

$\Rightarrow \quad I = \int_{0}^{\pi/4} |\sin x - \cos x|\, dx + \int_{\pi/4}^{\pi/2} |\sin x - \cos x|\, dx$

$\Rightarrow \quad I = \int_{0}^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx$

$\Rightarrow \quad I = |\sin x + \cos|_0^{\pi/4} + |-\cos x - \sin x|_{\pi/4}^{\pi/2}$

$\Rightarrow \quad I = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right) + (-1) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$

$\Rightarrow \quad I = 2\sqrt{2} - 2$

Example : 14

Given a function such that :

(i) it is integrable over every interval on the real line.

(ii) $f(t + x) = f(x)$ for every x and a real t, then show that the integral $\int_a^{a+t} f(x)\,dx$ is independent of a.

Solution

Let $I = \int_a^{a+t} f(x)\,dx$

$$\Rightarrow \quad \int_a^{t} f(x)\,dx + \int_t^{a+t} f(x)\,dx \quad \text{.............(i)}$$

Consider $I_1 = \int_t^{a+t} f(x)\,dx$

Put $x = y + t \quad \Rightarrow \quad dx = dy$

For $x = a + t$, $y = a$ and For $x = t$, $y = 0$

$$\Rightarrow \quad I_1 = \int_0^{a} f(y + t)\,dy$$

$$\Rightarrow \quad I_1 = \int_0^{a} f(y)\,dy \quad \text{(using property 3)}$$

$$\Rightarrow \quad I_1 = \int_0^{a} f(x)\,dx \quad [\text{using } f(x + T) = f(x)]$$

On substituting the value of I_1 in (i), we get :

$$\Rightarrow \quad I = \int_a^{t} f(x)\,dx + I_1$$

$$\Rightarrow \quad I = \int_a^{t} f(x)\,dx + \int_0^{a} f(x)\,dx$$

$$\Rightarrow \quad I = \int_0^{a} f(x)\,dx + \int_a^{t} f(x)\,dx$$

$$\Rightarrow \quad I = \int_0^{t} f(x)\,dx \quad \text{(using property – 1)}$$

$\Rightarrow$ I is independent of a.

Example : 18

Determine a positive integer $n \le 5$ such that : $\int_0^1 e^x (x-1)^n dx = 16 - 6e$

Solution

Let $I_n = \int_0^1 e^x (x-1)^n dx$

using integration by parts

$$I_n = \left[(x-1)^n \int e^x\, dx\right]_0^1 - \int_0^1 e^x n (x-1)^{n-1}\, dx$$

$$I_n = 0 - (-1)^n - n \int_0^1 e^x (x-1)^{n-1}\, dx$$

$$I_n = -(-1)^n - nI_{n-1} \qquad \text{...............(i)}$$

Also $I_0 = \int_0^t e^x (x-1)^0\, dx = e - 1$

$\Rightarrow \quad I_1 = 1 - I_0 = 1 - (e - 1) = 2 - e$
$\Rightarrow \quad I_2 = -1 - 2I_1 = -1 - 2(2 - e) = -5 + 2e$
$\Rightarrow \quad I_3 = 1 - 3I_2 = 1 - 3(-5 + 2e) = 16 - 6e$

$\Rightarrow$ Hence for n = 3, $\int_0^1 e^x (x-1)^n\, dx = 16 - 6e$

Example : 16

If $f(x) = \int_{x^2}^{x^3} \frac{1}{\log t} dt$ t > 0, then find $f'(x)$

Solution

Using the property – 12,

$$f'(x) = \frac{1}{\log(x^3)} \frac{d}{dx} (x^3) + \frac{1}{\log x^2} \frac{d}{dx} (x^2)$$

$$\Rightarrow \quad f'(x) = \frac{3x^2}{3\log x} - \frac{2x}{2\log x} = \frac{x^2 - x}{\log x}$$

Example : 17

Find the points of local minimum and local minimum of the function $\int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t} dt$

Solution

Let $y = \int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t} dt$

For the points of Extremes, $\frac{dy}{dx} = 0$

Using property – 12

$$\frac{x^4 - 5x^2 + 5}{2 + e^{x^2}}\ 2x = 0$$

$\Rightarrow$ x = 0 or $x^4 - 5x^2 + 4 = 0$

$\Rightarrow$ x = 0 or (x – 1) (x + 1) (x – 2) (x+ 2) = 0

$\Rightarrow$ x = 0, x = ± 1 and x = ±2

With the help of first derivative test, check yourself x = – 2, 0, 2 are points of local minimum and x = – 1, 1 are points of local maximum.

Example : 18

Evaluate : $\int_a^b x^2 dx$ using limit of a sum formula

Solution

Let $I = \int_a^b x^2 dx = \lim_{\substack{n\to\infty \\ h\to 0}} h\,[1 + h)^2 + (1 + 2h)^2 + \ldots\ldots.. + (a + nh)^2\,]$

$\Rightarrow \quad I = \lim_{\substack{n\to\infty \\ h\to 0}} h[na^2 + 2ah\,(1 + 2 + 3 + \ldots\ldots. + n) + h^2\,(1^2 + 2^2 + 3^2 + \ldots\ldots. + n^2)]$

$$\Rightarrow \quad I = \lim_{\substack{n\to\infty \\ h\to 0}} \left[nha^2 + \frac{2ah^2 n(n+1)}{2} + \frac{h^2 n(n+1)(2n+1)}{6}\right]$$

Using nh = b – a, we get

$$\Rightarrow \quad I = \lim_{n\to\infty} \left[a^2(b-a) + a(b-a)^2\left(1+\frac{1}{n}\right) + (b-a)^3\,\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

$$\Rightarrow \quad I = a^2\,(b-a) + a\,(b-a)^2 + \frac{(b-a)^2}{6}\ 2$$

$$\Rightarrow \quad I = (b-a)\left[a^2 + ab - a^2 + \frac{b^2 + a^2 - 2ab}{3}\right]$$

$$\Rightarrow \quad I = \frac{(b-a)}{3}\,[a^2 + b^2 + ab] = \frac{b^3 - a^3}{3}$$

Example : 19

Evaluate the following sum. $S = \lim_{n\to\infty}\left[\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots.. + \frac{1}{2n}\right]$

Solution

$$S = \lim_{n\to\infty}\left[\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots.. + \frac{1}{2n}\right]$$

$$\Rightarrow \quad S = \lim_{n\to\infty}\frac{1}{n}\left[\frac{n}{n+1} + \frac{n}{n+2} + \ldots. + \frac{n}{2n}\right]$$

$$\Rightarrow \quad S = \lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{1+1/n} + \frac{1}{1+2/n} + \ldots\ldots. + \frac{1}{1+n/n}\right]$$

$$\Rightarrow \quad S = \lim_{n\to\infty}\frac{1}{n}\left[\sum_{r=1}^{n}\frac{1}{1+r/n}\right]$$

$$\Rightarrow \quad \int_0^1 \frac{1}{1+x}\,dx$$

$$\Rightarrow \quad S = \left| \log(1+x) \right|_0^1 = \log 2$$

Example : 20

Find the sum of the series : $\lim_{n\to\infty} \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \text{.........} + \frac{1}{6n}$

Solution

Let $S = \lim_{n\to\infty} \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \text{.........} + \frac{1}{6n}$

Take 1/n common from the series i.e.

$$S = \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{1+1/n} + \frac{1}{1+2/n} + \text{.........} + \frac{1}{1+5n/n}\right] = \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{2n} \frac{1}{1+r/n}$$

For the definite integral,

$$\text{Lower limit} = a = \lim_{n\to\infty} \left(\frac{r}{n}\right) = \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\text{Upper limit} = b = \lim_{n\to\infty} \left(\frac{r}{n}\right) = \lim_{n\to\infty} \frac{5n}{n} = 5$$

Therefore, $S = \lim_{n\to\infty} \sum_{r=0}^{5n} \frac{1}{1+(r/n)} = \int_0^5 \frac{dx}{1+x} = \ell n \left| 1+x \right|_0^5 = \ell n = 6 - \ell n\ 1 = \ell n\ 6$

Example : 21

Show that : $1 \le \int_0^1 e^{x^2} dx \le e$

Solution

Using the result given in section 3.3,

$$m\,(1-0) \le \int_0^1 e^{x^2} dx \le M(1-0) \qquad \text{..........(i)}$$

let $\quad f(x) = e^{x^2}$

$\Rightarrow \quad f'(x) = 2x\, e^{x^2} = 0 \qquad \Rightarrow \qquad x = 0$

Apply first derivative test to check that there exists a local minimum at x = 0

$\Rightarrow$ f(x) is an increasing function in the interval [0, 1]

$\Rightarrow \quad m = f(0) = 1 \quad$ and $\quad M = f(1) = e^1 = e$

Substituting the value of m and M in (i), we get

$$(1-0) \le \int_0^1 e^{x^2} dx \le e(1-0)$$

$\Rightarrow \quad 1 \le \int_0^1 e^{x^2} dx \le e \qquad$ Hence proved.

Example : 22

Consider the integral : $I = \int_0^{2\pi} \frac{dx}{5 - 2\cos x}$

Making the substitution tan x/2 = t, we have :

$$\int_0^{2\pi} \frac{dx}{5-2\cos x} = \int_0^0 \frac{2dt}{(1+t^2)\left[5 - 2\frac{1-t^2}{1+t^2}\right]} = 0$$

This result is obviously wrong since the integrand is positive and consequently the integral of this function can not be equal to zero. Find the mistake in this evaluation.

Solution

The mistake lies in the substitution $\tan\frac{x}{2} = t$. Since the function $\tan\frac{x}{2}$ is discontinuous at $x = \pi$, a point in the interval $(0, 2\pi)$, we can not use this substitution for the changing the variable of integration.

Example : 23

Find the mistake in the following evaluation of the integral

$$\int_0^{\pi} \frac{dx}{1+2\sin^2 x} = \int_0^{\pi} \frac{dx}{\cos^2 x + 3\sin^2 x} = \int_0^{\pi} \frac{\sec^2 x\, dx}{1+3\sin^2 x} = \frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_0^{\pi} = 0$$

Solution

The Newton-Leibnitz formula for evaluating the definite integrals is not applicable here since the anti-derivative.

$F(x) = \frac{1}{\sqrt{3}} \tan^{-1}\left(\sqrt{3}\tan x\right)$ has a discontinuity at the point $x = \pi/2$ which lies in the interval $[0, \pi]$.

$$\underset{\text{at } x=\pi/2}{\text{LHL}} = \lim_{h\to 0} \frac{1}{\sqrt{3}} \tan^{-1}\left[\tan\left(\frac{\pi}{2} - h\right)\right]$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{3}} \tan^{-1}\left(\sqrt{3}\cot h\right)$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{3}} \tan^{-1}(\to \infty) = \frac{\pi}{2\sqrt{3}} \qquad \text{.............(i)}$$

$$\underset{\text{at } x=\pi/2}{\text{RHL}} = \lim_{h\to 0} \frac{1}{\sqrt{3}} \tan^{-1} \sqrt{3}\left[\tan\left(\frac{\pi}{2} + h\right)\right]$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{3}} \tan^{-1}\left(-\sqrt{3}\cot h\right)$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{3}} \tan^{-1}(\to -\infty) = -\frac{\pi}{2\sqrt{3}} \qquad \text{.............(ii)}$$

From (i) and (ii), LHL $\neq$ RHL at $x = \pi/2$

$\Rightarrow$ Anti-derivative, F(x) is discontinuous at $x = \pi/2$

PART - B   AREA UNDER CURVE

Example : 24

Find the area bounded by the curve $y = x^2 - 5x + 6$, X-axis and the lines $x = 1$ and $x = 4$.

Solution

For $y = 0$, we get $x^2 + 5x + 6 = 0$

$\Rightarrow \quad x = 2, 3$

$$\text{Hence Area} = \int_1^2 y\,dx + \left|\int_2^3 y\,dx\right| + \int_3^4 y\,dx$$

$$\Rightarrow \quad A = \int_1^2 (x^2 - 5x + 6)\,dx + \left|\int_2^3 (x^2 - 5x + 6)\,dx\right| + \int_5^4 (x^2 - 5x + 6)\,dx$$

$$\int_1^2 (x^2 - 5x + 6)\,dx = \frac{2^2 - 1^3}{3} - 5\left(\frac{2^2 - 1^2}{2}\right) + 6\,(2 - 1) = \frac{5}{6}$$

$$\int_2^3 (x^2 - 5x + 6)\,dx = \frac{3^3 - 2^3}{3} - 5\left(\frac{3^2 - 2^2}{2}\right) + 6\,(3 - 2) = -\frac{1}{6}$$

$$\int_3^4 (x^2 - 5x + 6)\,dx = \frac{4^3 - 3^3}{3} - 5\left(\frac{4^2 - 3^2}{2}\right) + 6\,(4 - 3) = \frac{5}{6}$$

$$\Rightarrow \quad A = \frac{5}{6} + \left|-\frac{1}{6}\right| + \frac{5}{6} = \frac{11}{6} \text{ sq. units.}$$

Example : 25

Find the area bounded by the curve : $y = \sqrt{4 - x}$, X-axis and Y-axis

Solution

Trace the curve $y = \sqrt{4 - x}$

1. Put $y = 0$ in the given curve to get $x = 4$ as the point of intersection with X-axis.
Put $x = 0$ in the given curve to get $y = 2$ as the point of intersection with Y-axis.

2. For the curve, $y = \sqrt{4 - x}$, $4 - x \ge 0$

$\Rightarrow \quad x \le 4$

$\Rightarrow$ curve lies only to the left of $x = 4$ line.

3. As y is positive, curve is above X-axis.

Using steps 1 to 3, we can draw the rough sketch of $y = \sqrt{4 - x}$.

In figure

$$\text{Bounded area} = \int_0^4 \sqrt{4 - x}\;dx = \left|\frac{-2}{3}(4 - x)\sqrt{4 - x}\right|_0^4 = \frac{16}{3} \text{ sq. units.}$$

Example : 26

Find the area bounded by the curves $y = x^2$ and $x^2 + y^2 = 2$ above X-axis.

Solution

Let us first find the points of intersection of curves.

Solving $y = x^2$ and $x^2 + y^2 = 2$ simultaneously, we get :

$x^2 + x^4 = 2$

$\Rightarrow \quad (x^2 - 1)(x^2 + 2) = 0$

$\Rightarrow \quad x^2 = 1 \quad \text{and} \quad x^2 = -2$ (reject)

$\Rightarrow \quad x = \pm 1$

$\Rightarrow$ A ≡ (–1, 0) and B ≡ (1, 0)

Shaded Area = $\int_{-1}^{+1}\left(\sqrt{2-x^2}-x^2\right)dx$

$= \int_{-1}^{+1}\left(\sqrt{2-x^2}\,dx\right)-\int_{-1}^{+1}x^2\,dx$

$= 2\int_{0}^{1}\sqrt{2-x^2}\,dx-2\int_{0}^{1}x^2dx$

$= 2\left|\frac{x}{2}\sqrt{2-x^2}+\frac{2}{2}\sin^{-1}\frac{x}{\sqrt{2}}\right|_0^1-2\left(\frac{1}{3}\right)$

$= 2\left(\frac{1}{2}+\frac{\pi}{4}\right)-\frac{2}{3}$

$= \frac{1}{3}+\frac{\pi}{3}$ sq. units.

Example : 27

Find the area bounded by $y = x^2 - 4$ and $x + y = 2$

Solution

After drawing the figure, let us find the points of intersection of

$y = x^2 - 4$ and $x + y = 2$

$\Rightarrow$ $x + x^2 - 4 = 2$ $\Rightarrow$ $x^2 + x - 6 = 0$ $\Rightarrow$ $(x + 3)(x - 2) = 0$

$\Rightarrow$ $x = -3, 2$

$\Rightarrow$ A ≡ (–3, 0) and B ≡ (2, 0)

Shaded area $= \int_{-3}^{2}\left[(2-x)-(x^2-4)\right]dx$

$= \int_{-3}^{2}(2-x)\,dx-\int_{-3}^{2}(x^2-4)\,dx$

$= \left|2x-\frac{x^2}{2}\right|_{-3}^{2}-\left|\frac{x^3}{2}-4x\right|_{-3}^{2}$

$= 2 \times 5 - \frac{1}{2}(4-9)-\frac{1}{3}(8+27)+4(5) = \frac{125}{6}$

Example : 28

Find the area bounded by the circle $x^2 + y^2 = a^2$.

Solution

$x^2 + y^2 = a^2$ $\Rightarrow$ $y = \pm\sqrt{a^2-x^2}$

Equation of semicircle above X-axis is $y = +\sqrt{a^2-x^2}$

Area of circle = 4 (shaded area)

$= 4\int_{0}^{a}\sqrt{a^2-x^2}\,dx$

$$= 4\left|\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right|_0^a = 4\ \frac{a^2}{2}\left(\frac{\pi}{3}\right) = \pi a^2$$

Example : 29

Find the area bounded by the curves $x^2 + y^2 = 4a^2$ and $y^2 = 3ax$.

Solution

The points of intersection A and B can be calculated by solving $x^2 + y^2 = 4a^2$ and $y^2 = 3ax$

$\Rightarrow \quad \left(\frac{y^2}{3a}\right)^2 = y^2 = 4a^2$

$\Rightarrow \quad y^4 + 9a^2 y^2 - 36a^4 = 0$

$\Rightarrow \quad (y^2 - 3a^2)(y^2 + 12a^2) = 0$

$\Rightarrow \quad y^2 = 3a^2$

$\Rightarrow \quad y^2 = -12a^2$ (reject)

$\Rightarrow \quad y^2 = 3a^2 \quad \Rightarrow \quad y = \pm\sqrt{3}a$

$\Rightarrow \quad y_A = \sqrt{3}a^2$ and $y_B = -\sqrt{3}a$

The equation of right half of $x^2 + y^2 = 4a^2$ is $x = \sqrt{4a^2 - y^2}$

$$\text{Shaded area} = \int_{-\sqrt{3}a}^{\sqrt{3}a}\left(\sqrt{4a^2 - y^2} - \frac{y^2}{3a}\right)dy$$

$$= 2\int_0^{\sqrt{3}a}\left(\sqrt{4a^2 - y^2} - \frac{y^2}{3a}\right)dy \qquad \text{(using property – 8)}$$

$$= 2\left|\frac{y}{2}\sqrt{4a^2 - y^2} + \frac{4a^2}{2}\sin^{-1}\frac{y}{2a}\right|_0^{\sqrt{3}a} - \frac{2}{3a}\left|\frac{y^3}{3}\right|_0^{\sqrt{3}a}$$

$$= \sqrt{3}a^2 + 4a^2\ \frac{\pi}{3} - \frac{2}{9a}\ 3\sqrt{3}\ a^3$$

$$= \left(\frac{1}{\sqrt{3}} + \frac{4\pi}{3}\right)a^2$$

Alternative Method :

shaded area = 2 × (area above X-axis)

x-coordinate of A = $\frac{y^2}{3a} = \frac{3a^2}{3a} = a$

The given curves are $y = \pm\sqrt{3ax}$ and $y = \pm\sqrt{4a^2 - x^2}$

But above the X-axis, the equations of the parabola and the circle are $\sqrt{3ax}$ and $y = \sqrt{4a^2 - x^2}$ respectively.

$$\Rightarrow \quad \text{shaded area} = 2\left[\int_0^a \sqrt{3ax}\,dx + \int_a^{2a}\sqrt{4a^2 - x^2}\,dx\right]$$

Solve it yourself to get the answer.

Example : 30

Find the area bounded by the curves : $y^2 = 4a(x + a)$ and $y^2 = 4b(b - x)$.

Solution

The two curves are :

$y^2 = 4a(x + a)$ ....(i)

and $y^2 = 4b(b - x)$ ....(ii)

Solving $y^2 = 4a(x + a)$ and $y^2 = 4b(b - x)$ simultaneously,

we get the coordinates of A and B.

Replacing values of x from (ii) into (i), we get :

$$y^2 = 4a\left(b - \frac{y^2}{4b} + a\right)$$

$\Rightarrow$ $y = \pm\sqrt{4ab}$ and $x = b - a$

$\Rightarrow$ $A \equiv -(b - a, \sqrt{4ab})$ and $B \equiv (b - a, -\sqrt{4ab})$

$$\text{shaded area} = \int_{-\sqrt{4ab}}^{\sqrt{4ab}} \left[\left(b - \frac{y^2}{4b}\right) - \left(\frac{y^2}{4b} - a\right)\right] dy$$

$\Rightarrow$ $A = 2(a + b)\sqrt{4ab} - \int_0^{\sqrt{4ab}} \left(\frac{y^2}{2b} + \frac{y^2}{2a}\right) dy$ (using property – 8)

$\Rightarrow$ $A = 2(a + b)\sqrt{4ab} - \frac{1}{2}\left[\frac{4ab\sqrt{4ab}}{3b} + \frac{4ab\sqrt{4ab}}{3a}\right]$

$\Rightarrow$ $A = 2(a + b)\sqrt{4ab} - \frac{2}{3}(a + b)\sqrt{4ab}$

$\Rightarrow$ $A = \frac{8}{3}(a + b)\sqrt{ab}$

Example : 31

Find the area bounded by the hyperbola : $x^2 - y^2 = a^2$ and the line $x = 2a$.

Solution

Shaded area = 2 × (Area of the portion above X-axis)

The equation of the curve above x-axis is : $y = \sqrt{x^2 - a^2}$

$\Rightarrow$ required area (A) $= 2\int_a^{2a} \sqrt{x^2 - a^2}\, dx$

$\Rightarrow$ $A = 2\left|\frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right|\right|_a^{2a}$

$\Rightarrow$ $A = 2\sqrt{3}\,a^2 - a^2\log|(2a + \sqrt{3}a| + a^2\log a$

$\Rightarrow$ $A = 2\sqrt{3}\,a^2 - a^2\log(2 + \sqrt{3})$

Alternative Method :

$$\text{Area (A)} = \int_{yB}^{yA} \left(2a - \sqrt{a^2 + y^2}\right) dy$$

$\Rightarrow \quad A = \int_{-\sqrt{3a}}^{\sqrt{3a}} \left(2a - \sqrt{a^2 + y^2}\right) dy$

Example : 32

Find the area bounded by the curves : $x^2 + y^2 = 25$, $4y = |4 - x^2|$ and $x = 0$ in the first quadrant.

Solution

First of all find the coordinates of point of intersection. A by solving the equations of two gives curves :

$\Rightarrow \quad x^2 + y^2 = 25 \quad$ and $\quad 4y = |4 - x^2|$

$\Rightarrow \quad x^2 + \frac{(4 - x^2)^2}{16} = 25$

$\Rightarrow \quad (x^2 - 4)^2 + 16x^2 = 400$

$\Rightarrow \quad (x^2 + 4)^2 = 400$

$\Rightarrow \quad x^2 = 16$

$\Rightarrow \quad x = \pm 4$

$\Rightarrow \quad y = \frac{|4 - x^2|}{4} = 3$

$\Rightarrow$ Coordinates of point are $A \equiv (4, 3)$

Shaded are = $\int_0^4 \left[\sqrt{25 - x^2} - \frac{|4 - x^2|}{4}\right] dx$

$\Rightarrow \quad A = \int_0^4 \sqrt{25 - x^2}\, dx - \frac{1}{4}\int_0^4 |4 - x^2|\, dx \quad$ ....(i)

Let $\quad I = \frac{1}{4} \int_0^4 |4 - x^2|\, dx$

$\Rightarrow \quad A = \frac{1}{4} \int_0^2 (4 - x^2)\, dx + \frac{1}{4}\int_2^4 (x^2 - 4)\, dx$

$\Rightarrow \quad A = \frac{1}{4} \left(\frac{64}{3} - 16\right) - \frac{1}{4} \left(\frac{8}{3} - 8\right) = 4$

On substituting the value of I in (i), we get :

$$A = \int_0^4 \sqrt{25 - x^2}\; dx - 4$$

$\Rightarrow \quad A = \left| \frac{x}{2}\sqrt{25 - x^2} + \frac{25}{2}\sin^{-1}\frac{x}{5} \right|_0^4 - 4$

$\Rightarrow \quad A = 6 + \frac{25}{2} \sin^{-1} \frac{4}{5} - 4 = 2 + \frac{25}{2} \sin^{-1} \frac{4}{5}$

Example : 33

Find the area enclosed by the loop in the curve : $4y^2 = 4ax^2 - x^3$.

Solution

The given curve is : $4y^2 = 4ax^2 - x^3$

To draw the rough sketch of the given curve, consider the following steps :

(1) On replacing y by –y, there is no change in function. It means the graph is symmetric about Y-axis

(2) For x = 4, y = 0 and for x = 0, y = 0

(3) In the given curve, LHS is positive for all values of y.

$\Rightarrow$ RHS ≥ 0 $\Rightarrow$ $x^2 (1 - x/4) \geq 0$ $\Rightarrow$ $x \leq 4$

Hence the curve lines to the left of x = 4

(4) As $x \to -\infty$, $y \to \pm \infty$

(5) Points of maximum/minimum :

$$8y \frac{dy}{dx} = 8x - 3x^2$$

$$\frac{dy}{dx} = 0 \quad \Rightarrow \quad x = 0, \frac{8}{3}$$

At x = 0, derivative is not defined

By checking for $\frac{d^2y}{dx^2}$, $x = \frac{8}{3}$ is a point of local maximum (above X-axis)

From graph

Shaded area (A) = 2 × (area of portion above X-axis)

$$\Rightarrow \quad A = 2 \int_0^4 \frac{x}{2}\sqrt{4-x}\, dx = \int_0^4 x\sqrt{4-x}\, dx$$

$$\Rightarrow \quad A = \int_0^4 (4-x)\sqrt{4-(4-x)}\, dx \qquad \text{(using property – 4)}$$

$$\Rightarrow \quad A = \int_0^4 (4-x)\sqrt{x}\, dx$$

$$\Rightarrow \quad A = 4 \left| \frac{2}{3}x\sqrt{x} \right|_0^4 - \left| \frac{2}{5}x^2\sqrt{x} \right|_0^4$$

$$\Rightarrow \quad A = \frac{128}{15} \text{ sq. units.}$$

Example : 34

Find the area bounded by the parabola $y = x^2$, X-axis and the tangent to the parabola at (1, 1)

Solution

The given curve is $y = a^2$

Equation of tangent at $A \equiv (1, 1)$ is :

$$y - 1\ y - 1 = \left.\frac{dy}{dx}\right]_{x=1} (x - 1) \qquad [\text{using : } y - y_1 = m(x - x_1)]$$

$\Rightarrow$ $y - 1 = 2 (x - 1)$

$\Rightarrow$ $y = 2x - 1$ ....(i)

The point of intersection of (i) with X-axis is B = (1/2, 0)

Shaded area = area (OACO) – area (ABC)

$\Rightarrow \quad \text{area} = \int_0^1 x^2\,dx - \int_{1/2}^1 (2x-1)\,dx$

$\Rightarrow \quad \text{area} = \frac{1}{3} - \left[1 - \frac{1}{4} - (1 - 1/2)\right]$

$\Rightarrow \quad \text{area} = \frac{1}{12}$

Example : 35

Evaluate : $\displaystyle\int_0^{\pi} \frac{x\sin(2x)\sin\left(\frac{\pi}{2}\cos x\right)dx}{2x-\pi}$

Solution

Let $\quad I = \displaystyle\int_0^{\pi} \frac{x\sin(2x)\sin\left(\frac{\pi}{2}\cos x\right)dx}{2x-\pi}$ ....(i)

Apply property – 4 to get

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)\sin(2\pi - 2x)\sin\left(\frac{\pi}{2}\cos(\pi - x)\right)dx}{2(\pi - x) - \pi}$$

$= \displaystyle\int_0^{\pi} \frac{(\pi - x)\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)dx}{2x-\pi}$ ....(ii)

Add (i) and (ii) to get

$$2I = \int_0^{\pi} \sin 2x \sin\left[\frac{\pi}{2}\cos x\right]dx$$

Let $\quad \frac{\pi}{2}\cos x = t \quad \Rightarrow \quad -\frac{\pi}{2}\sin x\,dx = dt$

$$\Rightarrow \quad I = -\frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2} t\sin t\,dt = \frac{8}{\pi^2}\int_0^{\pi/2} t\sin t\,dt$$

$$\Rightarrow \quad I = \frac{8}{\pi^2}\left[t\int_0^{\pi/2}\sin t\,dt + \int_0^{\pi/2}\cos t\,dt\right]$$

$$\Rightarrow \quad I = \frac{8}{\pi^2}\left[-\left|\, t\cos t\right|_0^{\pi/2} + (\sin t)_0^{\pi/2}\right] = \frac{8}{\pi^2}\,[0 + 1] = \frac{8}{\pi^2}$$

Example : 36

Prove that : $\int_0^{\pi} \theta^3 \log\sin\theta d\theta = \frac{3\pi}{2}\int_0^{\pi} \theta^2 \log\left(\sqrt{2}\ \sin\theta\right) d\theta$

Solution

Let $I = \int_0^{\pi} \theta^3 \log\sin\theta\ d\theta$

Using property – 4, we get :

$$I = \int_0^{\pi} (\pi-\theta)^3 \log(\pi-\theta)\ d\theta = \int_0^{\pi} [\pi^3 - \theta^3 - 3\pi^2\theta + 3\pi\theta^2] \log\sin\theta\ d\theta$$

$$\Rightarrow \quad I = \pi^3 \int_0^{\pi} \log\sin\theta - \int_0^{\pi} \theta^3 \log\sin\theta d\theta - 3\pi\int_0^{\pi} \theta\log\sin\theta\ d\theta + 3\pi\int_0^{\pi} \theta^2 \log\sin\theta d\theta$$

$$\Rightarrow \quad 2I = \pi^3 \int_0^{\pi} \log\sin\theta d\theta = 3\pi^2 I_1 + 3\pi\int_0^{\pi} \theta^2 \log\sin\theta d\theta \quad \text{............(i)}$$

Consider I_1 $\quad I_1 = \int_0^{\pi} \theta\log\sin\theta d\theta$

Using property – 4,

$$\text{we get } I_1 = \int_0^{\pi} (\pi-\theta)\log\sin\theta d\theta = \pi\int_0^{\pi} \log\sin\theta - \int_0^{\pi} \log\sin\theta$$

$$\Rightarrow \quad 2I_1 = \pi \int_0^{\pi} \log\sin\theta d\theta = 2\pi \int_0^{\pi/2} \log\sin\theta d\theta \qquad \text{[using property – 6]}$$

$$\Rightarrow \quad I_1 = -\frac{\pi^2}{2}\log 2 \qquad \text{using : } \int_0^{\pi/2} \log\sin\theta\ d\theta = \frac{-\pi}{2}\log 2$$

On Replacing value of I_1 in (i) we get,

$$2I = -\pi^4 \log 2 - 3\pi^2 \left(\frac{\pi^2}{2}\log 2\right) + 3\pi \int_0^{\pi} \theta^2 \ \log\sin\theta\ d\theta$$

$$= \frac{\pi^4}{2}\log 2 + 2\ 3\pi \int_0^{\pi} \theta^2 \log\sin\theta = 3\pi\int_0^{\pi} \left(\log\sqrt{2}\right)\theta^2 d\theta + 3\pi\int_0^{\pi} \theta^2 \log\sin\theta\ d\theta$$

$$= 3\pi \int_0^{\pi} \theta^2 \log\left(\sqrt{2}\sin\theta\right) d\theta$$

$$\Rightarrow \quad I = \frac{3}{2}\pi \int_0^{\pi} \theta^2 \log\sqrt{2}\sin\theta\ d\theta$$

Example : 37

Determine the value of $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}$

Solution

$$I = \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\, dx = 2\int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x} \qquad \left[\text{using}: \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)+f(-x)\,dx\right]$$

$\Rightarrow \quad I = 4\int_{0}^{\pi} \frac{x\sin x}{1+\cos^2 x}\, dx$

$\Rightarrow \quad 2I = 4\int_{0}^{\pi} \frac{\pi\sin x}{1+\cos^2 x}\, dx \qquad \left(\text{using} \int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx\right)$

$\Rightarrow \quad I = 4\pi\int_{0}^{\pi/2} \frac{\sin x\, dx}{1+\cos^2 x}\, dx \qquad \left(\text{using} \int_{0}^{2a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(2a-x)\,dx\right)$

Put $\cos x = t \quad \Rightarrow \quad -\sin x\, dx = dt$

For $x = 0$, $t = 1$ and for $x = \pi/2$, $t = 0$

$\Rightarrow \quad I = 4\pi\int_{0}^{1} \frac{dt}{1+t^2} = 4\pi \tan^{-1} t \Big|_{0}^{1} = 4\pi\, \frac{\pi}{4} = \pi^2$

Example : 38

Let A_n be the area bounded by the curve $y = (\tan x)^n$ and the lines $x = 0$, $y = 0$ and $x = \pi/4$. Prove that for $n > 2$, $A_n + A_{n-2} = \frac{1}{n-1}$ and deduce $\frac{1}{2n+2} < An < \frac{1}{2n-2}$

Solution

According to the function, A_n is the area bounded by the curve $y = (\tan x)^n$, $x = 0$, $y = 0$ and $x = \pi/4$.

So A_n = Shaded Area = $\int_{0}^{\pi/4} (\tan x)\, dx = \int_{0}^{\pi/4} \tan^2 x \tan^{n-2} x$

$\Rightarrow \quad A_n = \int_{0}^{\pi/4} (\sec^2 x - 1)\tan^{n-2} x = \int_{0}^{\pi/4} \sec^2 x \tan^{n-2} x - \int_{0}^{\pi/n} \sec^{n-2} x\, dx$

$\Rightarrow \quad A_n = \frac{\tan^{n-1} x}{n-1}\Big|_{0}^{\pi/4} - A_{n-2}$

$\Rightarrow \quad A_n + A_{n-2} = \frac{1}{n-1} \qquad$ ....(i)

Hence proved.

Replace n by n + 2 to get : $A_{n+2} + A_n = \frac{1}{n+1} \qquad$ ....(ii)

Observe that if n increases, $(\tan x)^n$ decreases because $0 \le \tan x \le 1$ [0, π/4]

$\Rightarrow$ As n is increased, A_n decreases.

$\Rightarrow$ $A_{n+2} < A_n < A_{n-2}$

Using (i) and (ii), replace values of A_{n-2} and A_{n+2} in terms of A_n to get,

$$\frac{1}{n+1} - A_n < A_n < \frac{1}{n-1} - A_n$$

$$\Rightarrow \quad \frac{1}{n+1} < 2A_n < \frac{1}{n-1} - A_n$$

$$\Rightarrow \quad \frac{1}{2n+1} < A_n < \frac{1}{2n-2}$$

Hence Proved.

Example : 39

Show that $\int_0^{n\pi+v} |\sin x|\, dx = 2n + 1 - \cos v$, where n is a +ve integer and $0 \le v \le \pi$

Solution

Let $I = \int_0^{n\pi+v} |\sin x| = \int_0^{n\pi} |\sin x| + \int_{n\pi}^{n\pi+v} |\sin x|$ (using property – 1)

$\Rightarrow \quad I = I_1 + I_2$ ....(i)

Consider I_1

$I_1 = \int_0^{n\pi} |\sin x| = n\int_0^{\pi} |\sin x|$ (using property – 9 and period of |sin x| is π)

$\Rightarrow \quad I_1 = n \int_0^{\pi} \sin x\, dx$ (As $\sin x \ge 0$ in $[0, \pi]$, $|\sin x| = \sin x$)

$\Rightarrow \quad I_1 = -n\, |\cos x|_0^{\pi} = -n\,[-1 - 1] = 2n$

Consider I_2

$I_2 = \int_{n\pi}^{n\pi+v} |\sin x|\, dx$

Put $x = n\pi + \theta \quad \Rightarrow \quad dx = d\theta$

when x is $n\pi$, $\theta = 0$ and when $x = n\pi + v$, $\theta = v$

$\Rightarrow \quad I_2 = \int_0^{v} |\sin(\pi x + \theta)|\, d\theta = \int_0^{v} |\sin\theta|\, d\theta$ ($\because$ period of |sin x| = π)

$\Rightarrow \quad I_2 = \int_0^{v} |\sin\theta|\, d\theta = \int_0^{v} \sin\theta\, d\theta$ ($\because$ for $0 \le \theta \le \pi$, $\sin\theta$ is positive)

$= -\, |\cos\theta|_0^{v} = 1 - \cos v$

On substituting the values of I_1 and I_2 in (i), we get

$I = 2n + 1 (1 - \cos v) = 2n + 1 - \cos v$

Hence proved.

Example : 40

It is known that f(x) is an odd function in the interval $\left[-\frac{T}{2}, \frac{T}{2}\right]$ and has a period equal to T. Prove that

$\int_a^x f(t)\,dt$ is also periodic function with the same period.

Solution

It is given that : f(x) = – f(x) ....(i)

and f(x + t) = f(x) ....(ii)

Let $g(x) = \int_a^x f(t)\,dt$

$$\Rightarrow \quad g(x+T) = \int_a^{x+T} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{T/2} f(t)\,dt + \int_{T/2}^{x+T} f(t)\,dt \qquad \text{(using property – 1)}$$

Put t y = y + T in the third integral on RHS.

⇒ dt = dy

when t = T/2, y = –T/2 and hwn t = x + T, y = x

$$\Rightarrow \quad g(x+T) = \int_a^x f(t)\,dt + \int_x^{T/2} f(t)\,dt + \int_{-T/2}^{x} f(y+T)\,dy$$

Using (i), we get $g(x+T) = \int_a^x f(t)\,dt + \int_x^{T/2} f(t)\,dt + \int_{-T/2}^{x} f(y)\,dy$

$$g(x+T) = \int_a^x f(t)\,dt + \int_{-T/2}^{T/2} f(t)\,dt \qquad \text{(using property – 1)}$$

$$\Rightarrow \quad g(x+T) = \int_a^x f(t)\,dt \qquad \text{(using property – 8)}$$

⇒ g(x + T) = g(x)

⇒ g(x) is also periodic function when period T.

Example : 41

Evaluate the integral $\int_{-1\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\cos^{-1}\frac{2x}{1+x^2}\,dx$

Solution

$$I = \int_{-1\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\cos^{-1}\frac{2x}{1+x^2}\,dx = \int_{-1\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\left[\frac{\pi}{2} - \sin^{-1}\frac{2x}{1+x^2}\right]dx \qquad \text{(using : } \sin^{-1}x + \cos^{-1}x = \pi/2\text{)}$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_{-1\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\,dx - \int_{-1\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\sin^{-1}\frac{2x}{1+x^2}\,dx$$

As integrand of second integral is an odd function, integral will be zero i.e.

$$\Rightarrow \quad I = \frac{\pi}{2}\int_{-1\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\,dx - 0 \qquad \text{[using property – 8]}$$

$$= -\frac{2\pi}{2}\int_0^{1/\sqrt{3}} \frac{x^4-1+1}{x^4-1} = -\pi\int_0^{1/\sqrt{3}}\left(1+\frac{1}{x^4-1}\right)dx$$

$$\Rightarrow \qquad I = \frac{-\pi}{\sqrt{3}} + \frac{(-\pi)}{2}\int_0^{1/\sqrt{3}} \frac{x^2+1-(x^2-1)}{(x^2+1)(x^2-1)}\,dx$$

$$= -\frac{-\pi}{\sqrt{3}} - \frac{\pi}{2}\left[\int_0^{1/\sqrt{3}} \frac{1}{x^2-1} - \int_0^{1/\sqrt{3}} \frac{1}{x^2+1}\,dx\right]$$

$$= -\frac{-\pi}{\sqrt{3}} - \frac{\pi}{2}\left[\frac{1}{2}\left|\log\frac{x-1}{x+1}\right|_0^{1/\sqrt{3}} - \left|\tan^{-1}x\right|_0^{1/\sqrt{3}}\right]$$

$$= -\frac{\pi}{\sqrt{3}} + \frac{\pi^2}{12} - \frac{\pi}{4}\log\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

Example : 42

If f is a continuous function $\int_0^x f(t)\,dt \to \infty$, then show that every line y = mx intersect the curve

$y^2 + \int_0^x f(t)\,dt = 2$

Solution

If y = mx and $y^2 + \int_0^X f(t)\,dt = 2$ have to intersect for all value of m, then

$m^2x^2 + \int_0^X f(t)\,dt = 2$ must posses atleast one solution (root) for all m. ....(i)

Let $g(x) = m^2x^2 + \int_0^x f(t)\,dt - 2$

For (i) to e true, g(x) should be zero for atleast one value of x.
As f(x) is a given continuous function and m^2x^2 is a continuous function,

$g(x) = m^2x^2 + \int_0^x f(t)\,dt$ is also a continuous function ....(iii)

($\because$ because sum of two continuous functions is also continuous)

g(0) = – 2 and $\lim_{x\to\infty} g(x) = \infty$ ....(iii)

Combining (ii) and (iii), we can say that :
for all values of m, the curve g(x), intersect the y = 0 line (i.e. X-axis) for atleast one value of x.
$\Rightarrow$ g(x) = 0 has atleast one solution for all values of m.
Hence proved

Example : 43

Let a + b = 4, where a < 2, and let g(x) be a differentiable function. If $\frac{dg}{dx}$ > 0 for all x, prove that

$\int_0^a g(x)\,dx + \int_0^b g(x)\,dx$ increases as (b – a) increases

Solution

Let $b - a = t$ ....(i)

It si given that a + b = 4 ....(ii)

Solving (i) and (ii), we get b = $\frac{t+4}{2}$ and a = $\frac{4-t}{2}$

As a < 2, $\frac{4-t}{2} < 2$

$\Rightarrow \quad 4 - t < 4 \quad \Rightarrow \quad t > 0$

Let $f(t) = \int_0^a g(x)\,dx + \int_0^b g(x)\,dx$

$\Rightarrow \quad f(t) = \int_0^{\frac{4-t}{2}} g(x)\,dx + \int_0^{\frac{t+4}{2}} g(x)\,dx$

$f'(t) = g\left(\frac{4-t}{2}\right)\left(-\frac{1}{2}\right) + g\left(\frac{4+t}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{2}\left[g\left(\frac{4+t}{2}\right) - g\left(\frac{4-t}{2}\right)\right]$ ....(i)

As $\frac{dg}{dx}$. 0, g(x) is an increasing function.

For t > 0, $\frac{4+t}{2} > \frac{4-t}{2}$

$\Rightarrow \quad g\left(\frac{4+t}{2}\right) > g\left(\frac{4-t}{2}\right)$ [$\because$ g(x) is an increasing function]

$\Rightarrow \quad f'(t) > 0 \ \forall\ t > 0$ [using (i)]

$\Rightarrow$ f(t) is an increasing function as t increases.

Hence Proved.

Example : 44

Find the area between the curve $y = 2x^4 - x^2$, the x-axis and the ordinates of two minima of the curve.

Solution

Using the curve tracing steps, draw the rough sketch of the function $y = 2x^4 - x^2$.

Following are the properties of the curve which can be used to draw its rough sketch

(i) The curve is symmetrical about y-axis

(ii) Point of intersection with x-axis are x = 0, x = ±1/√2. Only point of intersection with y-axis is y = 0.

(iii) For $x \in \left(-\infty - \frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}\ \infty\right)$, y > 0 i.e. curve lies above x-axis and in the other intervals it lies below x-axis.

(iv) Put $\frac{dy}{dx}$ = 0 to get x = ±1/2 as the points of local minimum. On plotting the above information on graph, we get the rough sketch of the graph. The shaded area in the graph is the required area

$$\text{Required Area} = 2\left|\int_0^{1/2}(2x^4 - x^2)\,dx\right| = 2\left|\left[\frac{2x^5}{5} - \frac{x^3}{3}\right]_0^{1/2}\right| = \frac{7}{120}$$

Example : 45

Consider a square with vertices at (1, 1), (–1, –1), (–1, 1) and (1, –1). Let S be the region consisting of all points inside the square which are nearer to the origin than to the edge. Sketch the region S and find its area.

Solution

Let ABCD be the square with vertices A(1, 1), B(–1, 1), C(1, –1) and D(1, –1). The origin O is the centre of this square. Let (x, y) be a moving point in the required region. Then :

$$\sqrt{x^2+y^2} < |1-x|,\ \sqrt{x^2+y^2} < |1+x|,\ \sqrt{x^2+y^2} < |1-y|,\ \sqrt{x^2+y^2} < |1+y|$$

i.e. $x^2 + y^2 < (1 - x)^2$, $x^2 + y^2 < (1 + x)^2$, $x^2 + y^2 < (1 - y)^2$, $x^2 + y^2 < (1 + y)^2$

$\Rightarrow$ $y^2 = 1 - 2x$ ....(i)

$y^2 = 1 + 2x$ ....(ii)

$x^2 = 1 - 2y$ ....(iii)

$x^2 = 1 + 2y$ ....(iv)

Plotting the curves (i) to (iv), we can identify that the area bounded by the curves is the shaded area (i.e. region lying inside the four curves).

Required Area = 4 × Area (OPQR) = 4 [Area (OSQRO) + Area (SPQS)]

= 4 [Area (OSQRO) + Area (SPQS)]

$$= 4\left[\int_0^{x_s}\frac{1}{2}(1-x^2)\,dx + \int_{x_s}^{1/2}\sqrt{1-2x}\,dx\right] \quad (x_s \text{ is the x-coordinate of point S}) \quad ....(v)$$

To find x_s, solve curves (i) and (iii)

$\Rightarrow$ $x^2 - y^2 = -2(y - x)$

$\Rightarrow$ $(x - y)[x + y - 2] = 0$ $\Rightarrow$ $x = y$

Replace x = y in (i) to get $x^2 + 2x - 1 = 0$ $\Rightarrow$ $x = \sqrt{2} \pm 1$

(Check yourself that for x + y = 2, these is no point of intersection between the lines)

As x < 1, S is ($\sqrt{2} - 1$, $\sqrt{2} - 1$)

replacing the value of x_s in (i), we get

$$\text{Required Area} = 4\left[\int_0^{\sqrt{2}-1}\frac{1}{2}(1-x^2) + \int_{\sqrt{2}-1}^{1/2}\sqrt{1-2x}\,dx\right]$$

$$= 4\left[\frac{1}{2}\left(x - \frac{x^3}{3}\right)\right]_0^{\sqrt{2}-1} - \frac{2}{3}\times\frac{1}{2}(1-2x)^{3/2}\Bigg]_{\sqrt{2}-1}^{1/2} = \frac{2}{3}\left(8\sqrt{2} - 10\right) \text{ sq. units}$$

Example : 46

Let O(0, 0), A(2, 0) and B (1, 1/$\sqrt{3}$) be the vertices of a triangle. Let R be the region consisting of all those points P inside ΔOAB which satisfy d(P, OA) ≤ min {d (P, AB)}, where d denotes the distance from the point to the corresponding line. Sketch the region R and find its area.

Solution

Let the coordinates of moving point P be (x, y)

Equation of line OA ≡ y = 0

Equation of line OB ≡ $\sqrt{3}$ = x

Equation of line AB ≡ $\sqrt{3}$y = 2 – x.

d(P, OA) = distance of moving point P from line OA = y

d(P, OB) = distance of moving point P from line OB = $\dfrac{|\sqrt{3}y - x|}{2}$

d(p, AB) = distance of moving point P from line AB = $\frac{|\sqrt{3}y + x - 2|}{2}$

It is given in the question that P moves inside the triangle OAB according to the following equation.
d (P, OA) ≤ min {d(P, OB), d (P, AB)}

$\Rightarrow \quad y \le \min\left\{\frac{|\sqrt{3}y - x|}{2}, \frac{|\sqrt{3}y + x - 2|}{2}\right\}$

$\Rightarrow \quad y \le \frac{|\sqrt{3}y - x|}{2}$ ....(i) and $y \le \frac{|\sqrt{3}y + x - 2|}{2}$ ....(ii)

Consider (i) $\quad y \le \frac{|\sqrt{3}y - x|}{2}$

$y \le \frac{x - \sqrt{3}y}{2}$ i.e. $x > \sqrt{3}y$ because P(x, y) moves inside the triangle, below the lines OB

$\Rightarrow \quad (2 + \sqrt{3})\, y \le x$

$\Rightarrow \quad y \le (2 - \sqrt{3})\, x$

$\Rightarrow \quad y \le \tan 15^o\, x$. (Note : y = tan 15ºx is an acute ∠ bisector of ∠ O] ....(iii)

Consider (ii) $\quad y \le \frac{|\sqrt{3}y + x - 2|}{2}$

$\Rightarrow \quad 2y \le 2 - x - \sqrt{3}y$

i.e. $\sqrt{3}y + x - 2 < 0$ because P(x, y) moves inside the triangle, below the line AB.

$\Rightarrow \quad (2 + \sqrt{3})\, y \le -(x - 2)$

$\Rightarrow \quad y \le -(2 - \sqrt{3})(x - 2)$

$\Rightarrow \quad y \le -\tan 15^o (x - 2)$ [Note : y = – tan 15º (x – 2) is an acute ∠ bisector of ∠A]

From (iii) and (iv), P moves inside the triangle as shown in figure. (shaded area).

Let D be the foot of the perpendicular from C to OA

As ∠COA = ∠OAC = 15º, ΔOCA is an isosceles Δ .

$\Rightarrow$ OD = AD = 1 unit.

Area of shaded region = Area of ΔOCA = 1/2 base × height = $\frac{1}{2}$ (2) [1 tan 15º] = tan 15º = $2 - \sqrt{3}$

Alternate Method

Let acute angle bisector fo angles O and A meet at point C inside the triangle ABC.

Consider OC

On Line OC, d(P, OA) = d(p, OB) [note if P moves on OC d(P, OB) < d (P, AB)]

$\Rightarrow$ Below the line OC, d(P, OA) < d(p, OB) < d (P, AB) ....(i)

On Line AC, d(P, OA) = d(P, AB) [note if P moves on AC d(P, AB) < d (P, OB)]

$\Rightarrow$ Below the line OC, d(P, OA) > d (P, AB) < d (P, OB) ....(ii)

On combining (i) and (ii), P moves inside the triangle OAC

Now the required area is the area of the triangle OAC = $2 - \sqrt{3}$ (refer previous method)

Example : 47

Sketch the smaller of the regions bonded by the curves $x^2 + 4y^2 - 2x - 8y + 1 = 0$ and $4y^2 - 3x - 8y + 7 = 0$. Also find its area.

Solution

Express the two curves in perfect square form to get :

$\frac{(x-1)^2}{4} + (y - 1)^2 = 1$ ....(i)

[i.e. ellipse centred at (1, 1)]

and $(y-1)^2 = \frac{3}{4}(x-1)$ ....(ii)

[i.e. parabola whose vertex is at (1, 1)]

To calculate the area bounded between curves (i) and (ii), it is convenient to shift the origin at (1, 1).
Replace x – 1 = X and y – 1 = Y in (i) and (ii).
The new equations of parabola and ellipse with shifted origin are :

$$\frac{X^2}{4} + Y^2 = 1 \quad ....(iii)$$

$$Y^2 = \frac{3}{4}X \quad ....(iv)$$

It can be easily observed that the area of the smaller region bounded by (i) and (ii) is the same as the area of the smaller region bounded by (iii) and (iv) on the X-Y plane i.e. Area bounded remains same in the two cases.
So area of region bounded by (iii) and (iv)

= shaded area shown in the figure

= 2 × shaded area lying in I[st] quadrant

$$= 2\left[\int_0^{x_A} \frac{\sqrt{3}}{2}\sqrt{X}\, dX + \frac{1}{2}\int_{x_A}^{2} \sqrt{4-x^2}\, dX\right] \quad ....(v)$$

Solve curves (iii) and (iv) to get point of intersection A = $\left(1, \frac{\sqrt{3}}{2}\right)$

⇒ $x_A = 1$

Replace x_A in (v) to get :

$$\text{Required Area} = 2\left[\int_0^{1} \frac{\sqrt{3}}{2}\sqrt{X}\, dX + \frac{1}{2}\int_{1}^{2} \sqrt{4-X^2}\, dx\right]$$

$$= \frac{2}{\sqrt{3}}X^{3/2}\Big|_0^1 + \left[\frac{X}{2}\sqrt{4-X^2} + 2\sin^{-1}\frac{X}{2}\right]_1^2 = \frac{\sqrt{3}}{6} + \frac{2\pi}{3}$$

Differential Equations

Example : 1

Solve the differential equation : $\frac{dy}{dx} = x$.

Solution

The given differential equation is : dy – xdx

$\Rightarrow \quad \int dy = \int x dx$

$\Rightarrow \quad y = \frac{x^2}{2} + C \qquad$ ....(i)

where C is an arbitrary constant.

Note that (i) is the general solution of the given differential equation.

Example : 2

Solve the differential equation : $\frac{dy}{dx} = x - 1$ if y = 0 for x = 1.

Solution

The given differential equation is : dy = (x – 1) dx

$\int dy = \int (x-1)\, dx \quad \Rightarrow \quad y = \frac{x^2}{2} - x + C \qquad$ (general solution)

This is the general solution. We can find value of C using y = 0 for x = 1.

$0 = \frac{1}{2} - 1 + C \quad \Rightarrow \quad C = \frac{1}{2}$

$y = \frac{x^2}{2} - x + \frac{1}{2}$ is the particular solution.

Example : 3

Solve the differential equation : (1 + x) y dx + (1 – y) x dy = 0

Solution

Separate the term of x and y to get : (1 + x) y dx = – (1 – y) x dy

$\Rightarrow \quad \frac{1+x}{x} dx = \frac{y-1}{y} dy$

$\Rightarrow \quad \int\left(\frac{1+x}{x}\right) dx = \int\left(\frac{y-1}{y}\right) dy$

$\Rightarrow \quad \log x + x = y - \log y + C$

$\Rightarrow \quad \log xy + x - y = C$ is the general solution.

Example : 4

Solve the differential equation : $xy^2 \frac{dy}{dx} = 1 - x^2 + y^2 - x^2y^2$

Solution

The given differential equation : $xy^2 \frac{dy}{dx} = 1 - x^2 + y^2 - x^2y^2$

$\Rightarrow \quad xy^2 \frac{dy}{dx} = (1 - x^2)(1 + y^2)$

$\Rightarrow \quad \frac{y^2 dy}{1+y^2} = \frac{(1-x^2)\, dx}{x}$

$\Rightarrow \quad \int \frac{y^2}{1+y^2}\, dy = \int \left(\frac{1}{x} - x\right) dx$

$\Rightarrow \quad y - \tan^{-1} y = \log x - \frac{x^2}{2} + C$ is the general solution of the given differential equation.

Example : 5

Solve $\frac{d^2y}{dx^2} = x + \sin x$ if y = 0 and $\frac{dy}{dx} = -1$ for x = 0

Solution

The given differential equation is : $\frac{d^2y}{dx^2} = x + \sin x$ ....(i)

It is an order 2 differential equation. But it can be easily reduced to order 1 differential equation by integrating both sides. On Integrating both sides of equation (i), we get

$\frac{dy}{dx} = \int (x + \sin x)\, dx$

$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2}{2} - \cos x + C_1$, where C_1 is an arbitrary constant ....(ii)

$\Rightarrow \quad dy = (x^2/2 - \cos x + C_1)\, dx$

$\Rightarrow \quad \int dy = \int \left(\frac{x^2}{2} - \cos x + C_1\right) dx$

$\Rightarrow \quad y = \frac{x^3}{6} - \sin x + C_1x + C_2$

This is the genral solution. For particular solution, we have to find C_1 and C_2

for x = 0, y = 0 $\Rightarrow$ $0 = \frac{0^3}{6} - \sin 0 + 0\, C_1 + C_2$ $\Rightarrow$ $C_2 = 0$

for x = 0, $\frac{dy}{dx} = -1$ $\Rightarrow$ $-1 = \frac{0^2}{2} - \cos 0 + C_1$ $\Rightarrow$ $C_1 = 0$ [put x = 0 and dy/dx = – 1 in (2)]

$\Rightarrow \quad y = \frac{x^3}{6} - \sin x$ is the particular solution of the given differential equation.

Example : 6

Solve the differential equation : $\frac{dy}{dx} - x \tan (y - x) = 1$

Solution

The given differential equation is : $\frac{dy}{dx} - x \tan (y - x) = 1$

Put $z = y - x$

$\Rightarrow \quad \frac{dz}{dx} = \frac{dy}{dx} - 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dz}{dx} + 1$

$\Rightarrow$ the given equation becomes : $\left(\frac{dz}{dx} + 1\right) - x \tan z = 1$

$\Rightarrow \quad \frac{dz}{dx} = x \tan z$

$\Rightarrow \quad \int \cot z \, dz = \int x \, dx$

$\Rightarrow \quad \log \sin z = \frac{x^2}{2} + C$

$\Rightarrow \quad \sin(y - x) = e^{x^2/2} \cdot e^C$

$\Rightarrow \quad \sin(y - x) = ke^{x^2/2}$ where k is an arbitrary constant.

Example : 7

Solve the differential equation : $\frac{dy}{dx} = \frac{2x - y}{x + y}$

Solution

The given differential equation is : $\frac{dy}{dx} = \frac{2x - y}{x + y}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{2 - y/x}{1 + y/x}$

Let $y = mx \quad \Rightarrow \quad \frac{dy}{dx} = m + x\frac{dm}{dx}$

$\Rightarrow \quad m + x\frac{dm}{dx} = \frac{2 - m}{1 + m}$

$\Rightarrow \quad x\frac{dm}{dx} = \frac{2 - 2m - m^2}{1 + m}$

$\Rightarrow \quad \frac{(1 + m)\, dm}{2 - 2m - m^2} = \frac{dx}{x}$

Integrate both sides :

$\Rightarrow \quad \frac{-1}{2} \int \frac{-2 - 2m}{2 - 2m - m^2} \, dm = \int \frac{dx}{x}$

$\Rightarrow \quad \frac{-1}{2} \log(2 - 2m - m^2) = \log x + \log C,$ where C is an arbitrary constant

$\Rightarrow \quad (2 - 2m - m^2) = \frac{1}{C^2 x^2}$

$\Rightarrow \quad \left(2 - \frac{2y}{x} - \frac{y^2}{x^2}\right) x^2 = K,$ where K is an arbitrary constant.

$\Rightarrow \quad 2x^2 - 2xy - y^2 = K$ is the required general solution.

Example : 8

Solve the differential equation : xdy – ydx = $\sqrt{x^2+y^2}$ dx

Solution

The given differential equation is : xdy – ydx = $\sqrt{x^2+y^2}$ dx

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$$

Let y = mx $\Rightarrow$ $\frac{dy}{dx} = m + x\frac{dm}{dx}$

$$\Rightarrow \quad \frac{dm}{\sqrt{1+m^2}} = \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{dm}{\sqrt{1+m^2}} = \int \frac{dx}{x}$$

$\Rightarrow \quad \log\left|m+\sqrt{1+m^2}\right| = \log x + \log C,$ where C is an arbitrary constant.

$$\Rightarrow \quad \frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = Cx$$

Example : 9

Solve the differential equation : (2x + y – 3) dy = (x + 2y – 3) dx

Solution

The given differential equation is : $\frac{dy}{dx} = \frac{x-2y-3}{2x+y-3}$

Solving $\left\{\begin{matrix} x+2y-3=0 \\ 2x+y-3=0 \end{matrix}\right\}$, we get : x = 1, y = 1

Put x = u + 1 and y = v + 1

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dv}{du}$$

$$\Rightarrow \quad \frac{dv}{du} = \frac{(1+u)+2(1+v)-3}{2(1+u)+(1+v)-3} = \frac{u+2v}{2u+v}$$

Now put v = mu $\Rightarrow$ $\frac{dv}{du} = m + u\frac{dm}{du}$

$$\Rightarrow \quad m + u\frac{dm}{du} = \frac{1+2m}{2+m}$$

$$\Rightarrow \quad \frac{2+m}{1-m^2}\,dm = \frac{du}{u}$$

$$\Rightarrow \quad \int \frac{2+m}{1-m^2}\,dm = \int \frac{du}{u}$$

$\Rightarrow \quad \int \left\{\frac{1/2}{1+m} + \frac{3/2}{1-m}\right\} dm = \int \frac{du}{u}$ (Resolving into partial fractions)

$\Rightarrow \quad \frac{1}{2} \log |1 + m| - \frac{3}{2} \log |1 - m| = \log u + \log C$

$\Rightarrow \quad (1 + m)(1 - m)^{-3} = u^2 C^2$ where $m = \frac{v}{u} = \frac{y-1}{x-1}$ and $u = x - 1$

$\Rightarrow \quad \left[1+\frac{y-1}{x-1}\right]\left[1-\frac{y-1}{x-1}\right]^{-3} = (x-1)^2 C^2$

$\Rightarrow \quad (x + y - 2) = (x - y)^3 C^2$ where c^2 is a constant

Example : 10

Solve the differential equation : $x\frac{dy}{dx} + y = x^3$.

Solution

The given equation is : $x\frac{dy}{dx} + y = x^3$.

Convert to standard from by dividing by x.

$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{x} y = x^2$

$\Rightarrow \quad P = \frac{1}{x}$ and $Q = x^2$

If $= e^{\int P\,dx} = e^{\int \frac{dx}{x}} = e^{\ln x} = x$

$\Rightarrow$ Solution is : $yx = \int x^2 (x)\,dx + C$ (using the formula)

$\Rightarrow \quad xy = \frac{x^4}{4} = C$ is the genral solution

Example : 11

Solve $\sin x \frac{dy}{dx} + y \cos x = 2 \sin^2 x \cos x$

Solution

The given differential equation is :

$\frac{dy}{dx} + \cot x\, y = 2 \sin x \cos x$

$\Rightarrow \quad P = \cot x$ and $Q = 2 \sin x \cos x$

$\int P\,dx = \int \cot x\,dx = \log \sin x$

$\Rightarrow \quad$ I.F. $= e^{\log \sin x} = \sin x$

Using the standard result, the solution is : $y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C$

$\Rightarrow \quad y \sin x = \int 2 \sin x \cos x \sin x\,dx + C$

$\Rightarrow \quad y \sin x = \frac{2}{3} \sin^3 x + C$ is the general solution.

Example : 12

Solve the differential equation : $x^2 \frac{dy}{dx} + xy = y^2$.

Solution

The differential equation is : $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$ (Bernoulli's Differential Equation)

$\Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \frac{1}{x^2}$ ....(i)

Let $\frac{1}{y} = t \quad \Rightarrow \quad \frac{-1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

On substituting in (i), we get

$\frac{dy}{dx} - \frac{t}{x} = \frac{-1}{x^2}$ i.e. linear differential equation.

I.F. $= e^{\int -\frac{1}{x} dx} = e^{-lux} = \frac{1}{x}$

Using the standard result, the solution of the differential equation is :

$$\frac{t}{x} = -\int \left(\frac{1}{x}\right) \frac{1}{x^2} \, dx + C$$

$\Rightarrow \quad \frac{1}{xy} = + \frac{1}{2x^2} + C$ is the general solution.

Example : 13

Solve the differential equation : $y^2 \frac{dy}{dx} = x + y^3$.

Solution

The given differential equation is : $y^2 \frac{dy}{dx} = x + y^3$

$\Rightarrow \quad \frac{dy}{dx} = \frac{x}{y^2} + y$

$\Rightarrow \quad \frac{dy}{dx} - y = xy^{-2}$ (Bernoulli's Differential Equation)

$\Rightarrow \quad y^2 \frac{dy}{dx} - y^3 = x$

Let $y^3 = t \quad \Rightarrow \quad 3y^2 \frac{dy}{dx} = \frac{dt}{dx}$

On substituting in the differential equation, it reduces to linear differential equation : i.e.

$\frac{dt}{dx} - dt = 3x$

I.F. $= e^{\int -3\, dx} = e^{-3x}$

Using the standard result, the solution of the differential equation is :

$e^{-3x} t = 3 \int xe^{-3x} \, dx + C$

$\Rightarrow \quad y^3 e^{-3x} = 3\left[x\int e^{-3x}\,dx + \frac{1}{3}\int e^{-3x}\,dx\right] + C$

$\Rightarrow \quad y^3 = -x - 1/3 + Ce^{3x}$

$\Rightarrow \quad 3(y^3 + x) + 1 = ke^{3x}$ is the general solution

Example : 14

Solve the differential equation : $xyp^2 - (x^2 - y^2)p - xy = 0$, where $\frac{dy}{dx} = p$.

Solution

The given differential equation is : $xyp^2 - x^2p + y^2p - xy = 0$

$\Rightarrow \quad (xyp^2 + y^2p) - (x^2p + xy) = 0$

$\Rightarrow \quad yp(xp + y) - x(xp + y) = 0$

$\Rightarrow \quad (xp + y)(yp - x) = 0$

Case – I $\quad x\frac{dy}{dx} + y = 0$

$\Rightarrow \quad xdy + ydx = 0 \quad \Rightarrow \quad d(xy) = 0$

On integrating, we get : $xy = k$

Case – II $\quad xp - x = 0$

$y\frac{dy}{dx} - x = 0$

integrating, we get $\frac{y^2}{2} - \frac{x^2}{2} = k$

or $\quad y^2 - x^2 - 2k = 0$

Hence the solution is $(xy - k)(y^2 - x^2 - 2k) = 0$

Example : 15

Solve the differential equation : $p(p + x) = y(x + y)$, where $p = \frac{dy}{dx}$

Solution

The given differential equation is : $p^2 + px - xy - y^2 = 0$

$\Rightarrow \quad (p^2 - y^2) + (px - xy) = 0$

$\Rightarrow \quad (p - y)(p + y) + x(p - y) = 0$

$\Rightarrow \quad (p - y)(p + x + y) = 0$

Case – I

$\Rightarrow \quad \frac{dy}{dx} - y = 0 \quad \Rightarrow \quad \frac{dy}{y} - dx = 0$

Integrating, we get : $\quad \log y = x + \log A = \log(Ae^x)$

or $\quad y = Ae^x$, where A is an arbitrary constant ....(i)

Case – II $\quad p + x + y = 0$

$\Rightarrow \quad \frac{dy}{dx} + x + y = 0$

$\Rightarrow \quad \frac{dy}{dx} + y - x \quad$ which is a linear equation.

I.F. = $\quad e^{\int dx} = e^x$

Using the standard result, the solution of the differential equation is :

$$y\,e^x = -\int xe^x\,dx + A$$

$\Rightarrow \quad y.e^x = e^x (1 - x) + A$

$\Rightarrow \quad y = 1 - x + Ae^{-x}$ ....(ii)

From (i) and (ii), we get the combined solution of the given equation as $(y - Ae^x)(y + x - 1 - Ae^{-x}) = 0$

Example : 16

Solve the differential equation : $y = (1 + p)x + ap^2$, where $p = \frac{dy}{dx}$

Solution

The given differential equation is : $y = (1 + p)x + ap^2$ [solvable for y, refer section 3.3] ....(i)

Differentiating the given equation w.r.t. x, we get

$$\frac{dy}{dx} = p = 1 + p + x\frac{dp}{dx} + 2ap\frac{dp}{dx}$$

$$\Rightarrow \quad 0 = 1 + \frac{dp}{dx}(x + 2ap)$$

$$\Rightarrow \quad \frac{dx}{dp} + x + 2ap = 0, \text{ which is a linear equation.}$$

I.F. = $e^{\int dp} = e^p$

Using the standard result, the solution of the differential equation is :

$$x\,e^p = -2a\int pe^p\,dp + C = -2a(p-1)\,e^p + C$$

$\Rightarrow \quad x = 2a(1 - p) + Ce^{-p}$ ....(ii)

The p-eliminant of (i) and (ii) is the required solution.

Example : 17

Solve the differential equation : $p^2y + 2px = y$

Solution

The given differential is : $x = \frac{y}{2p} - \frac{yp}{2}$ [solvable for x, refer section 3.4] ....(i)

Differentiating with respect to y, we get

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - \frac{p}{2} - \frac{y}{2}\frac{dp}{dy}$$

$$\Rightarrow \quad \frac{1}{2p} + \frac{p}{2} = -\frac{y}{2}\frac{dp}{dy}\left(\frac{1}{p^2} + 1\right)$$

$$\Rightarrow \quad \frac{1+p^2}{2p} = -\frac{y}{2}\frac{dp}{dy}\frac{1+p^2}{p^2}$$

$$\Rightarrow \quad 1 = -\frac{y}{p}\frac{dp}{dy} \quad \text{as } 1 + p^2 \neq 0$$

$\Rightarrow \quad d\,py + ydp = 0$

$\Rightarrow \quad d(py) = 0$

Integrating, we get $py = k \quad \Rightarrow \quad p = \frac{C}{y}$

Putting the value of p in (i), we get

$$y\,.\,\frac{C^2}{y^2} + 2x\,.\,\frac{C}{y} = y$$

$C^2 + 2Cx = y^2$

which si the required solution.

Example : 18

Solve the differential equation : $x = yp + ap^2$.

Solution

The given differential is : $x = yp + ap^2$

Differentiating with respect to y, we get

$$\frac{dx}{dy} = \frac{1}{p} = p + y\frac{dp}{dy} + 2ap\frac{dp}{dy}$$

i.e. $\frac{1}{p} - p = \frac{dp}{dy}(y + 2ap)$

i.e. $\frac{dy}{dp} = \frac{py}{1-p^2} + \frac{2ap^2}{1-p^2}$

i.e. $\frac{dy}{dp} - \frac{p}{1-p^2}y = \frac{2ap^2}{1-p^2}$ which is linear equation

I.F. = $e^{-\int\frac{p}{1-p^2}} = e^{\frac{1}{2}\log(1-p^2)}$

Using the standard result, the solution of the differential equation is :

$$y\sqrt{1-p^2} = 2a\int\frac{p^2}{1-p^2}\cdot\sqrt{1-p^2}\,dp$$

$$= 2a\int\frac{p^2dp}{1-p^2} = -2a\int\frac{(1-p^2)-1}{\sqrt{1-p^2}}\,dp$$

$$= -2a\int\sqrt{1-p^2}\,dp + 2a\int\frac{dp}{\sqrt{1-p^2}}$$

$$= -2a\left[\frac{1}{2}p\sqrt{1-p^2} + \frac{1}{2}\sin^{-1}p\right] + 2a\sin^{-1}p + k$$

$$= y\sqrt{1-p^2} = -ap\sqrt{1-p^2} + a\sin^{-1}p + k. \qquad \text{.........(ii)}$$

The p-eliminant of (i) and (ii) is the required solution.

Example : 19

Solve the differential equation : $p^3x - p^2y - 1 = 0$

Solution

The given differential equation is : $y = px - 1/p^2$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = p = p + x\frac{dp}{dx} + \frac{2}{p^3}\frac{dp}{dx}$$

$$\Rightarrow \quad \frac{dp}{dx}\left(x + \frac{2}{p^3}\right) = 0$$

$\Rightarrow \quad \frac{dp}{dx} = 0 \qquad$ ....(ii)

or $\quad p^3 = \frac{-2}{x} \qquad$ ....(iii)

Consider (2)

Integrate both sides to get : $\quad p = c$ where c is an arbitrary constant

put p = c in (i) to get the general solution of the differential equation i.e.

$y = cx - 1/c^2$ $\quad$ is the general solution

Consider (3)

Eliminate p between (iii) and (i) to get the singular solution i.e.

$$y = \frac{\left(\frac{-2}{x}\right)x - 1}{\left(\frac{-2}{x}\right)^{2/3}} = \frac{-3}{\left(\frac{-2}{x}\right)^{2/3}}$$

Take cube of both sides to get : $y^3 = \frac{-27}{4/x^2}$

$\Rightarrow \quad 4y^3 = -2yx^2$ is the singular solution.

Example : 20

Form the differential equation satisfied by the general circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Solution

In forming differential equations for curve, we have to eliminate the arbitrary constants (g, f, v) for n arbitrary constant, we get will finally get an nth order differential equation. Here we will get a third order differential equation in this example.

Differentiating once, $\quad 2x + 2yy' + 2g + 2fy' = 0 \qquad$ ....(i)

Differentiating again $\quad 1 + y'^2 + yy'' + fy'' = 0 \qquad$ ....(ii)

Differentiating again $\quad 2y'y'' + yy''' + y'y'' + fy''' = 0$

We can now eliminate from (i) and (ii)

$\Rightarrow \quad y''' (1 + yy'' + y'^2) - y'' (yy''' + 3y' y'') = 0$

$\Rightarrow \quad y''' (1 + y'^2) - 3y' y''^2 = 0$ is the required differential equation

Example : 21

Find the differential equation satisfied by : $ax^2 + by^2 = 1$

Solution

The given solution is : $ax^2 + by^2 = 1$

Differentiate the above solution to get :

$2ax + 2byy' = 0 \qquad$ ....(i)

Differentiating again, we get

$2a + 2b(y'^2 + y'') = 0 \qquad$ ....(ii)

Eliminating a and b from (i) and (ii), we will get the required differential equation

from (i), we have $\quad \frac{a}{b} = -\frac{yy'}{x} \quad$ and

from (ii), we have $\quad \frac{a}{b} = -(y'^2 + yy'')$

$\Rightarrow \quad -\frac{yy'}{x} = -(y'^2 + yy'')$

$\Rightarrow \quad yy' = xy'^2 + xyy''$

$\Rightarrow \quad xyy'' + xy'2 - yy' = 0$ is the required differential equation.

Example : 22

The slope of curve passing through (4, 3) at any point is reciprocal of twice the ordinate at that point. Show that the curve is a parabola.

Solution

The slope of the curve is the reciprocal of twice the ordinate at each point of the curve. Using this property, we can define the differential equation of the curve i.e.

$$\text{slope} = \frac{dy}{dx} = \frac{1}{2y}$$

Integrate both sides to get :

$$\int 2y\,dy = \int dx$$

$\Rightarrow \quad y^2 = x + C$

As the required curve passes through (4, 3), it lies on it.

$\Rightarrow \quad 9 = 4 + C \quad \Rightarrow \quad C = 5$

So the required curve is : $y^2 = x + 5$ which is a parabola

Example : 23

Find the equation of the curve passing through (2, 1) which has constant subtangent.

Solution

The length of subtangent is constant. Using this property, we can define the differential equation of the curve i.e.

$$\text{subtangent} = \frac{y}{y'} = k \quad \text{where k is a constant}$$

$$\Rightarrow \quad k\frac{dy}{dx} = y$$

Integrate both sides to get :

$$\int \frac{k\,dy}{y} = \int dx$$

$\Rightarrow \quad k \log y = x + C$ where C is an arbitrary constant.

As the required curve passes through (2, 1), it lies on it.

$\Rightarrow \quad 0 = 2 + k \quad \Rightarrow \quad C = -2$

$\Rightarrow$ the equation of the curve is : $k \log y = x - 2$.

Note that above equation can also be put in the form $y = Ae^{Bx}$.

Example : 24

Find the curve through (2, 0) so that the segment of tangent between point of tangency and y-axis has a constant length equal to 2

Solution

The segment of the tangent between the point of tangency and y-axis has a constant length = PT = 2. Using this property, we can define the differential equation of the curve i.e.

$$PT = x \sec\theta = x\sqrt{1+\tan^2\theta} = x\sqrt{1+y'^2}$$

$$\Rightarrow \quad x\sqrt{1+\left(\frac{dy}{dx}\right)^2} = 2$$

$$\Rightarrow \quad 1 = \left(\frac{dy}{dx}\right)^2 = \frac{4}{x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \pm\sqrt{\frac{4-x^2}{x^2}}$$

Integrate both sides to get :

$\Rightarrow \quad y = \pm \int \sqrt{\dfrac{4-x^2}{x^2}}\ dx + C_1$

Put $x = 2 \sin \theta \quad \Rightarrow \quad dx = 2 \cos \theta\ d\theta$

$\Rightarrow \quad y = 2 \pm \int \dfrac{\cos^2 \theta}{\sin\theta}\ d\theta + C_1 =$

$\pm 2 \int (\operatorname{cosec}\theta - \sin\theta)\ d\theta + C_1 = \pm (2 \log |\operatorname{cosec} \theta - \cot \theta| + 2 \cos \theta) + C$

$\Rightarrow \quad y = \pm 2 \log \left(\left| \dfrac{2-\sqrt{4-x^2}}{x} \right| + \sqrt{4-x^2} \right) + C$

As (2, 0) lies on the curve, it should satisfy its equation, i.e. $\quad C = 0$

$\Rightarrow$ the equation of the curve is : $y = \pm 2 \log \left(\left| \dfrac{2-\sqrt{4-x^2}}{x} \right| + \sqrt{4-x^2} \right)$

Example : 25

Find the equation of the curve passing through the origin if the mid-point of the segment of the normal drawn at any point of the curve and the X-axis lies on the parabola $2y^2 = x$.

Solution

$OB = OM + MB = x + y \tan \theta = x + yy'$

$\Rightarrow \quad B \equiv (x + yy', 0)$

$\Rightarrow \quad N$ (mid point of PB) $\equiv \left(x + \dfrac{yy'}{2}, \dfrac{y}{2} \right)$

N lies on $2y^2 = x$

$\Rightarrow \quad 2\left(\dfrac{y}{2}\right)^2 = x + \dfrac{yy'}{2}$

$\Rightarrow \quad yy' - y^2 = -2x$ (Divide both sides by y and check that it is a Bernoulli's differential equation)

Put $\quad y^2 = t \quad \Rightarrow \quad 2yy' = \dfrac{dt}{dx}$

$\Rightarrow \quad \dfrac{1}{2}\dfrac{dt}{dx} - t = -2x$

$\Rightarrow \quad \dfrac{dt}{dx} - 2t = -4x$ $\qquad$ which is a linear differential equation.

I.F. = Integrating factor = $e^{\int -2\, dx} = e^{-2x}$

Using the standard result, the solution of the differential equation is ;

$te^{-2x} = \int -4x^{-2x} dx$

$\Rightarrow \quad te^{-2x} = -\left(\dfrac{xe^{-2x}}{-2} + \int \dfrac{e^{-2x}}{2} dx \right)$

$\Rightarrow \quad te^{-2x} = -4 \left\{ -\dfrac{xe^{-2x}}{2} - \dfrac{e^{-2x}}{4} \right\} + C$

$\Rightarrow \quad t = 2x + 1 + Ce^{2x}$

$\Rightarrow \quad y^2 = 2x + 1 + Ce^{2x}$

As it passes through (0, 0), C = – 1

$\Rightarrow \quad y^2 = 2x + 1 - e^{2x}$ is the required curve.

Example : 26

Find equation of curves which intersect the hyperbola xy = 4 at an angle $\pi/2$.

Solution

Let $m_1 = \frac{dy}{dx}$ for the required family of curves at (x, y)

Let m_2 = value of $\frac{dy}{dx}$ for xy = 4 curve.

$\Rightarrow \quad m_2 = \frac{dy}{dx} = -\frac{4}{x^2}$

As the requied family is perpendicular to the given curve, we can have :

$m_1 \times m_2 = -1$

$\Rightarrow \quad \frac{dy}{dx} \times \left(-\frac{4}{x^2}\right) = -1$

$\Rightarrow$ for required family of curves : $\frac{dy}{dx} = \frac{x^2}{4}$

$\Rightarrow \quad dy = \frac{x^2 dx}{4}$

$\Rightarrow \quad y = \frac{x^3}{12} + C$ is the requied family which intersects xy = 4 curve at an angle $\pi/2$

Example : 27

Solve the differential equation : $(1 + e^{x/y})\, dx = e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$

Solution

The given differential equation is : $(1 + e^{x/y})\, dx = e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$ which is a homogenous differential equation.

Put $\quad x = my \Rightarrow \quad \frac{dx}{dy} = m + y\frac{dm}{dy}$

The given equation reduces to $(1 + e^m)\left(m + y\frac{dm}{dy}\right) + e^m (1 - m) = 0$

$(m + me^m + e^m - me^m) = -(1 + e^m)\, y\frac{dm}{dy} \quad \Rightarrow \quad \frac{dy}{y} = -\frac{1+e^m}{m+e^m}\, dm$

Integrating both sides, we get :

$\log y + \log (m + e^m) = C_1$

$\Rightarrow \quad \log y\left(\frac{x}{y} + e^{x/y}\right) = C_1 \quad \Rightarrow \quad x + ye^{x/y} = C$ is the requied general solution.

Example : 28

Solve the equation : $\left(1+x\sqrt{x^2+y^2}\right)$ dx + $\left(-1+\sqrt{x^2+y^2}\right)$ y dy = 0

Solution

The given differential equation can be written as :

$$dx - ydy + x\sqrt{x^2+y^2}\ dx + \sqrt{x^2+y^2}\ ydy = 0$$

$$\Rightarrow \quad dx - ydy + \sqrt{x^2+y^2}\ (xdx + ydy) = 0$$

$$\Rightarrow \quad dx - ydy + \frac{1}{2}\sqrt{x^2+y^2}\ d(x^2 + y^2) = 0$$

Integrating both the sides, we get :

$$x - \frac{y^2}{2} + \frac{1}{2}\int \sqrt{t}\,dt + C = 0 \qquad \text{where } t = x^2 + y^2$$

$$\Rightarrow \quad x - \frac{y^2}{2} + \frac{1}{3}(x^2 + y^2)^{3/2} = C$$

Example : 29

Determine the equation of the curve passing through the origin in the form y = f(x), which satisfies the differential equation dy/dx = sin (10 + 6y)

Solution

Let $\quad 10x + 6y = m \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{6}\left(\frac{dm}{dx} - 10\right)$

So, we get, $\frac{dm}{dx}$ = 2 (3 sin m = 5)

$$\Rightarrow \quad \int \frac{dm}{2(3\sin m + 5)} = \int dx$$

Put tan m/2 = t and solve integral on LHS to get :

$$\frac{1}{4}\tan^{-1}\left(\frac{5t+3}{4}\right) = x + C$$

As curve passes through (0, 0) $\;C = \frac{1}{4}\tan^{-1}\frac{3}{4}$

$$\Rightarrow \quad \tan(4x + \tan^{-1}3/4) = \frac{5\tan(5x+3y)+3}{4}$$

Simplify to get :

$$y = \frac{1}{3}\tan^{-1}\left(\frac{5\tan 4x}{4-3\tan 4x}\right) - \frac{5x}{3} \qquad \left[\text{use } \tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B}\right]$$

Example : 30

Solve the differential equation : $(xy^4 + y)\,dx - x\,dy = 0$

Solution

The given differential equation is : $(xy^4 + y)\,dx - x\,dy = 0$

$$\Rightarrow \quad x\frac{dy}{dx} = xy^4 + y$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = y^4 \qquad \text{(Bernoulli's differential equation)}$$

Divide both sides by y^4 to get :

$$\frac{1}{y^4}\ \frac{dy}{dx} - \frac{1}{y^3x} = 1 \qquad \text{.........(i)}$$

Let $\frac{1}{y^3} = t \qquad \Rightarrow \qquad \frac{-3}{y^4}\frac{dy}{dx} = \frac{dt}{dx}$

After substitution, (i) reduces to :

$\frac{dt}{dx} + \frac{3t}{x} = -3$ (linear differential equation)

I.F. $e^{\int P\,dx} = e^{\int \frac{3}{x} dx} = e^{3\ell nx} = x^3$

Using the standard result, the solution of differential equation is :

$$tx^3 = \int -3x^3 dx + C_1$$

$$\Rightarrow \qquad tx^3 = \frac{-3x^4}{4} + C$$

$$\Rightarrow \qquad \frac{x^3}{y^3} = -\frac{3}{4}x^4 + C$$

$\Rightarrow \qquad \frac{x^3}{3y^3} + \frac{1}{4}x^4 = C$ is the required general solution.

Alternate Method

Consider the given differential equation, $(xy^4 + y)\,dx - x\,dy = 0$

$\Rightarrow \qquad dy^4dx + ydx - xdy = 0$

Divide both sides by y^4 to get

$$xdx + \frac{ydx - xdy}{y^4} = 0$$

Multiply both sides by x^2 to get :

$$x^2\,dx + \left(\frac{x^2}{y^2}\right)\frac{ydx - xdy}{y^2} = 0$$

$$\Rightarrow \qquad x^3\,dx + \frac{x^2}{y^2}\ d\left[\frac{x}{y}\right] = 0$$

Integrate both sides

$$\int x^3\,dx + \int \frac{x^2}{y^2} d\left(\frac{x}{y}\right) = 0$$

$\Rightarrow \qquad \frac{x^4}{4} + \frac{x^3}{3y^3} = C$ is the requied general solution

Example : 31

Solve the following differential equation : $\frac{xdx + ydy}{xdy - ydx} = \frac{\sqrt{1-(x^2+y^2)}}{\sqrt{x^2+y^2}}$

Solution

The given differential equation can be written as

$$\frac{xdx + ydy}{\sqrt{1-(x^2+y^2)}} = \frac{xdy - ydx}{\sqrt{x^2+y^2}}$$

Divide both sides by $\sqrt{x^2+y^2}$ to get

$$\frac{xdx+ydy}{\sqrt{x^2+y^2}\sqrt{1-(x^2+y^2)}} = \frac{xdy-ydx}{x^2+y^2}$$

Using the fact that d [$x^2 + y^2$] = 2 (xdx + ydy) and d $\left[\tan^{-1}\frac{y}{x}\right] = \frac{xdy-ydx}{x^2+y^2}$, we get

$$\frac{\frac{1}{2}d(x^2+y^2)}{\sqrt{x^2+y^2}\sqrt{1-(x^2+y^2)}} = d\left[\tan^{-1}\frac{y}{x}\right]$$

Put $x^2 + y^2 = t^2$ in the LHS to get :

$$\frac{tdt}{t\sqrt{1-t^2}} = d\left(\tan^{-1}\frac{y}{x}\right)$$

Integrate both sides

$$\int\frac{tdt}{t\sqrt{1-t^2}} = \tan^{-1}\frac{y}{x} + C_1$$

$\Rightarrow$ $\sin^{-1} t = \tan^{-1} (y/x) + C$

so the general solution is : $\sin^{-1}\sqrt{x^2+y^2} = \tan^{-1}\frac{y}{x} + C$

Example : 32

Solve the differential equation : $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$.

Solution

The given differential equation is : $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

Dividing both sides by $\cos^2 y$, we get

$$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$$

Let $\tan y = t$ $\Rightarrow$ $\sec^2 y \frac{dy}{dx} = \frac{dt}{dx}$

On substitution, differential equation reduces to :

$\frac{dt}{dx} + 2xt = x^3$ (linear differential equation)

I.F. = $e^{\int 2x\,dx} = e^{x^2}$

Using the standard result, the general solution is :

$$te^{x^2} = \int x^3 e^{x^2} dx + C_1$$

Integrate RHS yourself to get the general solution :

$$te^{x^2} = \frac{1}{2}(x^2-1)\,e^{x^2} + C$$

Replace t by tan y, we get :

$\tan y = \frac{1}{2}(x^2-1)\,C e^{-x^2}$ which is the requied solution

Example : 33

A normal is drawn at a point P(x, y) of a curve. It meets the x-axis at Q. If PQ is of constant length k, then show that the differential equation describing such curves is $y\frac{dy}{dx} = \pm\sqrt{k^2 - y^2}$. Also find the equation of the curve if it passes through (0, k) point

Solution

Let M be the foot of the perpendicular drawn from P to the x-axis

In triangle PMQ,

PQ = k (given), QM = subnormal = y (dy/dx) and PM = y

Apply pythagoras theorem in triangle PMQ to get :

$PQ^2 = PM^2 + MQ^2$

$\Rightarrow \quad k^2 = y^2 + y^3 \left(\frac{dy}{dx}\right)^2$

$\Rightarrow \quad y\frac{dy}{dx} = \pm\sqrt{k^2 - y^2}$ which is requied to be shown

Solving the above differential equation, we get :

$$\int \frac{ydy}{\sqrt{k^2 - y^2}} = \pm \int dx$$

$\Rightarrow \quad -\sqrt{k^2 - y^2} = \pm x + C$

As (0, k) lies on it, 0 = 0 + C $\Rightarrow$ C = 0

$\Rightarrow$ equation of curve is : $-\sqrt{k^2 - y^2} = \pm x$

$\Rightarrow \quad x^2 + y^2 = k^2$ is the required equation of the curve.

Example : 34

A curve y = f(x) passes through the point P(1, 1). The normal to the curve at P is : a(y – 1) + (x – 1) = 0. If the slope of the tangent at any point on the curve is proportional to the ordinate of that point, determine the equation of the curve. Also obtain the area bounded by the y-axis, the curve and the normal to the curve at P.

Solution

It is given that equation of the normal at point P(1, 1) is ≡ ay + x = a + 1

$\Rightarrow$ slope of tangent at P = –1/(slope of normal at P)

$\Rightarrow \quad \left.\frac{dy}{dx}\right]_{at\ P} = a$ ....(i)

It is also given that slope of the tangent at any point of the curve is proportional to the ordinate i.e.

$\Rightarrow \quad \tan\theta = \frac{dy}{dx} = dy$

$\Rightarrow \quad \frac{dy}{dx} = ay$ [∵ from (i0, at P(1, 1), dy/dx = a]

On solving, we get : $\ell nx = ax + C$

As curve passes through (1, 1), 0 = a + C $\Rightarrow$ C = – a

$\Rightarrow$ equation of the curve is : $y = e^{x(x-1)}$

$$\text{requied Area} = \int_0^1 \left[\frac{1-x}{a} + 1 - e^{x(x-1)}\right] dx = \left|\frac{x}{a} - \frac{x^2}{2a} + x - \frac{e^{x(x-1)}}{a}\right|_0^1$$

$$= \left(\frac{1}{a} - \frac{1}{2a} + 1 - \frac{1}{a}\right) + \frac{e^{-a}}{a} = \frac{2e^{-a} - 1 + 2a}{2a}$$

Example : 35

Find the equation to the curve such that the distance between the origin and the tangent at an arbitrary point is equal to the distance between the origin and the normal at the same point.

Solution

Equation of tangent to the curve y = f(x) and any point (x, y) is :

$$Y - y = f'(x)(X - x) \quad \text{.............(i)}$$

The distance of the tangent from origin = $\dfrac{|y - f'(x)x|}{\sqrt{1 + (f'(x))^2}}$ ....(i)

Equation of norma to the curve y = f(x) and any point (x, y) is :

$$Y - y = -\frac{1}{f'(x)}(X - x)$$

The distance of the normal from origin = $\dfrac{\left|y + \dfrac{1}{f'(x)}x\right|}{\sqrt{1 + \left(\dfrac{1}{f'(x)}\right)^2}}$ ....(ii)

From (i) and (ii) and using the fact that the distance of the tangent and normal from origin is equal, we get:

$$y - f'(x)x = f'(x)\left|y + \frac{1}{f'(x)}x\right| = \pm[f'(x)y + x]$$

$\Rightarrow \quad y - x = (x + y)\dfrac{dy}{dx} \quad$ or $\quad x + y = (x - y)\dfrac{dy}{dx}$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{y - x}{y + x} \quad$ or $\quad \dfrac{dy}{dx} = \dfrac{x + y}{x - y}$

Consider case – I

$\dfrac{dy}{dx} = \dfrac{y - x}{y + x} = \dfrac{y/x - 1}{y/x + 1}$ which is a homogeneous equation.

Put y = mx $\Rightarrow$ dy/dx = m + x (dm/dx)

On substituting in the differential equation, we get :

$$m + x\frac{dm}{dx} = \frac{m - 1}{m + 1}$$

$$\Rightarrow \quad \frac{dx}{x} = -\left(\frac{1 + m}{1 + m^2}\right) dm$$

Integrate both sides, to get :

$$\int \frac{dx}{x} = \int\left(-\frac{1}{1 + m^2} - \frac{1}{2}\cdot\frac{2m}{1 + m^2}\right) dm$$

$\Rightarrow \quad \log x = -\tan^{-1} m - 1/2 \log(1 + m^2) + C$

$\Rightarrow \quad \log x(1 + m^2)^{1/2} = -\tan^{-1} m + C$

$$\Rightarrow \quad x\left(1 + \frac{y^2}{x^2}\right)^{1/2} = Ce^{-\tan^{-1} y/x}$$

$\Rightarrow \quad \sqrt{x^2 + y^2} = Ce^{-\tan^{-1} y/x}$ is the general solution

Consider case – II

$\dfrac{dy}{dx} = \dfrac{x + y}{x - y} = \dfrac{1 + y/x}{1 - y/x}$ which is a homogeneous equation.

On solving the above homogenous differential equation, we can get :

$$\sqrt{x^2+y^2} = Ce^{\tan^{-1} y/x} \text{ as the general solution}$$

Example : 36

Show that curve such that the ratio of the distance between the normal at any of its points and the origin to the distance between the same normal and the point (a, b) si equal to the constant k(k > 0) is a circle if k ≠ 1.

Solution

Equation of the normal at any point (x, y) to curve y = f(x) is

$$Y - y = -\frac{1}{f'(x)}(X - x)$$

$$\text{its distance from origin} = \frac{\left|y + \frac{x}{f'(x)}\right|}{\sqrt{1+\left(\frac{1}{f'(x)}\right)^2}}$$

$$\text{The distance of the normal from (a, b)} = \frac{\left|y - b\frac{1}{f'(x)}(x-a)\right|}{\sqrt{1+\left(\frac{1}{f'(x)}\right)^2}}$$

As the ratio of these distances is k, we get :

$$\left|y + \frac{x}{f'(x)}\right| = k\left|y - b + \frac{1}{f'(x)}(x-a)\right|$$

$$y + \frac{x}{f'(x)} = \pm k\left(y - b + \frac{1}{f'(x)}(x-a)\right)$$

$$(1-k)y + bk = (kx - x - ak)\frac{dx}{dy} \quad \text{and} \quad (1+k)y - bk = (-kx - x + ak)\frac{dx}{dy}$$

$$\Rightarrow \quad (1-k)ydy + bkdy = kxdx - xdx - akdx \quad \text{and} \quad (1+k)ydy - bkdy = -kxdx - xdx + akdx$$

Integrating both the sides

$$(1-k)\frac{y^2}{2} = bky = \left(k\frac{x^2}{2} - \frac{x^2}{2} - akx\right) + C_1 \quad \text{and} \quad (1+k)\frac{y^2}{2} - bky = \left(-k\frac{x^2}{2} - \frac{x^2}{2} + akx\right) + C_2$$

$$\frac{(1-k)}{2}x^2 + (1-k)\frac{y^2}{2} + bky + akx + C_1 = 0 \quad \text{and} \quad \frac{(1-k)}{2}x^2 + (1+k)\frac{y^2}{2} - bky - akx + C_2 = 0$$

If k ≠ 1, then both the above equations represent circle.

Example : 37

Let y = f(x) be a curve passing through (1, 1) such that the triangle formed by the coordinate axes and the tangent at any point of the curve lies in the first quadrant and has area 2. From the differential equation and determine all such possible curves.

Solution

$$\text{Equation of tangent at (x, y)} = Y' - y = \frac{dy}{dx}(X - x)$$

$$X_{intercept} = x - \frac{y}{dy/dx} \quad \text{and} \quad Y_{intercept} = y - x\frac{dy}{dx}$$

Area of the triangle = $\left|\frac{1}{2} X_{\text{int ercept}} \times Y_{\text{int ercept}}\right|$ = 2

Both X-intercept and Y-intercept are positive as the triangle lies in the first quadrant. So we can remove mod sign.

$\Rightarrow \quad \left(x - \frac{y}{y'}\right)(y - xy') = 4$

$\Rightarrow \quad (xy' - y)^2 = -4y'$

$\Rightarrow \quad xy' - y = -2\sqrt{-y'}$ $\qquad \left(\because\ y_{int} = y - \frac{xdy}{dx} > 0 \ \Rightarrow\ xy' - y < 0\right)$

$\Rightarrow \quad y = xy' + 2\sqrt{-y'}$ (Clairaut's differential equation) ....(i)

Differentiate both sides w.r.t. to x, to get :

$\Rightarrow \quad y' = xy'' + y' + \frac{2}{2\sqrt{-y'}}(-y'')$

$\Rightarrow \quad y'' = 0 \quad \text{or} \quad x = \frac{1}{\sqrt{-y'}}$

consider $y'' = 0$ integrate both sides to get : $y' = c$

Put $y' = c$ in (i) to get the general solution of the equation i.e.

$y = cx + 2\ 2\sqrt{-c}$

As the curve passes through (1, 1), c = – 1 (check yourself)

$\Rightarrow$ the equation of the curve is : x + y = 2

Consider : $x = \frac{1}{\sqrt{-y'}}$

$\Rightarrow \quad y' = \frac{-1}{x^2}$ ....(ii)

To find singular solution of the Clairaut's equation, eliminate y′ in (i) and (ii)

Replace y′ from (ii) into (i) to get :

$y = \frac{-x}{x^2} + 2\sqrt{\frac{1}{x^2}} = \frac{-1}{x} + \frac{2}{x} = \frac{1}{x}$

$\Rightarrow$ the requied curves are y = 1/x and x + y = 2.

Example : 38

Let u(x) and v(x) satisfy the differential equations $\frac{du}{dx}$ + P(x) u = f(x) and $\frac{dv}{dx}$ + P(x) v = g(x) where P(x), f(x) and g(x) are continuous function. If $u(x_1) > v(x_1)$ for some x_1 and f(x) > g(x) for all $x > x_1$, prove that any point (x, y) where $x > x_1$

Solution

The given differential equation are :

$\frac{du}{dx} = P(x) = u = f(x)$ ....(i)

$\frac{dv}{dx} = P(x)\, v = g(x)$ ....(ii)

On subtracting the two differential equations, we get

$\frac{d}{dx}(u - v) + P(x)(u - v) = f(x) - g(x)$

For $x > x_1$, $f(x) > g(x)$ $\Rightarrow$ $\frac{d}{dx}(u - v) + P(x)(u - v) > 0$

$\Rightarrow$ $\frac{d(u-v)}{u-v} > -P(x)\,dx$

Integrate both sides to get :

$$\ell n\,(u - v) + C > \int -P(x)\,dx$$

$\Rightarrow$ $u - v > e^{\int P(x)\,dx - C}$

As RHS > 0 for all x, u > v for all $x > x_1$

$\Rightarrow$ y = u(x) and y = v(x) have no solution (i.e. no point of intersection as one curve lies above the other)

Example : 39

A and B are two separate reservoirs of water. Capacity of reservoir A is double the capacity of reservoir B. Both the reservoirs are filled completely with water, their inlets are closed and then the water is released simultaneously from both the reservoirs. The rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservoir at that time. One hour after the water is released, the quantity of water in reservoir is $1\frac{1}{2}$ times the quantity of water in reservoir B. After how many hours do both the reservoir have the same quantity of water?

Solution

Let V_{Ai} and V_{Bi} be the initial amounts of water in reservoirs A and B respectively

As capacity of reservoir A si double that of B and both are completely filled initially, we can have:

$$V_{Ai} = 2V_{Bi}$$

Let V_A and V_n be the amount of water in reservoirs A and B respectively at any instant fo time t.

As the rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservoir at that time, we can have :

$$\frac{dV_A}{dt} = -k_1 V_A \qquad ....(i)$$

and $$\frac{dV_B}{dt} = -k_2 V_B \qquad ....(ii)$$

where k_1 and k_2 are proportionality constants.

Let V_{Af} and V_{Bf} be the amounts of water in reservoirs A and B respectively after 1 hour.

To find V_{Ar} and A_{bf} integrate (i) and (ii)

$\Rightarrow$ $\int_{V_{Ai}}^{V_{Af}} \frac{dV_A}{V_A} = -\int_0^1 k_1 dt$ $\Rightarrow$ $\ell n\left(\frac{V_{Af}}{V_{Ai}}\right) = -k_1$

Similarity we can get : $\ell n\left(\frac{V_{Bf}}{V_{Bi}}\right) = -k_2$ $\Rightarrow$ $V_{Ai}\,e^{-k_1} = \frac{3}{2}\,V_{Bi}\,e^{-k_2}$

$\Rightarrow$ $k_1 - k_2 = \ell n\left(\frac{4}{3}\right)$ ....(iii)

After time t $V_A = V_B$

$\Rightarrow$ $V_{Ai}e^{-k_1 t} = V_{Bi}e^{-k_2 t}$

$\Rightarrow$ $2e^{-k_1 t} = e^{-k_2 t}$

$\Rightarrow$ $(k_1 - k_2)\,t = \ell n\,2$ ....(iv)

Solving (iii) and (iv), we get : $t = \frac{\ell n 2}{\ell n\left(\frac{4}{3}\right)}$

Differentiability

Example : 1

Discuss the differentiability of f(x) at x = – 1, if f (x) = $\begin{cases} 1-x^2 & ; \quad x \le -1 \\ 2x+2 & ; \quad x > -1 \end{cases}$

Solution

$f(-1) = 1 - (1)^2 = 0$

Right hand derivative at x = – 1 is

$$Rf'(-1) = \lim_{h\to 0} \frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0} \frac{2(-1+h)+2-0}{h} = \lim_{h\to 0} \frac{2h}{h} = 2$$

Left hand derivative at x = – 1 is

$$Lf'(-1) = \lim_{h\to 0} \frac{f(-1-h)-f(-1)}{-h} = \lim_{h\to 0} \frac{1-(-1-h)^2-0}{-h} = \lim_{h\to 0} \frac{-h^2-2h}{-h} = \lim_{h\to 0} (h+2) = 2$$

Hence Lf′ (–1) = Rf′ (–1) = 2

⇒ the function is differentiable at x = – 1

Example : 2

Show that the function : $f(x) = |x^2 - 4|$ is not differentiable at x = 2

Solution

$$f(x) = \begin{cases} x^2-4 & ; \quad x \le -2 \\ 4-x^2 & ; \quad -2 < x < 2 \\ x^2-4 & ; \quad x \ge 2 \end{cases} \quad \Rightarrow \quad f(2) = 2^2 - 4 = 0$$

$$Lf'(2) = \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h} = \lim_{h\to 0} \frac{4-(2-h)^2-0}{-h} = \lim_{h\to 0} \frac{4h-h^2}{-h} = \lim_{h\to 0} (h-4) = -4$$

$$Rf'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{[(2+h)^2-4]-0}{h} = \lim_{h\to 0} \frac{h^2+4h}{h} = \lim_{h\to 0} (h+4) = 4$$

⇒ $Lf'(2) \ne Rf'(2)$

Hence f(x) is not differentiable at x = 2

Example : 3

Show that f(x) = x |x| is differentiable at x = 0

Solution

$$f(x) = \begin{cases} -x^2 & ; \quad x \le 0 \\ x^2 & ; \quad x > 0 \end{cases}$$

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{-(-h)^2-0}{-h} \lim_{h\to 0} h = 0$$

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)^2-f(0)}{h} \lim_{h\to 0} \frac{h^2-0}{h} = 0$$

⇒ $Lf'(0) = Rf'(0)$

Hence f(x) is differentiable at x = 0

Example : 4

Prove that following theorem :

"If a function y = f(x) is differentiable at a point, then it must be continuous at that point."

Solution

Let the function be differentiable at x = a.

⇒ $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ and $\lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}$ are finite numbers which are equal

L.H.L. = $\lim_{h\to0}$ f (a – h)

= $\lim_{h\to0}$ [f(a – h) – f(a)] + f(a)

= $\lim_{h\to0}$ (–h) $\left[\lim_{h\to0}\frac{f(a-h)-(a)}{-h}\right]$ + f(a)

= 0 × [Lf′(a)] + f(a) = f(a)

R.H.L. = $\lim_{h\to0}$ f(a + h)

= $\lim_{h\to0}$ [f(a + h) – f(a)] + f(a)

= $\lim_{h\to0}$ h $\left[\lim_{h\to0}\frac{f(a+h)-(a)}{h}\right]$ + f(a) = 0 × [Rf′(a)] + f (a) = f(a)

Hence the function is continuous at x = a

Note : that the converse of this theorem is not always true. If a function is continuous at a point, if may or may not be differentiable at that point.

Example : 5

Discuss the continuity and differentiability of f(x) at x = 0 if f(x) = $\begin{cases} x^2\sin\frac{1}{x} & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$

Let us check the differentiability first. Lf′(0) = $\lim_{h\to0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to0}\frac{(-h)^2\sin\left(\frac{1}{-h}\right)-0}{-h}$

= $\lim_{h\to0}$ h sin $\frac{1}{h}$ = $\lim_{h\to0}$ h × $\lim_{h\to0}$ sin $\frac{1}{h}$

= 0 × (number between – 1 and + 1) = 0

Rf′(0) = $\lim_{h\to0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to0}\frac{h^2\sin\frac{1}{h}-0}{h} = \lim_{h\to0}$ h sin $\frac{1}{h}$ = $\lim_{h\to0}$ h × $\lim_{h\to0}$ sin $\frac{1}{h}$

= 0 × (number between – 1 and + 1) = 0

Hence Lf′(0) = Rf′(0) = 0

⇒ function is differentiable at x = 0

⇒ if must be continuous also at the same point.

Example : 6

Show that the function f(x) is continuous at x = 0 but its derivative does not exists at x = 0 if

f(x) = $\begin{cases} x\sin(\log x^2) & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$

Solution

Test for continuity :

LHL = $\lim_{h\to0}$ f(0 – h) = $\lim_{h\to0}$ (–h) sin log $(-h)^2$ = – $\lim_{h\to0}$ h sin log h^2

as h → 0, log h^2 → – ∞

Hence sin log h^2 oscillates between – 1 and + 1

⇒ LHL = – $\lim_{h\to0}$ (h) × $\lim_{h\to0}$ (sin log h^2) = – 0 × (number between – 1 and + 1) = 0

R.H.L. = $\lim_{h\to0}$ f(0 + h) = $\lim_{h\to0}$ = $\lim_{h\to0}$ h sin logh^2

$= \lim_{h\to 0} h \,.\, \lim_{h\to 0} \sin \log h^2 = 0 \times$ (oscillating between – 1 and + 1) = 0

f(0) = 0 (Given)

$\Rightarrow$ LHL = RHL = f(0)

Hence f(x) is continuous at x = 0

Test for differentiability :

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{-h\sin\log(-h)^2 - 0}{-h} = \lim_{h\to 0} \sin(\log h^2)$$

As the expression oscillates between – 1 and + 1, the limit does not exists.

$\Rightarrow$ Left hand derivative is not defined.

Hence the function is not differentiable at x = 0

Note : As LHD is undefined there is no need to check RHD for differentiability as for differentiability both LHD and RHD should be defined and equal

Example : 7

Discuss the continuity of f, f′ and f′′ on [0, 2] if $f(x) = \begin{cases} \dfrac{x^2}{2} & ; \ 0 \le x < 1 \\ 2x^2 - 3x + \dfrac{3}{2} & ; \ 1 \le x \le 2 \end{cases}$

Solution

Continuity of f(x)

For $x \ne 1$, f(x) is a polynomial and hence is continuous

At x = 1, LHL = $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^2}{2} = \frac{1}{2}$

RHL = $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \left(2x^2 - 3x + \frac{3}{2}\right) = 2 - 3 + \frac{3}{2} = \frac{1}{2}$

$f(1) = 2(1)^2 - 3(1) + \frac{3}{2} = \frac{1}{2}$

$\Rightarrow$ LHL = RHL = f(1)

Therefore, f(x) is continuous at x = 1

Continuity of f′(x)

Let g(x) = f′(x)

$\Rightarrow$ $g(x) = \begin{cases} x & ; \ 0 \le x < 1 \\ 4x - 3 & ; \ 1 \le x \le 2 \end{cases}$

For $x \ne 1$, g(x) is linear polynomial and hence continuous.

At x = 1, LHL = $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} x = 1$

RHL = $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} (4x - 3) = 1$

g(1) = 4 – 3 = 1

$\Rightarrow$ LHL = RHL = g(1)

$\therefore$ g(x) = f′(x) is continuous at x = 1

Continuity of f′′(x)

Let h(x) = f′′(x) = $\begin{cases} 1 & ; \ 0 \le x < 1 \\ 4 & ; \ 1 \le x \le 2 \end{cases}$

For $x \ne 1$, h(x) is continuous because it is a constant function.

At x = 1, LHL = $\lim_{x\to 1^-} h(x) = 1$

RHL = $\lim_{x\to 1^+} h(x) = 4$

Thus LHL ≠ RHL

∴ h(x) is discontinuous at x = 1

Hence f(x) and f′(x) are continuous on [0, 2] but f′′(x) is discontinuous at x = 1.

Note : Continuity of f′(x) is same as differentiability of f(x)

Example : 8

Show that $\lim_{x\to a} \frac{f(x)g(a)-g(x)f(a)}{x-a}$ = f′(a) g(a) – g′(a) f(a) if f(x) and g(x) are differentiable at x = a.

Solution

$$\lim_{x\to a} \frac{f(x)g(a)-g(x)f(a)}{x-a} = \lim_{x\to a} \frac{f(x)g(x)-f(a)g(a)+f(a)g(a)-g(x)f(a)}{x-a}$$

$$= \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\right] g(a) - \lim_{x\to a}\left[\frac{g(x)-g(a)}{x-a}\right] f(a)$$

= f′(a) g(a) – g′ (a) f(a)

Example : 9

Let f(x) be defined in the interval [–2, 2] such that f(x) = $\begin{cases} -1 & ; \ -2 \le x \le 0 \\ x-1 & ; \ 0 < x \le 2 \end{cases}$ and g(x) = f(|x|) + |f(x)|. Test the differentiability of g(x) in (–2, 2).

Solution

Consider f(|x|)

The given interval is – 2 ≤ x ≤ 2

Replace x by |x| to get :

– 2 ≤ |x| ≤ 2 ⇒ 0 ≤ |x| ≤ 2

Hence f(|x|) can be obtained by substituting |x| in place of x in x – 1 [see definition of f(x)].

⇒ f(|x|) = |x| – 1 ; – 2 ≤ x ≤ 2 ....(i)

Consider |f(x)|

Now |f(x)| = $\begin{cases} |-1| & ; \ -2 \le x \le 0 \\ |x-1| & ; \ 0 < x \le 2 \end{cases}$ ⇒ |f(x)| = $\begin{cases} 1 & ; \ -2 \le x \le 0 \\ |x-1| & ; \ 0 < x \le 2 \end{cases}$

adding (i) and (ii)

$$f(|x|) + |f(x)| = \begin{cases} |x|-1+1 & ; \ -2 \le x \le 0 \\ |x|-1+|x-1| & ; \ 0 < x \le 2 \end{cases}$$

$$\Rightarrow \quad g(x) = \begin{cases} |x| & ; \ -2 \le x \le 0 \\ |x|-1+|x-1| & ; \ 0 < x \le 2 \end{cases}$$

on further simplification,

$$g(x) = \begin{cases} -x & ; \ -2 \le x \le 0 \\ x-1+1-x & ; \ 0 < x < 1 \\ x-1+x-1 & ; \ 1 \le x \le 2 \end{cases} \qquad g(x) = \begin{cases} -x & ; \ -2 \le x \le 0 \\ 0 & ; \ 0 < x < 1 \\ 2x-2 & ; \ 1 \le x \le 2 \end{cases}$$

For x ≠ 0 and x ≠ 1, g(x) is a differentiable function because it is a linear polynomial

At x = 0

$$Lg'(0) = \lim_{h\to 0} \frac{g(0-h)-g(0)}{-h} = \lim_{h\to 0} \frac{-(-h)-0}{-h} = -1$$

$$Rg'(0) = \lim_{h\to 0} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

⇒ Lg′ (0) ≠ Rg′ (0). Therefore g(x) is not differentiable at x = 0

At x = 1

$$Lg'(1) = \lim_{h\to 0} \frac{g(1-h)-g(1)}{-h} = \lim_{h\to 0} \frac{0-0}{-h} = 0$$

$Rg'(1) = \lim_{h\to 0} \dfrac{g(1+h)-g(1)}{h} = \lim_{h\to 0} \dfrac{2(1+h)-2-0}{h} = 2$

$\Rightarrow$ $Lg'(1) \neq Rg'(1)$. Therefore g(x) in not differential at x = 1

Hence g(x) is not differentiable at x = 0, 1 in (–2, 2)

Example : 10

Find the derivative of y = log x wrt x from first principles.

Solution

Let f(x) = log x

Using definition of derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h\to 0} \frac{\log(x+h)-\log x}{h} = \lim_{h\to 0} \frac{\log\left(1+\frac{h}{x}\right)}{h} = \lim_{h\to 0} \frac{\log\left(1+\frac{h}{x}\right)}{h/x}\,\frac{1}{x} = \frac{1}{x}$$

$$\left[\text{using } \lim_{t\to 0} \frac{\log(1+t)}{1} = 1\right]$$

Example : 11

Evaluate the derivative f (x) = x^n wrt x from definition of derivative. Hence find the derivative of $\sqrt{x}$, 1/x, $1/\sqrt{x}$, $1/x^p$ wrt x.

Solution

Using definition of derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{(x+h)-x} = \lim_{t\to x} \frac{t^n - x^n}{t-x} \qquad \text{(putting } t = x + h)$$

$$= nx^{n-1} \left[\text{using } \lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}\right]$$

Taking n = $\frac{1}{2}$, $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

taking n = – 1, $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$

taking n = $\frac{-1}{2}$, $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{-1}{2x\sqrt{x}}$

taking n = –p, $\frac{d}{dx}\left(\frac{1}{x^p}\right) = \frac{-p}{x^{p+1}}$

Example : 12

Find the derivative of sin x wrt x from first principles.

Solution

Let f(x) = sin x

Using the definition of derivative,

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0} \frac{2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{2\ \frac{h}{2}} = \cos x \ .\ \lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} = \cos x$$

$$\left(\text{using} \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right)$$

Hence f′(x) = cos x

Example : 13

Differentiate a^x wrt x from first principles

Solution

Let f(x) = a^x

Using the definition of derivatives $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$\Rightarrow \qquad f'(x) = \lim_{h\to 0} \frac{a^{x+h}-a^x}{h} = a^x \ .\ \lim_{h\to 0} \frac{a^h-1}{h} = a^x \log a \qquad \left(\text{using} \lim_{t\to 0} \frac{a^t-1}{t} = \log a\right)$$

Hence f′(x) = a^x log a

Example : 14

Differentiate sin (log x) wrt x from first principles

Solution

Let f(x) = sin (log x)

Using the definition of derivatives

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\sin\log(x+h)-\sin\log x}{h} = \lim_{h\to 0} \frac{2\cos\left(\frac{\log(x+h)+\log x}{2}\right)\sin\left(\frac{\log(x+h)-\log x}{2}\right)}{h}$$

$$= \lim_{h\to 0} 2\cos\left(\frac{\log(x+h)+\log x}{2}\right) \times \lim_{h\to 0}\left(\frac{\sin\left(\frac{\log(x+h)-\log x}{2}\right)}{h}\right)$$

$$= 2\cos\log x \lim_{h\to 0} \frac{\sin\left(\frac{\log(x+h)-\log x}{2}\right)}{\frac{\log(x+h)-\log x}{2}} \times \lim_{h\to 0} \frac{\log(x+h)-\log x}{2h}$$

$$= 2\cos\log x \ .\ 1 \ .\ \lim_{h\to 0} \frac{\log(1+h/x)}{2h} \qquad \left[\because \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right]$$

$$= \cos\log x \ .\ \lim_{h\to 0} \frac{\log(1+h/x)}{h/x} \ .\ \frac{1}{x} = \frac{\cos\log x}{x} \qquad \left[\because \lim_{t\to 0} \frac{\log(1+t)}{t} = 1\right]$$

Example : 15

Differentiate $x^2 \tan x$ wrt x from first principles

Solution

Let $f(x) = x^2 \tan x$

Using the definition of derivative,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 \tan(x+h) - x^2 \tan x}{h} = \lim_{h\to 0} \frac{x^2 \tan(x+h) - x^2 \tan x + (h^2+2hx)\tan(x+h)}{h}$$

$$= x^2 \lim_{h\to 0} \frac{\tan(x+h)-\tan x}{h} + \lim_{h\to 0} \frac{h(h+2)\tan(x+h)}{h}$$

$$= x^2 \lim_{h\to 0} \frac{\sin(x+h-x)}{h\cos x\cos(x+h)} + \lim_{h\to 0} (h + 2x) \tan (x + h)$$

$$= \frac{x^2}{\cos^2 x} + 2x \tan x \qquad \left[\text{using } \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\right]$$

Example : 16

Differentiate $\sin^{-1}x$ from first principles

Solution

Let $y = \sin^{-1}x \quad \Rightarrow \quad x = \sin y$

From first principles

$$\frac{dx}{dy} = \lim_{h\to 0} \frac{f(y+h)-f(y)}{h} \quad \Rightarrow \quad \frac{dx}{dy} = \lim_{h\to 0} \frac{\sin(y+h)-\sin y}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\left(\frac{2y+h}{2}\right)\sin\frac{h}{2}}{h} = \frac{dx}{dy} = \cos y \lim_{h\to 0} \frac{\sin\frac{h}{2}}{h/2} = \cos y$$

As (dy/dx) × (dx/dy) = 1, we get

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\pm\sqrt{1-\sin^2 y}} = \frac{1}{\pm\sqrt{1-x^2}} \qquad (\because x = \sin y)$$

But the principal value of y $\sin^{-1}x$ lies between $-\pi/2$ and $\pi/2$ and for these values of y, cos y is positive.
($\because$ cosine of an angle in the first or fourth quadrant is positive)

Therefore rejecting the negative sign, we have $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

Example : 17

Differentiate $\sqrt{\tan\sqrt{x}}$ from first principles.

Solution

Let $f(x) = \sqrt{\tan\sqrt{x}}$

From first principles,

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{\tan\sqrt{x+h}} - \sqrt{\tan\sqrt{x}}}{h}$$

Rationalise to get,

$$f'(x) = \lim_{h\to 0} \frac{\tan\sqrt{x+h} - \tan\sqrt{x}}{h\left(\sqrt{\tan\sqrt{x+h}} + \sqrt{\tan\sqrt{x}}\right)} \quad \Rightarrow \quad f'(x) = \frac{1}{2\sqrt{\tan\sqrt{x}}} \lim_{h\to 0} \frac{\sin\left(\sqrt{x+h}-\sqrt{x}\right)}{h\cos\sqrt{x+h}\cos\sqrt{x}}$$

$$\Rightarrow \quad f'(x) = \frac{1}{2\sqrt{\tan\sqrt{x}}\cos^2\sqrt{x}} \times \lim_{h\to 0} \frac{\sin\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}-\sqrt{x}\right)}{\left(\sqrt{x+h}-\sqrt{x}\right)h}$$

$$\Rightarrow \quad f'(x) = \frac{1}{2\sqrt{\tan\sqrt{x}}\cos^2\sqrt{x}} \times \lim_{h\to 0} \frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \qquad \left(\text{using } \lim_{t\to 0}\frac{\sin t}{t}=1\right)$$

$$\Rightarrow \quad f'(x) = \frac{1}{2\sqrt{\tan\sqrt{x}}\cos^2\sqrt{x}} \ \frac{1}{2\sqrt{x}}$$

Example : 18

If y = f ($\sin^2$x) and f′(x) = $\frac{1+x}{1-x}$, then show that $\frac{dy}{dx}$ = 2 tan x (1 + $\sin^2$x)

Solution

Let u $\sin^2$x

Using chain rule : $\frac{dy}{dx}$ = f′(u) $\frac{du}{dx}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1+u}{1-u}\ \frac{d}{dx}\ (\sin^2 x) = \frac{1+\sin^2 x}{1-\sin^2 x}\ (2\sin x\cos x) = 2\tan x\,(1+\sin^2 x)$$

Example : 19

A function f : R → R satisfy the equation f(x + y) = f(x) f(y) for all x, y in R and f(x) ≠ 0 for any x in R. Let the function be differentiable at x = 0 and f′(0) = 2. Show that f′(x) = 2f(x) for all x in R. Hence determine f(x).

Solution

In f(x + y) = f(x) f(y) substitute y = 0

⇒ f(x + 0) = f(x) f(0)

⇒ f(x) = f(x) f(0)

⇒ f(0) = 1 $\quad (\because f(x) \neq 0)$ ....(i)

Consider f′(0) = $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$

$$\Rightarrow \quad 2 = \lim_{h\to 0}\frac{f(h)-1}{h} \qquad \text{............(ii)}$$

Consider f′(x) = $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h}$ = f(x) $\lim_{h\to 0}\frac{f(h)-1}{h}$

= f(x) (2) [using (2)]

⇒ f′(x) = 2f(x)

$$\Rightarrow \quad \frac{f'(x)}{f(x)} = 2$$

$$\Rightarrow \quad \frac{d}{dx}[\log f(x)] = \frac{d}{dx}(2x)$$

⇒ log f(x) = 2x ⇒ f(x) = e^{2x}

Example : 20

(Logarithmic differentiation) Find dy/dx for the functions.

(i) $y = \left(1+\frac{1}{x}\right)^x + x^{1+\frac{1}{x}}$ (ii) $y = \frac{(2x+1)^3\sqrt{1-x^2}}{(3x-2)^2\ 2^x}$ (iii) $y = \log_x (\log x)$

Solution

(i) Let $u = \left(1+\frac{1}{x}\right)^x$ and $v = x^{1+\frac{1}{x}}$

$\Rightarrow$ $y = u + v$

$\Rightarrow$ $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ....(i)

(ii) $y = \frac{(2x+1)^3\sqrt{1-x^2}}{(3x-2)^2\ 2^x}$

Now $u = \left(1+\frac{1}{x}\right)^x$

$\Rightarrow$ $\log u = x \log\left(1+\frac{1}{x}\right) = x \log (x + 1) - x \log x$

$\Rightarrow$ $\frac{1}{u}\frac{du}{dx} = \frac{x}{x+1} + \log (x + 1) - \left(\frac{x}{x} + \log x\right)$

$\Rightarrow$ $\frac{du}{dx} = u\left(\log\frac{x+1}{x} - \frac{1}{x+1}\right)$ ....(ii)

consider $v = x^{1+\frac{1}{x}}$

$\Rightarrow$ $\log v = \left(1+\frac{1}{x}\right) \log x$

$\Rightarrow$ $\frac{1}{v}\frac{dv}{dx} = \left(1+\frac{1}{x}\right)\frac{1}{x} + \log x\left(-\frac{1}{x^2}\right)$

$\Rightarrow$ $\frac{dv}{dx} = \frac{v}{x^2}(x + 1 - \log x)$ ....(iii)

Substituting from (ii) and (iii) into (i)

$$\frac{dy}{dx} = \left(1+\frac{1}{x}\right)^x\left(\log\frac{x+1}{x} - \frac{1}{x+1}\right) + \frac{x^{1+\frac{1}{x}}}{x^2} \times (x + 1 - \log x)$$

(ii) Taking log on both sides :

$\log y = 3 \log (2x + 1) + 1/2 \log (1 - x^2) - 2 \log (3x - 2) - x \log 2$

Differentiating with respect to x, $\frac{1}{y}\frac{dy}{dx} = \frac{3(2)}{2x+1} + \frac{-2x}{2(1-x^2)} = \frac{2(3)}{3x-2} - \log 2$

$$\frac{dy}{dx} = \frac{(2x+1)^3\sqrt{1-x^2}}{(3x-2)^2 2^x} \times \left[\frac{6}{2x+1} - \frac{x}{1-x^2} - \frac{6}{3x-2} - \log 2\right]$$

(iii) $y = \log_x (\log x)$

$$y = \frac{\log \log x}{\log x} \qquad \left(\text{using } \log_a b = \frac{\log_m b}{\log_m a}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\log x\left(\frac{1}{\log x}\frac{1}{x}\right) - (\log\log x)\frac{1}{x}}{(\log x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x(\log x)^2}\ (1 - \log \log x)$$

Example : 21

(Implicit function) Find the expression for $\frac{dy}{dx}$ for the following implicit function.

(a) $x^{\sin y} = y^{\sin x}$ (b) $x^3 + y^3 - 3xy = 1$

Solution

(a) $x^{\sin y} = y^{\sin x}$

$\Rightarrow$ $\sin y \log x = \sin x \log y$

Differentiating with respect to x :

$$\sin y\ \frac{1}{x} + \log x \cos y\ \frac{dy}{dx} = \sin x\ \frac{1}{y}\ \frac{dy}{dx} + \log y \cos x$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\log x \cos y - \frac{\sin x}{y}\right) = \cos x \log y - \frac{\sin y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{xy \cos x \log y - y \sin y}{xy \log x \cos y - x \sin x} \qquad \frac{1+x^2}{1-x^2}$$

(b) $x^3 + y^3 - 3xy = 1$

Differentiating with respect to x;

$$3x^2 + 3y^2\ \frac{dy}{dx} - 3\left[x\frac{dy}{dx} + y, 1\right] = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

Example : 22

(Inverse circular functions) Find $\frac{dy}{dx}$ if

(1) $y = \tan^{-1}\left(\frac{a\cos - b\sin x}{b\cos x + a\sin x}\right)$ (2) $y = \tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)$

(2) $y = \tan^{-1}\frac{4x}{1+5x^2} + \tan^{-1}\frac{2+3x}{3-2x}$ (4) $y = \sin^{-1}\frac{2x}{1+x^2} + \sec^{-1}$

Solution

(1) $y = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right) \Rightarrow y = \tan^{-1}\left(\frac{a/b - \tan x}{1 + a/b \tan x}\right)$

$= \tan^{-1} (a/b) - \tan^{-1} \tan x = \tan^{-1} (a/b) - x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}\frac{a}{b} - x\right) = 0 - 1 = -1$$

(2) $y = \tan^{-1}\left(\dfrac{x}{1+\sqrt{1-x^2}}\right)$

Substitute $x = \sin\theta$ ....(i)

$y = \tan^{-1}\left(\dfrac{\sin\theta}{1+\sqrt{1-\sin^2\theta}}\right)$

$y = \tan^{-1}\left(\dfrac{\sin\theta}{1+\cos\theta}\right)$

$y = \tan^{-1}\left(\dfrac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}\right)$

$y = \tan^{-1}\tan\theta/2 = \dfrac{\theta}{2}$

using (i), $y = \dfrac{1}{2}\sin^{-1}x$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{1-x^2}}$

(3) $y = \tan^{-1}\dfrac{4x}{1+5x^2} + \tan^{-1}\dfrac{2+3x}{3-2x}$

$y = \tan^{-1}\dfrac{5x-x}{1+5x\,x} + \tan^{-1}\left(\dfrac{\frac{2}{3}+x}{1-\frac{2}{3}x}\right)$ $\dfrac{1+x^2}{1-x^2}$

$y = \tan^{-1}5x - \tan^{-1}x + \tan^{-1}2/3 + \tan^{-1}x$

$y = \tan^{-1}5x + \tan^{-1}2/3$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{5}{1+25x^2}$

(4) $y = \sin^{-1}\dfrac{2x}{1+x^2} + \sec^{-1}$

Substitute $x = \tan\theta$ ....(i)

$y = \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right) + \sec^{-1}\left(\dfrac{1+\tan^2\theta}{1-\tan^2\theta}\right)$

$y = \sin^{-1}\sin 2\theta + \cos^{-1}\cos 2\theta$

$y = 2\theta + 2\theta$

$y = 4\theta = 4\tan^{-1}x$ (using (i))

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{4}{1+x^2}$

Example : 23

Show that = 1 if $y = \cos^{-1}\left(\frac{\cos x + 4\sin x}{\sqrt{17}}\right)$

Solution

We can write

$$\cos x + 4 \sin x = \sqrt{17}\left[\frac{1}{\sqrt{17}}\cos x + \frac{4}{\sqrt{17}}\sin x\right] = \sqrt{17}\ \cos (x - \tan^{-1}4)$$

$$\text{Hence } y = \cos^{-1}\left(\frac{\sqrt{17}\cos(x - \tan^{-1}4)}{\sqrt{17}}\right) \Rightarrow y = x - \tan^{-1} 4$$

$$\Rightarrow \frac{dy}{dx} = 1$$

Example : 24

If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x - y)$, Show that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$

Solution

Substitute x = sin α and y = sin β ....(i)

$\Rightarrow \sqrt{1-\sin^2\alpha} + \sqrt{1-\sin^2\beta} = a (\sin \alpha - \sin \beta)$

$\Rightarrow \cos \alpha + \cos \beta = a (\sin \alpha - \sin \beta)$

$$\Rightarrow \frac{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\frac{\alpha-\beta}{2}}{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)} = a \qquad \frac{dy}{dx}$$

$$\Rightarrow \cot\left(\frac{\alpha-\beta}{2}\right) = a$$

$\Rightarrow \alpha - \beta = 2 \cot^{-1} a$

$\Rightarrow \sin^{-1} x - \sin^{-1} y = 2 \cot^{-1} a$, [using (i)]

differentiating with respect to x.

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$$

Example : 25

If x = a (cos t + log tan t/2), y = a sin t find d^2y/dx^2 at t = $\pi/4$

Solution

$$\frac{dy}{dt} = a \cos t$$

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan t/2}\frac{\sec^2 t/2}{2}\right) = a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$\Rightarrow \frac{dx}{dt} = \frac{a\cos^2 t}{\sin t}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{a\cos^2 t/\sin t} \quad \Rightarrow \quad \frac{dy}{dx} = \tan t$$

$$\text{Now} \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}(dy/dx)}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx^2} = \frac{\sec^2 t}{a\cos^2 t/\sin t} = \frac{\sin t}{a\cos^4 t}$$

$$\left.\frac{d^2y}{dx^2}\right]_{t=\pi/4} = \frac{\sin \pi/4}{a\cos^4 \pi/4} = \frac{2\sqrt{2}}{a}$$

Example : 25

If x = a (cos t + log tan t/2), y = a sin t find d^2y/dx^2 at t = π/4.

Solution

$$\frac{dy}{dt} = a \cos t$$

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan t/2}\frac{\sec^2 t/2}{2}\right) = a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{a\cos^2 t}{\sin t}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{a\cos^2 t/\sin t} \quad \Rightarrow \quad \frac{dy}{dx} = \tan t$$

$$\text{Now} \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}(dy/dx)}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx^2} = \frac{\sec^2 t}{a\cos^2 t/\sin t} = \frac{\sin t}{a\cos^4 t}$$

$$\left.\frac{d^2y}{dx^2}\right]_{t=\pi/4} = \frac{\sin \pi/4}{a\cos^4 \pi/4} = \frac{2\sqrt{2}}{a}$$

Example : 26

If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, then shown that $\frac{dy}{dx} = \frac{-1}{(1+x)^2}$

Solution

$x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring, we get :

$x^2 (1 + y) = y^2 (1 + x)$

$\Rightarrow \quad x^2 + x^2y - y^2 - xy^2 = 0$

$\Rightarrow \quad (x^2 - y^2) + xy (x - y) = 0$

$\Rightarrow \quad (x - y) (x + y + xy) = 0$

$\Rightarrow \quad y = x$ or $x + y + xy = 0$

Since y = x does not satisfy the give function, we reject it.

$\therefore \quad x + y + xy = 0$

$\Rightarrow \quad y = \frac{-x}{1+x}$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{(1+x)-x,1}{(1+x)^2} = \frac{-1}{(1+x)^2}$

Example : 27

If $y = \frac{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}$, then show that $\frac{dy}{dx} = -\frac{2a^2}{x^3}\left(1+\frac{a^2}{\sqrt{a^4-x^4}}\right)$

Solution

On rationalising the denominator, we get :

$$y = \frac{\left(\sqrt{a^2+x^2}+\sqrt{a^2-x^2}\right)^2}{2x^2}$$

$$y = \frac{2a^2+2\sqrt{a^4-x^4}}{2x^2}$$

$$y = \frac{a^2}{x^2} + \frac{\sqrt{a^4-x^4}}{x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2a^2}{x^3} + \frac{x^2\frac{-4x^3}{2\sqrt{a^4-x^4}} - \sqrt{a^4-x^4}(2x)}{x^4}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2a^2}{x^3} + \frac{-2x^4-2(a^4-x^4)}{x^3\sqrt{a^4-x^4}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2a^2}{x^3}\left[1+\frac{a^2}{\sqrt{a^4-x^4}}\right]$$

Example : 28

If $y = a^{x^{a^{x^{\cdots\cdots\infty}}}}$, then find $\frac{dy}{dx}$

Solution

$y = a^{x^{a^{x^{\cdots\cdots\infty}}}}$ can be written as $y = a^{x^y}$

$\Rightarrow \quad \log y = x^y \log a$

$\Rightarrow \quad \log \log y = y \log x + \log \log a$

differentiating with respect to x;

$$\frac{1}{\log y}\,\frac{1}{y}\,\frac{dy}{dx} = y \,.\, \frac{1}{x} + \log x\,\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{1}{y\log y} - \log x\right) = \frac{y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2 \log y}{x(1-y\log x \log y)}$$

Example : 29

If $y = \cos^{-1}\dfrac{a+b\cos x}{b+a\cos x}$, b > a, then show that $\dfrac{dy}{dx} = \dfrac{\sqrt{b^2-a^2}}{b+a\cos x}$

Solution

Differentiating y with respect to x

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{a+b\cos x}{b+a\cos x}\right)^2}} \times \frac{(b+a\cos x)(-b\sin x)-(a+b\cos x)(-a\sin x)}{(b+a\cos x)^2}$$

$$= \frac{-(b+a\cos x)}{\sqrt{(b^2-a^2)-(b^2-a^2)\cos^2 x}} \ \frac{-b^2\sin x+a^2\sin x}{(b+a\cos x)^2} = \frac{(b^2-a^2)\sin x}{\sqrt{b^2-a^2}\sqrt{1-\cos^2 x}(b+a\cos x)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{b^2-a^2}}{b+a\cos x}$$

Example : 30

If sin y = x sin (a + y), then show that :

(i) $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$ (ii) $\dfrac{dy}{dx} = \dfrac{\sin a}{1+x^2-2x\cos x}$

Solution

(i) As $\dfrac{dy}{dx}$ should not contain x, we write $\dfrac{\sin y}{\sin(a+y)}$ = x and differentiating with respect to x;

$$\left[\frac{\sin(a+y)\cos y-\sin y\cos(a+y)}{\sin^2(a+y)}\right]\frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{\sin(a+y-y)}{\sin^2(a+y)}\frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

(ii) As $\dfrac{dy}{dx}$ should not contain y, we try to express y explicitly in terms of x.

sin y = x (sin a cos y + cos a sin y)

$$\Rightarrow \quad \tan y = \frac{x\sin a}{1-x\cos a} \qquad \Rightarrow \quad y = \tan^{-1}\left(\frac{x\sin a}{1-x\cos a}\right)$$

Now differentiate with respect x;

$$\frac{dy}{dx} = \frac{1}{1+\frac{x^2\sin^2 a}{(1-x\cos a)^2}} \ \frac{(1-x\cos a)\sin a - x\sin a(-\cos a)}{(1-x\cos a)^2} = \frac{\sin a}{(1-x\cos a)^2+x^2\sin^2 a}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin a}{1+x^2-2x\cos a}$$

Example : 31

If $y = e^{mx}(ax + b)$, where a, b, m are constants, show that : $\frac{d^2y}{dx^2} - 2m\frac{dy}{dx} + m^2y = 0$

Solution

$y = e^{mx}(ax + b)$ ....(i)

$\frac{dy}{dx} = (a)\, e^{mx} + m\,(ax + b)\, e^{mx}$

using (i), $\frac{dy}{dx} = a\, e^{mx} + my$ ....(ii)

Again differentiating with respect to x;

$\frac{d^2y}{dx^2} = ame^{mx} + m\frac{dy}{dx}$

Substituting for a e^{mx} from (ii), we get

$\frac{d^2y}{dx^2} = m\left(\frac{dy}{dx} - my\right) + m\frac{dy}{dx}$

$\Rightarrow \quad \frac{d^2y}{dx^2} - 2m\frac{dy}{dx} + m^2y = 0$

Example : 32

If $y = x\log\frac{x}{a+bx}$, the show that : $x^3\frac{d^2y}{ax^2} = \left(x\frac{dy}{dx} - y\right)^2$

Solution

$y = x\log x - x\log(a + bx)$ ....(i)

$\Rightarrow \quad \frac{dy}{dx} = x\,\frac{1}{x} + \log x - \frac{xb}{a+bx} - \log(a + bx)$

$\Rightarrow \quad \frac{dy}{dx} = [\log x - \log(a + bx)] + \frac{a}{a+bx}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} + \frac{a}{a+bx}$ [using (i)]

$x\frac{dy}{dx} - y = \frac{ax}{a+bx}$ ....(ii)

Again differentiating with respect to x, we get ;

$\left(x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right) - \frac{dy}{dx} = \frac{(a+bx)a - ax.b}{(a+bx)^2}$

$\Rightarrow \quad x\frac{d^2y}{dx^2} = \frac{a^2}{(a+bx)^2}$

$\Rightarrow \quad x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2}$

$\Rightarrow \quad x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$ [using (ii) in R,H.S)

Example : 33

If x = sin t and y = cos pt, show that : $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$

Solution

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-p\sin pt}{\cos t}$$

As the equation to be derived does no contain t, we eliminate t using expressions for x and y.

$$\frac{dy}{dx} = \frac{-p\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \sqrt{1-x^2}\,\frac{dy}{dx} = -p\sqrt{1-y^2}$$

As the equation to be derived does not contain any square root, we square and then differentiate.

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = p^2(1-y^2)$$

$$(1-x^2)\,2\frac{dy}{dx}\frac{d^2y}{dx^2} + (-2x)\left(\frac{dy}{dx}\right)^2 = p^2\left(-2y\frac{dy}{dx}\right)$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -p^2y$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$$

Example : 34

If $x = at^3$, $y = bt^2$ (t a parameter), find

(i) $\frac{d^3y}{dx^3}$ (ii) $\frac{d^3x}{dy^3}$

Solution

(i) $x = at^3 \Rightarrow \frac{dx}{dt} = 3at^2$

$y = bt^2 \Rightarrow \frac{dy}{dt} = 2bt$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2bt}{3at^2} = \frac{2b}{3at}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{2b}{3a}\frac{d}{dx}\left(\frac{1}{t}\right) = \frac{2b}{3a}, \frac{-1}{t^2}\cdot\frac{dt}{dx} = \frac{-2b}{3at^2}\cdot\frac{1}{3at^2} = \frac{-2b}{9a^2t^4}$$

Again differentiating both sides w.r.t. x,

$$\frac{d^3y}{dx^3} = \frac{d}{dt}\left(\frac{d^2y}{dx^2}\right)\frac{dt}{dx} = -\frac{2b}{9a^2}\frac{d}{dt}\left(\frac{1}{t^4}\right)\frac{dt}{dx} = \frac{-2b}{9a^2}\cdot\frac{-4}{t^5}\cdot\frac{1}{3at^2} = \frac{8b}{27a^3t^7}$$

(ii) $x = at^3$, $y = bt^2$

$\frac{dx}{dt} = 3a^2$; $\frac{dy}{dt} = 2bt$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{dx/dt}{dy/dt} = \frac{3at^2}{2bt} = \frac{3at}{2b}$$

$$\Rightarrow \quad \frac{d^2x}{dy^2} = \frac{3a/2b}{dy/dt} = \frac{3a}{2b} \cdot \frac{1}{2bt} = \frac{3a}{4b^2t}$$

$$\Rightarrow \quad \frac{d^3x}{dy^3} = \frac{d}{dy}\left(\frac{3a}{4b^2t}\right) = \frac{3a}{4b^2} \cdot \frac{d}{dt}\left(\frac{1}{t}\right) \cdot \frac{1}{dy/dt} = \frac{3a}{4b^2}\left(-\frac{1}{t^2}\right)\left(\frac{1}{2bt}\right) = \frac{-3a}{8b^3t^3}$$

Example : 35

If y = f(x), express $\frac{d^2x}{dy^2}$ in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Solution

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \qquad \left(\frac{dy}{dx} \neq 0\right)$$

$$\Rightarrow \quad \frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{1}{dy/dx}\right) = \frac{d}{dx}\left(\frac{1}{dy/dx}\right) \cdot \frac{dx}{dy} = -\frac{1}{\left(\frac{dy}{dx}\right)^2} \frac{d}{dx}\left(\frac{dy}{dx}\right) \cdot \frac{dx}{dy}$$

$$= -\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3} \qquad \because \quad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

Example : 36

Change the independent variable x to θ in the equation $\frac{d^2y}{dx^2} + \frac{2x}{1+x^2} \cdot \frac{dy}{dx} + \frac{y}{(1+x^2)^2} = 0$, by means of the transformation x = tan θ

Solution

$x = \tan\theta \quad \Rightarrow \quad \frac{dx}{d\theta} = \sec^2\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \cos^2\theta \cdot \frac{dy}{d\theta} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = -2\cos\theta \cdot \sin\theta \cdot \frac{d\theta}{dx} \cdot \frac{dy}{d\theta} + \cos^2\theta \cdot \frac{d^2y}{d\theta^2} \cdot \frac{d\theta}{dx}$

$= -2\cos\theta \cdot \sin\theta \cdot \cos^2\theta \cdot \frac{dy}{d\theta} + \cos^2\theta \cdot \frac{d^2y}{d\theta^2} \cdot \cos^2\theta$

$= -2\sin\theta \cdot \cos^3\theta \cdot \frac{dy}{d\theta} + \cos^4\theta \cdot \frac{d^2y}{d\theta^2}$

Putting the values of x, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation, $\frac{d^2y}{dx^2} + \frac{2x}{1+x^2} \frac{dy}{dx} + \frac{y}{(1+x^2)^2} = 0$

we get $-2\sin\theta\cos^3\theta \frac{dy}{d\theta} + \cos^4\theta \frac{d^2y}{d\theta^2} + \frac{2\tan\theta}{1+\tan^2\theta} \cos^2\theta \frac{dy}{d\theta} + \frac{y}{(1+\tan^2\theta)^2} = 0$

$-2\sin\theta\cos^3\theta \frac{dy}{d\theta} + \cos^4\theta \frac{d^2y}{d\theta^2} + 2\sin\theta\cos^3\theta \frac{dy}{d\theta} + y\cos^4\theta = 0$

i.e. $\quad \frac{d^2y}{d\theta^2} + y = 0$

Example : 37

Differentiate x^x (x > 0) from first principles.

Solution

Let $f(x) = x^x = e^{x \ell nx}$

From first principles,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{e^{(x+h)\,\ell n(x+h)} - e^{x\ell nx}}{h} = \lim_{h\to 0} \frac{e^{x\ell nx}\left[e^{(x+h)\ell n(x+h)-x\ell nx}-1\right]}{h}$$

$$= \lim_{h\to 0} \frac{e^{x\ell n\,x}\left[e^{(x+h)\ell n(x+h)-x\ell nx}-1\right]}{(x+h)\ell n\,(x+h)-x\ell nx} \cdot \lim_{h\to 0} \frac{(x+h)\ell n(x+h)-x\ell nx}{h}$$

$$= e^{x\ell n\,x} \cdot \lim_{h\to 0} \frac{(x+h)\ell n(x+h)-x\ell n(x+h)+x\ell n(x+h)-x\ell nx}{h} \qquad \left[\text{using}: \lim_{h\to 0}\frac{e^h-1}{h}=1\right]$$

$$= e^{x\ell n\,x} \cdot \left[\lim_{h\to 0}\frac{\ell n(x+h)[x+h-x]}{h} + \lim_{h\to 0}\frac{x\ell n(x+h)-\ell nx}{h}\right]$$

$$= e^{x\ell n\,x} \cdot \left[\lim_{h\to 0}\ell n(x+h) + \lim_{h\to 0}\frac{x\left[\ell n\left(\frac{x+h}{x}\right)\right]}{h}\right] = e^{x\ell n\,x} \cdot \left[\ell n\,x + \lim_{h\to 0}\left[\ell n\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right]\right]$$

$$= ex\,\ell n\,x \,.\, (\ell n\,x + 1) \qquad \left[\text{using}: \lim_{t\to 0}\ell n\,(1+t)^{\frac{1}{t}} = \ell n\,e = 1\right]$$

$\Rightarrow \qquad f'(x) = x^x\,(1 + \ell n\,x)$

Example : 38

If $y = \log_u |\cos 4x| + |\sin x|$, where $u = \sec 2x$, find dy/dx at $x = -\pi/6$

Solution

In the sufficiently closed neighbourhood of $-\pi/6$ both cos 4x and sin x are negative. So for differentiating y, we can take $|\cos 4x| = -\cos 4x$ and $|\sin x| = -\sin x$.

Thus

$y = \log_u (-\cos 4x) + (-\sin x) = \log_{\sec 2x} (-\cos 4x) + (-\sin x)$

$$= \frac{\log(-\cos 4x)}{\log\sec 2x} - \sin x$$

On differentiating wit respect to x, we get

$$\frac{dy}{dx} = \frac{\frac{(4\sin 4x)x\log\sec 2x}{-\cos 4x} - \log(-\cos 4x)\frac{\sec 2x\times\tan 2x}{\sec 2x}\times 2}{(\log\sec 2x)^2} - \cos x$$

$$= \frac{-4\tan 4x\times\log\sec 2x - 2\tan 2x\times\log(-\cos 4x(}{(\log\sec 2x)^2} - \cos x$$

Taking derivative at $x = -\pi/6$, we get

$$\left[\frac{dy}{dx}\right]_{x=-\pi/6} = \frac{-4\tan(-2\pi/3)\times\log\sec\pi/3 - 2\tan(-\pi/3)\times\log(-\cos(-2\pi/3))}{(\log 2)^2} - \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} - \frac{6\sqrt{3}}{\log 2}$$

Example : 39

Test the differentiability of the following function at x = 0.

$$f(x) = \begin{cases} e^{-1/x^2} \sin\left(\frac{1}{x}\right) & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$$

Solution

Checking differentiability at x = 0

Right hand derivative = $\lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{e^{\frac{-1}{(0+h)^2}} \sin\left(\frac{1}{0+h}\right) - 0}{h}$

$$= \lim_{h\to 0} \frac{\sin\left(\frac{1}{h}\right)}{he^{\frac{1}{h^2}}} = \lim_{h\to 0} \frac{\sin\left(\frac{1}{h}\right)}{h\left(1+\frac{1}{h^2}+\frac{1}{h^4 2!}+....\right)}$$

$$= \lim_{h\to 0} \frac{\sin\left(\frac{1}{h}\right)}{\left(h+\frac{1}{h}+\frac{1}{h^3 2!}+....\right)} = \frac{\text{a finite quantity}}{0+\infty} = 0$$

(because sin (1/h) is finite and oscillates between – 1 to + 1).

Left Hand Derivative = $\lim\limits_{h\to 0} \dfrac{f(0-h)-f(0)}{-h} = \lim\limits_{h\to 0} \dfrac{e^{\frac{-1}{(\theta-h)^2}} \sin\left(\frac{1}{\theta-h}\right) - 0}{-h}$

$$= \lim_{h\to 0} \frac{\sin\left(\frac{1}{h}\right)}{he^{\frac{1}{h^2}}} = \lim_{h\to 0} \frac{\sin\left(\frac{1}{h}\right)}{h\left(1+\frac{1}{h^2}+\frac{1}{h^4 2!}+....\right)} = \frac{\text{a finite quantity}}{0+\infty} = 0$$

(because sin (1/h) is finite and oscillates between – 1 to + 1)

As Left Hand Derivative = Right Hand Derivative, the function f(x) is differentiable at x = 0.

Example : 40

The function f is defined by y = f(x) where x = 2t – |t|, y = t^2 + t|t|, t ∈ R. Draw the graph of f(x) for the interval – 1 ≤ x ≤ 1. Also discuss the continuity and differentiability at x = 0.

Solution

It is given that : x = x = 2t – |t| and y = t^2 + t|t|.

Consider t ≥ 0 x = 2t – t = t ....(i)

and y = t^2 + t × t = $2t^2$ ....(ii)

Eliminating t from (i) and (ii), we get y = $2x^2$

So y = $2x^2$ for x > 0 (because t ≥ 0 ⇒ x ≥ 0)

Consider t < 0 x = 2t + t = 3t ....(iii)

and y = t^2 + t × (–t) = 0 ....(iv)

Eliminating t from (iii) and (iv), we get y = 0

So y = 0 for x < 0 (because t < 0 ⇒ x < 0)

In the closed interval – 1 ≤ x ≤ 1, the function f(x) is :

$$f(x) = \begin{cases} 2x^2 & ; \quad x \geq 0 \\ 0 & ; \quad x < 0 \end{cases}$$

Checking differentiability at x = 0

$$\text{LHD} = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{0-0}{-h} = 0$$

$$\text{RHD} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{2(0+h)^2-0}{h} = \lim_{h\to 0} 2h = 0$$

As LHD = RHD, f(x) is continuous and differentiable at x = 0 (because if function is differential, it must be continuous)

Example : 41

If $x < 1$, prove that : $\frac{1-2x}{1-x+x^2} + \frac{2x-4x^3}{1-x^2+x^4} + \frac{4x^3-8x^7}{1-x^4+x^8} + \ldots\ldots \infty = \frac{1+2x}{1+x+x^2}$

Solution

$(1 + x + x^2)(1 - x + x^2) = (1 + x^2)^2 - x^2 = 1 + x^2 + x^4$

$(1 + x + x^2)(1 - x + x^2)(1 - x^2 + x^4) = (1 + x^2 + x^4)(1 - x^2 + x^4) = (1 + x^4)^2 - x^4 = 1 + x^4 + x^8$

Continuing the same way, we can obtain :

$$(1 + x + x^2)(1 - x + x^2)(1 - x^2 + x^4) \ldots\ldots\ldots\ldots \left(1-x^{2^{n-1}}+x^{2^n}\right) = 1+x^{2^n} +$$

Taking limit $n \to \infty$, we get

$$(1 + x + x^2)(1 - x + x^2)(1 - x^2 + x^4) \ldots\ldots\ldots\ldots = 1 \qquad (\because x < 1)$$

Take log of both sides to get

$$\log(1 + x + x^2) + \log(1 - x + x^2) + \log(1 - x^2 + x^4) + \ldots\ldots\ldots\ldots = 0$$

Differentiate both sides with respect to x :

$$\frac{1+2x}{1+x+x^2} + \frac{-1+2x}{1-x+x^2} + \frac{-2x+4x^3}{1-x^2+x^4} + \ldots\ldots = 0$$

$$\Rightarrow \quad \frac{1-2x}{1-x+x^2} + \frac{2x-4x^3}{1-x^2+x^4} + \ldots\ldots = \infty \ \frac{1+2x}{1+x+x^2}$$

Hence proved

Example : 42

Find the derivative with respect to x of the function:

$$(\log_{\cos x} \sin x)(\log_{\sin x} \cos x)^{-1} + \sin^{-1}\left(\frac{2x}{1+x^2}\right) \text{ at } x = \pi/4.$$

Solution

Let $y = (\log_{\cos x} \sin x)(\log_{\sin x} \cos x)^{-1} + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, $u = (\log_{\cos x} \sin x)(\log_{\sin x} \cos x)^{-1}$ and $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$\Rightarrow \quad y = u + v \qquad \ldots\ldots\ldots\ldots(i)$

consider u

$u = (\log_{\cos x} \sin x)(\log_{\sin x} \cos x)^{-1} = (\log_{\cos x} \sin x)(\log_{\cos x} \sin x) = (\log_{\cos x} \sin x)^2$

$$\frac{du}{dx} = \frac{d}{dx}(\log_{\cos x} \sin x)^2 = \frac{d}{dx}\left(\frac{\log_e \sin x}{\log_e \cos x}\right)^2 = 2\left(\frac{\log_e \sin x}{\log_e \cos x}\right)\frac{d}{dx}\left(\frac{\log_e \sin x}{\log_e \cos x}\right)$$

$$= 2(\log_{\cos x} \sin x) \times \left(\frac{\log_e \cos x \dfrac{\cos x}{\sin x} - \log_e \sin x \dfrac{-\sin x}{\cos x}}{(\log_e \cos x)^2}\right)$$

$$= 2(\log_{\cos x} \sin x) \times \left(\frac{\cot x \log_e \cos x + \tan x \log_e \sin x}{(\log_e \cos x)^2}\right)$$

consider y

$$v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

put x = tan θ ⇒ v = $\sin^{-1}$ (sin 2θ) = 2θ = 2 $\tan^{-1}x$

[for –π/2 ≤ 2θ ≤ π/2 ⇒ –π/4 ≤ θ ≤ π/4 ≤ θ ≤ π/4 ⇒ – 1 ≤ x ≤ 1

⇒ we can use this definition for x = π/4]

$$\Rightarrow \quad \frac{dv}{dx} = 2\frac{d}{dx}\tan^{-1}x = \frac{2}{1+x^2}$$

Differentiating (i) with respect to x at x = π/4, we get

$$= \left[\frac{du}{dx}\right]_{x=\pi/4} + \left[\frac{dv}{dx}\right]_{x=\pi/4}$$

On substituting the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$, we get

$$\left[\frac{dy}{dx}\right]_{x=\pi/4} = 2\log_{1/\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)\left[\frac{1\times\log\frac{1}{\sqrt{2}}+1\times\log\frac{1}{\sqrt{2}}}{\left(\log\frac{1}{\sqrt{2}}\right)}\right] + \frac{2}{1+\left(\frac{\pi}{4}\right)^4} = \frac{-8}{\log 2} + \frac{32}{16+\pi^2}$$

Example : 43

If y = e^{-xz} ($x\sqrt{z}$) and $z^4 + x^2z = x^5$, find dy/dx in terms of x and z.

Solution

Consider $z^4 + x^2z – x^5$

Differentiating with respect to x, we get :

$$\left[\frac{dy}{dx}\right]_{x=\pi/4}$$

$$4z^3\frac{dz}{dx} + x^2\frac{dz}{dx} + 2xz = 5x^4$$

$$\Rightarrow \quad \frac{dz}{dx} = \frac{5x^4-2xz}{4z^3+x^2} \quad \text{..........(i)}$$

Consider y = $e^{-xz}\sec^{-1}(x\sqrt{z})$

Differentiating with respect to x, we get :

$$\frac{dy}{dx} = e^{-xz}\frac{1}{|x|\sqrt{z}\left(\sqrt{x^2z-1}\right)}\frac{d}{dx}(x\sqrt{z}) + \sec^{-1}x\sqrt{z} \times e^{-xz}\frac{d}{dx}(-xz)$$

$$= e^{-xz}\frac{1}{|x|\sqrt{z}\left(\sqrt{x^2z-1}\right)}\left(\sqrt{z}+x\frac{1}{2\sqrt{z}}\frac{dz}{dx}\right) + e^{-xz}\sec^{-1}x\sqrt{z}\left(-z-x\frac{dz}{dx}\right)$$

On substituting the value of dz/dx from (i), we get

$$= e^{-xz}\left(\frac{1}{|x|\sqrt{x^2z-1}} + \frac{x}{2|x|z\sqrt{x^2z-1}}\frac{x(5x^3-2z)}{4z^3+x^2} - z\sec^{-1}x\sqrt{z} - \sec^{-1}x\sqrt{z}\frac{x^2(5x^3-2z}{4z^3+x^2}\right)$$

Example : 44

Find f′(x) if $f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$

Solution

$$f'(x) = \begin{vmatrix} \frac{d}{dx}(x) & \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ \frac{d}{dx}(1) & \frac{d}{dx}(2x) & \frac{d}{dx}(3x^2) \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ \frac{d}{dx}(0) & \frac{d}{dx}(2) & \frac{d}{dx}(6x) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2x & 3x^2 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 0 & 2 & 6x \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix}$$

$$= 0 + 0 + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix} = 6\,(2x^2 - x^2) = 6x^2$$

Example : 45

Differentiate $y = \cos^{-1}\frac{1-x^2}{1+x^2}$ with respect to $z = \tan^{-1}x$. Also discuss the differentiability of this function.

Solution

The given function is $y = \cos^{-1}\frac{1-x^2}{1+x^2}$

Substitute $x = \tan\theta$

$\Rightarrow \quad y = \cos^{-1}\frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos^{-1}(\cos 2\theta)$

$\Rightarrow \quad y = 2\theta = 2\tan^{-1}x \quad$ for $\quad 0 \le 2\theta \le \pi$

$\Rightarrow \quad 0 \le \theta \le \pi/2$

$\Rightarrow \quad 0 \le x < \infty$

and $\quad y = -2\theta = -2\tan^{-1}x \quad$ for $\quad -\pi < 2\theta < 0$

$\Rightarrow \quad -\pi/2 < \theta < 0$

$\Rightarrow \quad -\infty < x < 0$

So the given function reduces to :

$$y = \begin{cases} 2\tan^{-1}x & , \quad x \ge 0 \\ -2\tan^{-1}x & , \quad x < 0 \end{cases}$$

Differentiating with respect to $\tan^{-1}x$, we get

$$\frac{dy}{d(\tan^{-1}x)} = \begin{cases} 2 & x \ge 0 \\ -2 & x < 0 \end{cases}$$

Alternate Method

$$y = \cos^{-1}\frac{1-x^2}{1+x^2}$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}} \; \frac{(1+x^2)(-2x)-2x(1-x^2)}{(1+x^2)^2} = \frac{4x}{\sqrt{4x^2}} \; \frac{1}{1+x^2}$$

In the question, $z = \tan^{-1} x$. On differentiating with respect to x, we get

$$= \frac{1}{1+x^2}$$

On applying chain rule,

$$= \frac{dy/dx}{dz/dx} = \frac{4x}{\sqrt{4x^2}} = \frac{4x}{|2x|} = \begin{cases} 2 & x \geq 0 \\ -2 & x < 0 \end{cases}$$

Differentiability x = 0

LHD = –2 and RHD = 2

As LHD $\neq$ RHD, f(x) is not differentiable at x = 0

Example : 46

Find dy/dx at x= – 1 when $\sin y^{(\sin(\pi x/2)} + \frac{\sqrt{3}}{2} \sec^{-1}(2x) + 2^x \tan[\ell n(x+2)] = 0$

Solution

The given curve is : $\sin y^{\sin(\pi x/2)} + \frac{\sqrt{3}}{2} \sec^{-1}(2x) + 2^x \tan[\ell n(x+2)] = 0$

Let $A = \sin y^{\sin(px/2)}$; $B = \frac{\sqrt{3}}{2} \sec^{-1}(2x)$ and $C = 2^x \tan[\ell n(x+2)]$

$\Rightarrow$ A + B + C = 0 ....(i)

Consider A

Taking log and then differentiating A w.r.t. x, we get

$$\frac{1}{A}\,\frac{dA}{dx} = \left[\frac{\pi}{2}\cos\frac{\pi x}{2}\,\ell n(\sin y) + \sin\frac{\pi x}{2}\cot y\frac{dy}{dx}\right]$$

At x = –1

$$\left[\frac{dA}{dx}\right]_{x=-1} = (\sin y)^{-1}\left[0+(-1)\frac{\cos y}{\sin y}\left(\frac{dy}{dx}\right)_{x=-1}\right] = -\frac{\cos y}{\sin^2 y}\left(\frac{dy}{dx}\right)_{x=-1}$$

Consider B

$$B = \frac{\sqrt{3}}{2}\sec^{-1} 2x$$

Differentiating with respect to x, we get $\frac{dB}{dx} = \frac{\sqrt{3}}{2|x|\sqrt{4x^2-1}}$

At x = – 1 $\left[\frac{dB}{dx}\right]_{x=-1} = \frac{1}{2}$

Consider C

$C = 2^x \tan[\ell n(x+2)]$

Differentiating with respect to x, we get

$$\frac{dC}{dx} = 2^x\,\frac{\sec^2[\ell n(x+2)]}{x+2} + 2^x\,\ell n\,2\tan[\ell n(x+2)]$$

At x = – 1 $\left[\frac{dC}{dx}\right]_{x=-1} = \frac{1}{2}$

Differentiate (i) to get :

$$\frac{dA}{dx} + \frac{dB}{dx} + \frac{dC}{dx} = 0$$

On substituting the values of dA/dx, dB/dx and dC/dx at x = – 1, we get

$$\left[\frac{dy}{dx}\right]_{x=-1} = \frac{\sin^2 y}{\cos y} = \frac{\sin^2 y}{\pm\sqrt{1-\sin^2 y}} \qquad \text{...........(ii)}$$

Finding the value of sin y

Consider the given curve and put x = –1 in it to get

$$(\sin y)^{-1} + \frac{\sqrt{3}}{2}\sec^{-1}(-2) = 0$$

$\Rightarrow$ $\sin y = -\dfrac{2}{\sqrt{3}\sec^{-1}(-2)} = -\dfrac{\sqrt{3}}{\pi}$ [using $\sec^{-1}(-2) = \cos^{-1}(-1/2) = p - \cos^{-1}(1/2) = 2\pi/3$]

Substituting the value of sin y in (2), we get :

$$\left[\frac{dy}{dx}\right]_{x=-1} = \pm\frac{\left(\frac{-\sqrt{3}}{\pi}\right)^2}{\sqrt{1-\left(\frac{-\sqrt{3}}{\pi}\right)^2}} = \pm\frac{3}{\pi\sqrt{\pi^2-3}}$$

Example : 47

If g is the inverse function of f and $f'(x) = \dfrac{1}{1+x^n}$, prove that $g'(x) = 1 + [g(x)]^n$.

Solution

$\dfrac{1}{f'[(g(x)]}$

As g is inverse function of f(x), we can take : $g(x) = f^{-1}(x)$

$\Rightarrow$ $f[g(x)] = x$

Differentiating with respect to x, we get : $f'[g(x)]\, g'(x) = 1$

$$\Rightarrow \quad g'(x) = \qquad = \frac{1}{\frac{1}{1+[g(x)]^n}}$$

$\Rightarrow$ $g'(x) = 1 + [g(x)]^n$

Example : 48

If $y = 1 + \dfrac{z_1}{x-c_1} + \dfrac{c_2x}{(x-c_1)(x-c_2)} + \dfrac{c_3x^2}{(x-c_1)(x-c_2)(x-c_3)}$, then

Show that $\dfrac{dy}{dx} = \dfrac{y}{x}\left[\dfrac{c_1}{c_1-x} + \dfrac{c_2}{c_2-x} + \dfrac{c_3}{c_3-x}\right]$

Solution

$$y = \frac{x}{x-c_1} + \frac{c_2x}{(x-c_1)(x-c_2)} + \frac{c_3x^2}{(x-c_1)(x-c_2)(x-c_3)}$$

$$\Rightarrow \quad y = \frac{x(x-c_2)+c_2x}{(x-c_1)(x-c_2)} + \frac{c_3x^2}{(x-c_1)(x-c_2)(x-c_3)}$$

$$\Rightarrow \quad y = \frac{x^2}{(x-c_1)(x-c_2)} + \frac{c_3x^2}{(x-c_1)(x-c_2)(x-c_3)}$$

$\Rightarrow \quad y = \dfrac{x^3}{(x-c_1)(x-c_2)(x-c_3)}$

Take log on both sides and then differentiate to get

$\log y = 3 \log x - \log (x - c_1) - \log (x - c_2) - \log (x - c_3)$

$$\Rightarrow \quad \frac{dy}{dx} = y\left[\frac{3}{x} - \frac{1}{x-c_1} - \frac{1}{x-c_2} - \frac{1}{x-c_3}\right] = \frac{y}{x}\left[\left(1-\frac{x}{x-c_1}\right)+\left(1-\frac{x}{x-c_2}\right)+\left(1-\frac{x}{x-c_3}\right)\right]$$

$$= \frac{y}{x}\left[\frac{c_1}{c_1-x}+\frac{c_2}{c_2-x}+\frac{c_3}{c_3-x}\right]$$

Example : 49

If $p^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$, the prove that $p + \dfrac{d^2p}{d\theta^2} = \dfrac{a^2b^2}{p^3}$

Solution

$p^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$

$\Rightarrow \quad 2p^2 = a^2 + b^2 + (a^2 - b^2) \cos 2\theta$

$\Rightarrow \quad 2p^2 - a^2 - b^2 = (a^2 - b^2) \cos 2\theta$ ....(ii)

Also $2pp_1 = a^2 (-\sin 2\theta) + b^2 (\sin 2\theta)$ (by taking $p_1 = dp/d\theta$)

$\Rightarrow \quad 2pp_1 = (b^2 - a^2) \sin 2\theta$ ....(ii)

Square (i) and (ii) and add,

$\Rightarrow \quad [2p^2 - (a^2 + b^2)]^2 + 4p^2 p_1^2 (a^2 - b^2)^2$

$\Rightarrow \quad 4p^4 + (a^2 + b^2)^2 - (a^2 - b^2) + 4p^2 p_1^2 = 4p^2 (a^2 + b^2)$

$\Rightarrow \quad p^4 + a^2b^2 + p^2 \; p_1^2 = p^2 (a^2 + b^2)$

$\Rightarrow \quad p^2 + \dfrac{a^2b^2}{p^2} + p_1^2 = a^2 + b^2$

On differentiating w.r.t. θ, we get

$pp_1 - \dfrac{2a^2b^2}{p^3} \; p_1 + 2p_1 p_2 = 0$ (by taking $p_2 = d^2p/d\theta^2$)

$\Rightarrow \quad p + p_2 = \dfrac{a^2b^2}{p^3}$

Example : 50

If $y^{1/m} + y^{-1/m} = 2x$, then prove that $(x^2 - 1) \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} - m^2y = 0$

Solution

$y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$ ....(i)

Using $\left(y^{\frac{1}{m}} + y^{-\frac{1}{m}}\right)^2 - \left(y^{\frac{1}{m}} + y^{-\frac{1}{m}}\right)^2 = 4$

we get $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2\sqrt{x^2-1}$ ....(ii)

Adding (i) and (ii), we get $y^{\frac{1}{m}} = x + \sqrt{x^2-1}$ $\Rightarrow$ $y = \left(x+\sqrt{x^2-1}\right)^m$ ....(iii)

Differentiating wrt, we get

$$y' = m\left(x+\sqrt{x^2-1}\right)^{m-1}\left(1+\frac{x}{\sqrt{x^2-1}}\right)$$

$\Rightarrow$ $$y'\sqrt{x^2-1} = m\left(x+\sqrt{x^2-1}\right)^{m}$$

On squaring above and using (iii), we get $(x^2 - 1)\, y'^2 = m^2y^2$

Differentiate again to get

$$2xy'^2 + (x^2 - 1)\, 2y'y'' = 2m^2\, y\, y'$$

$\Rightarrow$ $(x^2 - 1)\, y'' + xy' = m^2\, y$

Hence proved

Example : 51

Evaluate $\lim\limits_{x\to a}\dfrac{x^n-a^x}{x^x-a^a}$ using LH rule $\left(\dfrac{0}{0}\text{type of indeterminate form}\right)$

Solution

Let L = $\lim\limits_{x\to a}\dfrac{x^a-a^x}{x^x-a^a}$ ....(i)

Note that the expression assumes $\dfrac{0}{0}$ type of indeterminate form at x = a.

As the expression satisfies all the conditions of LH rule, we can evaluate this limit by using LH rule.

Apply LH rule on (i) to get :

$$L = \lim_{x\to a}\frac{ax^{n-1}-a^x\,.\log a}{x^x(1+\log x)-0}$$

$\Rightarrow$ $$L = \frac{a^a-a^a\log a}{a^a(1+\log a)} = \frac{1-\log a}{1+\log a} = \frac{\log\left(\frac{c}{a}\right)}{\log(ae)}$$

Example : 52

Evaluate $\lim\limits_{x\to a}\left(2-\dfrac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$ using LH rule (1^∞ type of indeterminate form)

Solution

Let y = $\lim\limits_{x\to a}\left(2-\dfrac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$ (1^∞ form)

Taking log of both sides, we get :

$$\log y = \lim_{x\to a}\tan\left(\frac{\pi x}{2a}\right)\log\left(2-\frac{a}{x}\right) \qquad (\infty \times 0 \text{ form})$$

$$= \lim_{x\to a}\frac{\log\left(2-\frac{a}{x}\right)}{\cot\left(\frac{\pi x}{2a}\right)} \qquad \left(\frac{0}{0}\text{form}\right)$$

Applying LH rule, we get

$$L = \lim_{x\to a} \frac{\left(\frac{a}{x^2}\right)}{\left(2-\frac{a}{x}\right)\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)\left(\frac{\pi}{2a}\right)} = \frac{\frac{1}{a}}{(-1)\operatorname{cosec}^2\left(\frac{\pi}{2}\right)\left(\frac{\pi}{2a}\right)} = -\frac{2}{\pi}$$

$\therefore \qquad y = e^{-2/\pi}$

Example : 54

Evaluate $\lim\limits_{x\to 0} \dfrac{\sin 3x^2}{\ell n\cos(2x^2-1)}$ using LH rule $\qquad \left(\frac{0}{0}\text{type of indeterminate form}\right)$

Solution

Let $L = \lim\limits_{x\to 0} \dfrac{\sin 3x^2}{\ell n\cos(2x^2-x)}$ (0/0 form)

Apply LH rule to get :

$$L = \lim_{x\to 0} \frac{-6x\cos 3x^2\cos(2x^2-x)}{(4x-1)\sin(2x^2-x)} = -6\lim_{x\to 0}\frac{3x^2\cos(2x^2-x)}{4x-1}\lim_{x\to 0}\frac{x}{(2x^2-x)}$$

The limit of the first factor is computed directly, the limit of the second one, which represents an indeterminate form of the type $\frac{0}{0}$ is found with the aid of the L′Hospital's rule. Again consider,

$$L = -6\lim_{x\to 0}\frac{\cos 3x^2\cos(2x^2-x)}{4x-1}\lim_{x\to 0}\frac{x}{\sin(2x^2-x)}$$

$$\Rightarrow \qquad L = -6 \cdot \frac{1.1}{-1}\lim_{x\to 0}\frac{1}{(4x-1)\cos(2x^2-x)}$$

$$\Rightarrow \qquad L = -6\,\frac{1}{-1.1} = -6$$

Example : 55

Evaluate $\lim\limits_{x\to\infty} \dfrac{\log_a x}{x^k}$ (k > 0) using LH rule $\qquad \left(\frac{\infty}{\infty}\text{type of indeterminate form}\right)$

Solution

Let $L = \lim\limits_{x\to\infty} \dfrac{\log_a x}{x^k}$ (∞/∞ form)

Apply LH rule to get :

$$L = \lim_{x\to+\infty}\frac{\frac{1}{x}\log_a e}{kx^{k-1}}$$

$$\Rightarrow \qquad L = \log_a e \lim_{x\to+\infty}\frac{1}{kx^k} = 0$$

Example : 56

Evaluate $\lim_{x\to1}\left(\frac{1}{\ell nx}-\frac{1}{x-1}\right)$ using LH rule $\left(\frac{\infty}{\infty}\text{type of indeterminate form}\right)$

Solution

Let $L = \lim_{x\to1}\left(\frac{1}{\ell nx}-\frac{1}{x-1}\right)$ $(\infty - \infty$ form)

Let us reduce it to an indeterminate form of the type $\frac{0}{0}$

$$L = \lim_{x\to1}\frac{x-1-\ell nx}{(x-1)\ell nx} \qquad (0/0 \text{ form})$$

Apply LH rule to get :

$$L = \lim_{x\to1}\frac{1-1/x}{\ell nx+1-1/x}$$

$$\Rightarrow \quad L = \lim_{x\to1}\frac{x-1}{x\ell nx+x-1}$$

Apply LH rule again

$$\Rightarrow \quad L = \lim_{x\to1}\frac{1}{\ell n\,x+2} = \frac{1}{2}$$

Example : 57 $(\infty^0$ type of indeterminate form)

Evaluate $\lim_{x\to0}\left[\ell n\left(+1\sin^2 x\right)\cot\ell n^2(1+x)\right]$ using LH ule.

Solution

Let $L = \lim_{x\to0}\left[\ell n\left(+1\sin^2 x\right)\cot\ell n^2(1+x)\right]$

We have an indeterminate form of the type 0 . ∞. Let us reduce it to an indeterminate form of the type $\frac{0}{0}$.

$$\Rightarrow \quad L = \lim_{x\to0}\frac{\ell n(1+\sin^2 x)}{\tan\ell n^2(1+x)} \qquad (0/0 \text{ form})$$

Apply LH rule to get :

$$L = \lim_{x\to0}\frac{\frac{1}{1+\sin^2 x}\sin 2x}{2\sec^2[\ell n^2(1+x)]\ell n(1+x)\,.\,\frac{1}{1+x}}$$

Simplify to get :

$$L = \lim_{x\to0}\frac{\sin x}{\ell n(1+x)}$$

Apply LH rule again to get :

$$L = \lim_{x\to0}\frac{\sin x}{\ell n(1+x)} = \lim_{x\to0}\frac{\cos x}{\frac{1}{1+x}} = 1$$

Example : 58

Evaluate : $\lim_{x\to 0} (1/x)^{\sin x}$ using LH rule. (∞^0 type of indeterminate form)

Solution

We have an indeterminate form of the type ∞^0

Let $y = (1x)^{\sin x}$;

Taking log on both sides, we get :

$$\ell n\, y = \sin x\, \ell n\,(1/x)$$

$\Rightarrow \quad \lim_{x\to +0} \ell n\, y = \lim_{x\to +0} \sin x\, \ell n\,(1/x)$ (0, ∞ form)

Let us transform it to $\frac{\infty}{\infty}$ to apply LH rule.

$$\lim_{x\to +0} \ell n\, y = \lim_{x\to +0} \frac{-\ell n x}{1/\sin x}$$

Apply LH rule to get :

$$\lim_{x\to +0} \ell n\, y = \lim_{x\to 0} \frac{-1/x}{-(\cos x)/\sin^2 x} = \lim_{x\to 0} \frac{\sin^2 x}{x \cos x} = 0$$

$$\Rightarrow \quad \lim_{x\to +0} y = e^0 = 1$$

Example : 59

Find the values of a, b, c so that $\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2$

Solution

Let $L = \lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x}$ ....(i)

Here as $x \to 0$, denominator approaches 0. So for L to be finite, the numerator must tend to 0.

$a - b + c = 0$ ....(ii)

Apply LH rule on (i) to get :

$$L = \lim_{x\to 0} \frac{ae^x + b\sin x - ce^{-x}}{\sin x + x\cos x}$$

Here as $x \to 0$, the denominator tends to 0 and numerator tends to a – c. For L to be finite,

$a - c = 0$ ....(iii)

Apply LH rule again on L to get :

$$L = \lim_{x\to 0} \frac{ae^x + b\cos x + ce^{-x}}{2\cos x - x\sin x}$$

$\Rightarrow \quad \frac{a+b+c}{2} = 2 \quad \Rightarrow \quad a + b + c = 4$ ....(iv)

Solving equations (ii), (iii) and (iv), we get a = 1, b = 2, c = 1

Example : 60

Evaluate : $\lim_{x\to 0} \frac{1-\cos x}{x\log(1+x)}$

Solution

Let L = $\lim_{x\to 0} \frac{1-\cos x}{x\log(1+x)}$ (0/0 form)

Using the expansions of cos x and log (1 + x), we get :

$$L = \lim_{x\to 0} \frac{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-.........\right)}{x\left(x-\frac{x^2}{2}+\frac{x^3}{3}-..........\right)} = \lim_{x\to 0} \frac{\frac{x^2}{2!}-\frac{x^4}{4!}+..........}{x^2-\frac{x^3}{2}+\frac{x^4}{3}-........}$$

Dividing both numerator and denominator by x^2 , we get :

$$L = \lim_{x\to 0} \frac{\frac{1}{2!}-\frac{x^2}{4!}+.......}{1-\frac{x}{2}+\frac{x^2}{3}-........} = \frac{\frac{1}{2}+0-0+........}{1+0-0+.......} = \frac{1}{2}$$

Example : 61

Evaluate : $\lim_{x\to 0} \frac{e^x \sin x - x - x^2}{x^2}$

Solution

Let L = $\lim_{x\to 0} \frac{e^x \sin x - x - x^2}{x^2}$ (0/0 form)

Using the expansions of sin x and e^x , we get :

$$L = \lim_{x\to 0} \frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+.....\right)\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}.......\right)-x-x^2}{x^3}$$

$$= \lim_{x\to 0} \frac{x+x^2+\left(\frac{1}{2!}-\frac{1}{3!}\right)x^3+\left(\frac{1}{3!}-\frac{1}{3!}\right)x^4+\left(\frac{1}{4!}-\frac{1}{2!\,.\,3!}+\frac{1}{5!}\right)x^5+.....-x-x^2}{x^3}$$

$$= \lim_{x\to 0} \frac{\frac{1}{3}x^3-\frac{1}{30}x^5+..........}{x^3} = \lim_{x\to 0} \left(\frac{1}{3}-\frac{1}{30}x^2+.......\right) = \frac{1}{2}$$

Example : 1

Find all the tangents to the curve y = cos (x + y), $-2\pi \le x \le 2\pi$ that are parallel to the line x + 2y = 0.

Solution

Slope of tangent (x) = slope of line = – 1/2

$\Rightarrow \quad \frac{dy}{dx} = \frac{-1}{2}$

Differentiating the given equation with respect to x,

$\Rightarrow \quad \frac{dy}{dx} = -\sin(x+y)\left(1+\frac{dy}{dx}\right) = \frac{-\sin(x+y)}{1+\sin(x+y)} = \frac{-1}{2}$

$\Rightarrow \quad 2\sin(x+y) = 1 + \sin(x+y)$

$\Rightarrow \quad \sin(x+y) = 1$

$\Rightarrow \quad x + y = n\pi + (-1)^n \pi/2,\ n \in I$ in the given interval, we have $x + y = \frac{-3\pi}{2}, \frac{\pi}{2}$

(because $-(2\pi + 1) \le x + y \le 2\pi + 1$)

Substituting the value of (x + y) in the given curve i.e. y = cos (x + y), we get

y = 0 and $x = \frac{-3\pi}{2}, \frac{\pi}{2}$

Hence the points of contact are $\left(\frac{-3\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, 0\right)$ and the slope is $\left(\frac{-1}{2}\right)$

$\Rightarrow$ Equations of tangents are $y - 0 = \frac{-1}{2}\left(x + \frac{3\pi}{2}\right)$ and $y - 0 = \frac{-1}{2}\left(x - \frac{\pi}{2}\right)$

$\Rightarrow \quad 2x + 4y + 3\pi = 0$ and $2x + 4y - \pi = 0$

Example : 2

Find the equation of the tangent to $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$ at the point (x_0, y_0)

Solution

Differentiating wrt x,

$\Rightarrow \quad \frac{mx^{m-1}}{a^m} + \frac{my^{m-1}}{b^m}\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{b^m}{a^m}\left(\frac{x}{y}\right)^{m-1}$

$\Rightarrow$ at the given point (x_0, y_0), slope of tangent is $\left.\frac{dy}{dx}\right]_{(x_0, y_0)} = -\left(\frac{b}{a}\right)^m \left(\frac{x_0}{y_0}\right)^{m-1}$

$\Rightarrow$ the equation of tangent is $y - y_0 = -\left(\frac{b}{a}\right)^m \left(\frac{x_0}{y_0}\right)^{m-1}(x - x_0)$

$a^m yy_0^{m-1} - a^m y_0^m = -b^m xx_0^{m-1} + b^m x_0^m$

$a^m yy_0^{m-1} + b^m x x_0^{m-1} = a^m y_0^m + b^m x_0^m$

using the equation of given curve, the right side can be replaced by $a^m b^m$

$\therefore \quad a^m yy_0^{m-1} + b^m x x_0^{m-1} = a^m b^m$

$\Rightarrow$ the equation of tangent is

$\frac{x}{a}\left(\frac{x_0}{a}\right)^{m-1} = \frac{y}{b}\left(\frac{y_0}{b}\right)^{m-1} = 1$

Note : The result of this example can be very useful and you must try remember it

Example : 3

Find the equation of tangent to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ at $(x_0 - y_0)$. Hence prove that the length of the portion of tangent intercepted between the axes is constant.

Solution

Method 1 :

$$\frac{2}{3} x^{\frac{-1}{3}} + \frac{2}{3} y^{\frac{-1}{3}} \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right]_{(x_0, y_0)} = -\left(\frac{y_0}{x_0}\right)^{\frac{1}{3}}$$

$$\Rightarrow \quad \text{equation is } y - y_0 = -\left(\frac{y_0}{x_0}\right)^{\frac{1}{3}} (x - x_0)$$

$\Rightarrow \quad x_0^{1/3} y - y_0 x_0^{1/3} = - xy_0^{1/3} + x_0 y_0^{1/3}$

$\Rightarrow \quad x y_0^{1/3} + yx_0^{1/3} = x_0 y_0^{1/3} + y_0 x_0^{1/3}$

$$\Rightarrow \quad \frac{xy_0^{1/3}}{x_0^{1/3}y_0^{1/3}} + \frac{yx_0^{1/3}}{x_0^{1/3}y_0^{1/3}} = x_0^{2/3} + y_0^{2/3}$$

$$\Rightarrow \quad \text{equation of tangent is : } \frac{x}{x_0^{1/3}} + \frac{y}{y_0^{1/3}} = a^{2/3}$$

Length intercepted between the axes :

$$\text{length} = \sqrt{(x\ \text{intercept})^2 + (y\ \text{intercept})^2}$$

$$= \sqrt{\left(x_0^{1/3}a^{2/3}\right)^2 + \left(y_0^{1/3}a^{2/3}\right)^2}$$

$$= \sqrt{x_0^{2/3}a^{4/3} + y_0^{2/3}a^{4/3}}$$

$$= a^{2/3} \sqrt{x_0^{2/3} + y_0^{2/3}}$$

= a i.e. constant

Method 2

Express the equation in parametric form

$x = a \sin^2 t, \qquad y = a \cos^3 t$

Equation of tangent is :

$$y - a \cos^3 t = \frac{-3a\cos^2 t \sin t}{3a\sin^2 t \cos t} (x - a \sin^3 t)$$

$\Rightarrow \quad y \sin - a \sin t \cos^3 t = - x \cos t - a \sin^3 t \cos t$

$\Rightarrow \quad x \cos t + y \sin t = a \sin t \cos t$

$$\Rightarrow \quad \frac{x}{\sin t} + \frac{y}{\cos t} = a$$

in terms of (x_0, y_0) equation is :

$$\frac{x}{(x_0/a)^{1/3}} + \frac{y}{(y_0/a)^{1/3}} = a$$

Length of tangent intercepted between axes = $\sqrt{(x_{int})^2 + (y_{int})^2} = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} = a$

Note :

1. The parametric form is very useful in these type of problems

2. Equation of tangent can also be obtained by substituting b = a and m = 2/3 in the result of example 2

Example : 4

For the curve $xy = c^2$, prove that

(i) the intercept between the axes on the tangent at any point is bisected at the point of contact.

(ii) the tangent at any point makes with the co-ordinate axes a triangle of constant area.

Solution

Let the equation of the curve in parametric form by x = ct, y = c/t

Let the point of contact be (ct, c/t)

Equation of tangent is :

$$y - c/t = \frac{-c/t^2}{c}(x - ct)$$

$\Rightarrow$ $t^2y - ct = -x + ct$

$\Rightarrow$ $x + t^2y = 2ct$ ....(i)

(i) Let the tangent cut the x and y axes at A and B respectively

Writing the equations as : $\frac{x}{2ct} + \frac{y}{2ct/t} = 1$

$\Rightarrow$ $x_{intercept} = 2ct$, $y_{intercept} = 2c/t$

$\Rightarrow$ A ≡ (2ct = 0, and B ≡ $\left(0, \frac{2c}{1}\right)$

mid point of AB ≡ $\left(\frac{2ct + 0}{2}, \frac{0 + 2c/t}{2}\right)$ ≡ (ct, c/t)

Hence, the point of contact bisects AB

(ii) If O is the origin, Area of triangle ΔOAB = 1/2 (OA) (OB) = $\frac{1}{2}(2ct)\frac{(2c)}{1} = 2c^2$

i.e. constant for all tangents because it is independent of t.

Example : 5

Find the abscissa of the point on the curve $ay^2 = x^3$, the normal at which cuts of equal intercept from the axes.

Solution

The given curve is $ay^2 = x^3$ ....(i)

Differentiate to get :

$$2ay\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2}{2ay}$$

The slope of normal = $\frac{1}{-\frac{dy}{dx}} = -\frac{2ay}{3x^2}$

since the normal makes equal intercepts on the axes, its inclination to axis of x is either 45º or 135º.

So two normal are possible with slopes 1 and – 1

$$\Rightarrow \quad -\frac{2ay}{3x^2} = \pm 1$$

On squaring $4a^2y^2 = 9x^4$

Using (i), we get : $4a\,x^3 = 9x^4$

$\Rightarrow$ x = 4a/9

Example : 6

Show that two tangents can be drawn from the point A(2a, 3a) to the parabola $y^2 = 4ax$. Find the equations of these tangents.

Solution

The parametric form for $y^2 = 4ax$ is $x = at^2$, $y = 2at$

Let the point $P(at^2, 2at)$ on the parabola be the point of contact for the tangents drawn form A

i.e. $y - 2at = \frac{2a}{2at}(x - at^2)$

$\Rightarrow \quad ty - 2at^2 = x - at^2$

$\Rightarrow \quad x - ty + at^2 = 0$ ....(i)

it passes through A(2a, 3a)

$\Rightarrow \quad 2a - 3at + at^2 = 0$

$\Rightarrow \quad t^2 - 3t + 2 = 0$

$\Rightarrow \quad t = 1, 2$

Hence there are two points of contact P_1 and P_2 corresponding to $t_1 = 1$ and $t_2 = 2$ on the parabola. This means that two tangents can be drawn.

Using (i), the equations of tangents are :

$x - y + a = 0$ and $x - 2y + 4a = 0$

Example : 7

Find the equation of the tangents drawn to the curve $y^2 - 2x^3 - 4y + 8 = 0$ from the point (1, 2)

Solution

Let tangent drawn from (1, 2) to the curve

$y^2 - 2x^3 - 4y + 8 = 0$ meets the curve in point (h, k)

Equation of tangents at (h, k)

Slope of tangent at (h, k)

$$= \left.\frac{dy}{dx}\right]_{(h,k)} = \left.\frac{3x^2}{y-2}\right]_{(h,k)} = \frac{3h^2}{k-2}$$

Equation of tangent is $y - k = \frac{3h^2}{k-2}(x - h)$

As tangent passes through (1, 2), we can obtain $2 - k = \frac{3h^2}{k-2}(1 - h)$

$\Rightarrow \quad 3h^3 - 3h^2 - k^2 + 4k - 4 = 0$ ....(i)

As (h, k) lies on the given curve, we can make

$k^2 - 2h^3 - 4k + 8 = 0$ ....(ii)

Adding (i) and (ii), we get $h^3 - 3h^2 + 4 = 0$

$\Rightarrow \quad (h + 1)(h - 2)^2 = 0$

$\Rightarrow \quad h = -1$ and $h = 2$

For $h = -1$, k is imaginary

So consider only h = 2.

Using (ii) and h = 2, we get $k = 2 \pm 2\sqrt{3}$.

$(2, 2 + 2\sqrt{3}) =$ and $(2, 2) - 2\sqrt{3})$

Equation of tangents at these points are :

$y - (2 + 2\sqrt{3}) = 2\sqrt{3}\,(x - 2)$

and $\quad y - (2 - \sqrt{3}) = -2\sqrt{3}\,(x - 2)$

Example : 8

Find the equation of the tangent to $x^3 = ay^3$ at the point A (at^2, at^3). Find also the point where this tangent meets the curve again.

Solution

Equation of tangent to : $x = at^2$, $y = at^3$ is

$$y - at^3 = \frac{3at^3}{2at}(x - at^2)$$

$\Rightarrow$ $2y - 2at^3 = 3tx - 3at^3$

i.e. $3tx - 2y - at^3 = 0$

Let B (at_1^2 , at_1^3) be the point where it again meets the curve.

$\Rightarrow$ slope of tangent at A = slope of AB $\frac{3at^2}{2at} = \frac{a(t^3 - t_1^3)}{a(t^3 - t_1^2)}$

$\Rightarrow$ $\frac{3t}{2} = \frac{t^2 + t_1^2 + tt_1}{t + t_1}$

$\Rightarrow$ $3t^2 + 3tt_1 = 2t^2 + 2t_1^2 + 2t\,t_1$

$\Rightarrow$ $2t_1^2 - t\,t_1 - t^2 = 0$

$\Rightarrow$ $(t_1 - t)(2t_1 + t) = 0$

$\Rightarrow$ $t_1 = t$ or $t_1 = -1/2$

The relevant value is $t_1 = -t/2$

Hence the meeting point B is = $\left[a\left(\frac{-t}{2}\right)^2, a\left(\frac{-t}{2}\right)^3\right] = \left[\frac{at^2}{4}, \frac{-at^3}{8}\right]$

Example : 9

Find the condition that the line $x \cos\alpha + y \sin\alpha = P$ may touch the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution

Let (x_1, y_1) be the point of contact

$\Rightarrow$ the equation of tangent is $y - y_1 \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$

$\Rightarrow$ $y - y_1 = \frac{-b^2x_1}{a^2y_1}(x - x_1)$

$\Rightarrow$ $a^2 yy_1 - a^2 y_1^2 = -b^2 x x_1 + b^2 x_1^2$

$\Rightarrow$ $b^2 x x_1 + a^2 y y_1 = b^2 x_1^2 + a^2 y_1^2$

Using the equation of the curve : $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ is the tangent

If this tangent and the given line coincide, then the ratio of the coefficients of x and y and the constant terms must be same

Comparing $x \cos\alpha + y \sin\alpha = P$ and $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

we get $\frac{\cos\alpha}{x_1/a^2} = \frac{\sin\alpha}{y_1/b^2} = \frac{P}{1}$

$\Rightarrow$ $Px_1 = a^2 \cos\alpha$, $Py_1 = b^2 \sin\alpha$ and also we have $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$

From these three equations, we eliminate x_1, y_1 to get the required condition.

$$\frac{1}{a^2}\left(\frac{a^2\cos\alpha}{P}\right)^2 + \frac{1}{b^2}\left(\frac{b^2\sin\alpha}{P}\right)^2 = 1$$

$\Rightarrow$ $a^2\cos^2\alpha + b^2\sin^2\alpha = P^2$

Example : 10

Find the condition that the curves ; $ax^2 + by^2 = 1$ $a'x^2 + b'y^2 = 1$ may cut each other orthogonally (at right angles)

Solution

Condition for orthogonality implies that the tangents to the curves at the point of intersection are perpendicular. If (x_0, y_0) is the point of intersection, and m_1, m_2 are slopes of the tangents to the two curves at this point, the $m_1\, m_2 = -1$.

Let us find the point of intersection. Solving the equations simultaneously,

$ax^2 + by^2 - 1 = 0$

$a'x^2 + b'y^2 - 1 = 0$

$\Rightarrow$ $$\frac{x^2}{-b+b'} = \frac{y^2}{-a+a'} = \frac{1}{ab'-a'b}$$

$\Rightarrow$ the point of intersection $(x_0 - y_0)$ is given by

$$x_0^2 = \frac{b'-b}{ab'-a'b} \text{ and } y_0^2 = \frac{a-a'}{ab'-a'b}$$

The slope of tangent to the curve $ax^2 + by^2 = 1$ is

$m_1 = \dfrac{dy}{dx} = \dfrac{-ax_0}{by_0}$ and the slope of tangent to the curve $a'x^2 + b'y^2 = 1$ is $m_2 = \dfrac{-a'x_0}{b'y_0}$

for orthogonality, $m_1m_2 = \dfrac{aa'}{bb'}\,\dfrac{x_0^2}{y_0^2} = -1$

Using the values of x_0 and y_0, we get

$\Rightarrow$ $$\frac{aa'}{b'b}\,\frac{b'-b}{a-a'} = -1$$

$\Rightarrow$ $$\frac{b'-b}{bb'} = \frac{a'-a}{aa'}$$

$\Rightarrow$ $$\frac{1}{b} - \frac{1}{b'} = \frac{1}{a} - \frac{1}{a'}$$ is the required condition

Example : 11

The equation of two curves are $y^2 = 2x$ and $x^2 = 16y$

(a) Find the angle of intersection of two curves

(b) Find the equation of common tangents to these curves.

Solution

(a) First of all solve the equation of two curves to get their points of intersection.

The two curves are $y^2 = 2x$ ....(i)

and $x^2 = 16y$ ....(ii)

On solving (i) and (ii) two points of intersection are (0, 0) and (8, 4)

At (0, 0)

The two tangents to curve $y^2 = 2x$ and $x^2 = 16y$ are $x = 0$ and $y = 0$ respectively.

So angle between curve = angle between tangents = $\pi/2$

At (8, 4)

Slope of tangent to $y^2 = 2x$ is $m_1 = \left.\dfrac{dy}{dx}\right]_{at\,(8,\,4)} = \dfrac{1}{y}$

$\Rightarrow$ $m_1 = 1/4$

Similarly slope of tangent to $x^2 = 16y$ is $m_2 = 1$

Acute angle between the two curve at (8, 4)

$$= \left|\tan^{-1}\left[\frac{m_1 - m_2}{1 + m_1 m_2}\right]\right| = \left|\tan^{-1}\frac{\frac{1}{4} - 2}{1 + \frac{2}{4}}\right| = \tan^{-1}\frac{4}{5}$$

(b) Let common tangent meets $y^2 = 2x$ in point P whose coordinates are $(2t^2, 2t)$

Equation of tangent at P is $y - 2t = \frac{1}{2t}(x - 2t^2)$

$\Rightarrow$ $2ty - x = 2t^2$

On solving equation of second curve and tangent (i), we get :

$2t(x^2/16) - x = 2t^2$

$\Rightarrow$ $tx^2 - 8x = 16t^2$

This quadratic equation in x should have equal roots because tangent (i) is also tangent to second curve and hence only one point of intersection.

$\Rightarrow$ $D = 0$ $\Rightarrow$ $64 + 64t^3 = 0$

$\Rightarrow$ $t = -1$

So equation of common tangent can be obtained by substituting $t = -1$ in (i) i.e.

$-2y - x = 2$ $\Rightarrow$ $2y + x + 2 = 0$

Example : 12

Find the intervals where $y = \frac{3}{2}x^4 - 3x^2 + 1$ is increasing or decreasing

Solution

$dy/dx = 6x^3 - 6x = 6x(x - 1)(x + 1)$

This sign of dy/dx is positive in the interval :

$(-1, 0) \cup (1, \infty)$ and negative in the interval : $(-\infty, -1) \cup (0, 1)$

Hence the function is increasing in $[-1, 0] \cup [1, \infty)$ and decreasing $(-\infty, -1] \cup [0, 1]$

Example : 13

Find the intervals where y = cos x is increasing or decreasing

Solution

$$\frac{dy}{dx} = -\sin x$$

Hence function is increasing in the intervals where sin x is negative and decreasing where sin x is positive

$\frac{dy}{dx} < 0$ if $2n\pi < x < (2n + 1)\pi$

and $\frac{dy}{dx} > 0$ if $(2n + 1)\pi < x < (2n + 2)\pi$

where n is an integer

Hence the function is increasing in $[(2n + 1)\pi, (2n + 2)\pi]$

and decreasing in $[2n\pi, (2n + 1)\pi]$

Example : 14

Show that $\sin x < x < \tan x$ for $0 < x < \pi/2$.

Solution

We have to prove two inequalities; $x > \sin x$ and $\tan x > x$.

Let $f(x) = x - \sin x$

$f'(x) = 1 - \cos x = 2\sin^2 x/2$

$\Rightarrow$ $f'(x)$ is positive

$\Rightarrow$ $f(x)$ is increasing

By definition, $x > 0$

$\Rightarrow$ $f(x) > f(0)$

$\Rightarrow$ $x - \sin x > 0 - \sin 0$

$\Rightarrow$ $x - \sin x > 0$

$\Rightarrow$ $x > \sin x$ ....(i)

Now, let $g(x) = \tan x - x$

$g'(x) = \sec^2 x - 1 = \tan^2 x$ which is positive

$\Rightarrow$ g(x) is increasing

By definition, $x > 0$ $\Rightarrow$ $g(x) > g(0)$

$\Rightarrow$ $\tan x - x > \tan 0 - 0$

$\Rightarrow$ $\tan x - x > 0$

$\Rightarrow$ $\tan x > x$ ....(ii)

Combining (i) and (ii), we get $\sin x < x < \tan x$

Example : 15

Show that $x / (1 + x) < \log (1 + x) < x$ for $x > 0$.

Solution

Let $f(x) = \log (1 + x) - \dfrac{x}{1+x}$

$$f'(x) = \frac{1}{1+x} - \frac{(1+x)-x}{(1+x)^2}$$

$$f'(x) = \frac{x}{(1+x)^2} \; . \; 0 \text{ for } x > 0$$

$\Rightarrow$ f(x) is increasing

Hence $x > 0$ $\Rightarrow$ $f(x) > f(0)$ by the definition of the increasing function.

$\Rightarrow$ $\log (1 + x) - \dfrac{x}{1+x} > \log (1 + 0) - \dfrac{0}{1+0}$

$\Rightarrow$ $\log (1 + x) - \dfrac{x}{1+x} > 0$

$\Rightarrow$ $\log (1 + x) > \dfrac{x}{1+x}$ ....(i)

Now let $g(x) = x - \log (1 + x)$

$g'(x) = 1 - \dfrac{1}{1+x} = \dfrac{x}{1+x} > 0$ for $x > 0$

$\Rightarrow$ g(x) is increasing

Hence $x > 0$ $\Rightarrow$ $g(x) > g(0)$

$\Rightarrow$ $x - \log (1 + x) > 0 - \log (1 + 0)$

$\Rightarrow$ $x - \log (1 + x) > 0$

$\Rightarrow$ $x > \log (1 + x)$ ....(ii)

Combining (i) and (ii), we get

$$\frac{x}{1+x} < \log (1 + x) < x$$

Example : 16

Show that : $x - \frac{x^3}{6} < \sin x$ for $0 < x < \frac{\pi}{2}$

Solution

Let $f(x) = \sin x - x + \frac{x^3}{6}$

$f'(x) = \cos x - 1 + \frac{x^2}{2}$

$f''(x) = -\sin x + x$

$f'''(x) = -\cos x + 1 = 2\sin^2 \frac{x}{2} > 0$

$\Rightarrow$ $f''(x)$ is increasing

Hence $x > 0$ $\Rightarrow$ $f''(x) > f''(0)$

$\Rightarrow$ $-\sin x + x > -\sin 0 + 0$

$\Rightarrow$ $-\sin x + x > 0$

$\Rightarrow$ $f''(x) > 0$

$\Rightarrow$ $f'(x)$ is increasing

Hence $x > 0$ $\Rightarrow$ $f'(x) > f'(0)$

$\Rightarrow$ $\cos x - 1 + x^2/2 > \cos 0 - 1 + 0/2$

$\Rightarrow$ $\cos x - 1 + x^2/2 > 2$

$\Rightarrow$ $f'(x) > 0$

$\Rightarrow$ $f(x)$ is increasing

Hence $x > 0$ $\Rightarrow$ $f(x) > f(0)$

$\Rightarrow$ $\sin x - x + x^3/6 > \sin 0 - 0 + 0/6$

$\Rightarrow$ $\sin x - x + x^3/6 > 0$

$\Rightarrow$ $\sin x > x - x^3/6$

Example ; 17

Show that $x \geq \log(1 + x)$ for all $x \in (-1, \infty)$

Solution

Let $f(x) = x - \log(1 + x)$

Differentiate f(x) w.r.t. x to get,

$f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x}$

Note that $x = 0$ is a critical point of $f'(x)$ in $(-1, \infty)$.

So divide the interval about $x = 0$ and make two cases

Case – I $x \in (-1, 0)$

In this interval, $f'(x) < 0$

$\Rightarrow$ f(x) is a decreasing function

Therefore, $-1 < x < 0$ $\Rightarrow$ $f(x) \geq f(0) = 0$

Hence $x - \log(1 + x) \geq 0$ for all $x \in (-1, 0)$ ....(i)

Case – II $x \in [0, \infty)$

In this interval, $f'(x) > 0$

$\Rightarrow$ f(x) is an increasing function.

Therefore, $0 \leq x < \infty$ $\Rightarrow$ $f(x) \geq f(0) = 0$

Hence $x - \log(1 + x) \geq 0$ for all $x \in [0, \infty)$ ....(ii)

Combining (i) and (ii), $x \geq \log(1 + x)$ for all $x \in (-1, \infty)$

Example : 18

Find the intervals of monotonicity of the function f(x) = $\frac{|x-1|}{x^2}$

Solution

The given function f(x) can be written as :

$$f(x) = \frac{|x-1|}{x^2} = \begin{cases} \frac{1-x}{x^2} & ; \quad x < 1, x \neq 0 \\ \frac{x-1}{x^2} & ; \quad x \geq 1 \end{cases}$$

Consider x < 1 $f'(x) = \frac{-2}{x^3} + \frac{1}{x^2} = \frac{x-2}{x^3}$

For increasing, f′(x) > 0 $\Rightarrow \frac{x-2}{x^3} > 0$

$\Rightarrow$ x(x – 2) > 0 ($\because$ x^4 is always positive)

$\Rightarrow$ $x \in (-\infty, 0) \cup (2, \infty)$

Combining with x < 1, we get f(x) is increasing in x < 0 and decreasing in $x \in (0, 1)$ ....(i)

Consider $x \geq 1$

$$f'(x) = \frac{-1}{x^2} + \frac{2}{x^3} = \frac{2-x}{x^3}$$

For increasing f′(x) > 0

$\Rightarrow$ (2 – x) > 0 ($\because$ x^3 is positive)

$\Rightarrow$ (x – 2) < 0

$\Rightarrow$ x < 2

combining with x > 1, f(x) is increasing in $x \in (1, 2)$ and decreasing in $x \in (2, \infty)$ ....(ii)

Combining (i) and (ii), we get

f(x) is strictly increasing on $x \in (-\infty, 0) \cup (1, 2)$ and strictly decreasing on $x \in (0, 1) \cup (2, \infty)$

Example : 19

Prove that $(a + b)^n \leq a^n + b^n$, a > 0, b > 0 and $0 \leq n \leq 1$

Solution

We want to prove that $(a + b)^n \leq a^n + b^n$ i.e. $\left(\frac{a}{b}+1\right)^n \leq \left(\frac{a}{b}\right)^n + 1$

i.e. $(x + 1)^n \leq 1 + x^n$ where x = a/b and x > 0,

since a and b both are positive.

To prove above inequality, consider

$f(x) = (x + 1)^n - x^n - 1$

Differentiate to get,

$$f'(x) = n(x = 1)^{n-1} - nx^{n-1} = \left[\frac{1}{(x+1)^{1-n}} - \frac{1}{x^{1-n}}\right] \quad ....(i)$$

consider x + 1 > x

$\Rightarrow$ $(x + 1)^{1-n} > x^{1-n}$ ($\because$ 1 – n > 0)

$\Rightarrow$ $\frac{1}{(x+1)^{1-n}} , \frac{1}{x^{1-n}}$

Combining (i) and (ii), we can say f′(x) < 0

$\Rightarrow$ f(x) is a decreasing function $\forall$ x > 0

Consider $x \geq 0$

$f(x) \leq f(0)$ $\because$ f(x) is a decreasing function

$\Rightarrow$ $f(x) \leq 0$

$\Rightarrow$ $(x + 1)^n - x^n - 1 \leq 0$

$\Rightarrow$ $(x + 1)^n \leq x^n + 1$ Hence proved

Example : 20

Find the local maximum and local minimum values of the function $y = x^x$.

Solution

Let $f(x) = y = x^x$

$\Rightarrow \quad \log y = x \log x$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = x\,\frac{1}{x} + \log x$$

$$\Rightarrow \quad \frac{dy}{dx} = x^x (1 + \log x)$$

$f'(x) = 0 \quad \Rightarrow \quad x^x (1 + \log x) = 0$

$\Rightarrow \quad \log x = -1 \quad \Rightarrow \quad x = e^{-1} = 1/e$

Method – I

$f'(x) = x^x (1 + \log x)$

$f'(x) = x^x \log ex$

$x < 1/e \;\; ex < 1 \quad \Rightarrow \quad f'(x) < 0$

$x > 1/e \;\; ex > 1 \quad \Rightarrow \quad f'(x) > 0$

The sign of $f'(x)$ changes from –ve to +ve around $x = 1/e$. In other words $f(x)$ changes from decreasing to increasing at $x = 1/e$

Hence $x = 1/e$ is a point of local maximum

Local minimum value $= (1/e)^{1/e} = e^{-1/e}$.

Method – II

$$f''(x) = (1 + \log x)\,\frac{d}{dx}\,x^x + x^x\left(\frac{1}{x}\right) = x^x (1 + \log x)^2 + x^{x-1}$$

$f''(1/e) = 0 + (e)^{(e-1)/e} > 0.$

Hence $x = 1/e$ is a point of local minimum

Local minimum value is $(1/e)^{1/e} = e^{-1/e}$.

Note ; We will apply the second derivative test in most of the problems.

Example : 21

Let $f(x) = \sin^3 x + \lambda \sin^2 x$ where $-\pi/2 < x < \pi/2$. Find the interval in which λ should lie in order that $f(x)$ has exactly one minimum and exactly one maximum.

Solution

$f(x) = \sin^3 x + \lambda \sin^2 x.$

$f'(x) = 3 \sin^2 x \cos x + 2 \sin x \cos x \times \lambda$

$$f'(x) = 0 \quad \Rightarrow \quad 3 \sin x \cos x \left(\sin x + \frac{2\lambda}{3}\right) = 0$$

$$\Rightarrow \quad \sin x = 0 \quad \text{or} \quad \cos x = 0 \quad \text{or} \quad \sin x = \frac{-2\lambda}{3}$$

$\cos x = 0$ is not possible in the given interval.

$\Rightarrow \quad x = 0$ and $x = \sin^{-1}(-2\lambda/3)$ are two possible values of x.

These represent two distinct values of x if :

(i) $\quad \lambda \neq 0$ because otherwise $x = 0$ will be the only value

(ii) $\quad -1 < -2\lambda/3 < 1 \quad \Rightarrow \quad 3/2 > \lambda > -3/2$

for exactly one maximum and only one minimum these conditions must be satisfied by λ

$$\text{i.e.} \quad \lambda \in \left(-\frac{3}{2}, 0\right) \cup \left(0, \frac{3}{2}\right)$$

Since f(x) is continuous and differentiable function, these can not be two consecutive points of local maximum or local minimum. These should be alternate.

Hence $f'(x) = 0$ at two distinct points will mean that one is local maximum and the other is local minimum.

Example : 23

A windows is in the form of a rectangle surmounted by a semi-circle. The total area of window is fixed. What should be the ratio of the areas of the semi-circular part and the rectangular part so that the total perimeter is minimum?

Solution

Let A be the total area of the window. If 2x be the width of the rectangle and y be the height.

Let 2x be the width of the rectangle and y be the height. Let the radius of circle be x.

$\Rightarrow \quad A = 2xy + \pi/2\ x^2$

Perimeter (P) = $2x + 2y + \pi x$

A is fixed and P is to be minimised

Eliminating y,

$$P(x) = 2x + \pi x + \frac{1}{x}\left(A - \frac{\pi x^2}{2}\right)$$

$$P'(x) = 2 + \pi - A/x^2 - \pi/2$$

$$P'(x) = 0 \quad \Rightarrow \quad x = \sqrt{\frac{2A}{\pi + 4}}$$

$$P''(x) = 2A/x^3 > 0$$

$\Rightarrow \quad$ Perimeter is minimum for $x = \sqrt{\dfrac{2A}{\pi + 4}}$

for minimum perimeter,

$$\text{area of semicircle} = \frac{\pi(2A)}{(\pi + 4)2} = \frac{\pi A}{\pi + 4}$$

$$\text{area of rectangle} = A - \frac{\pi A}{\pi + 4} = \frac{4A}{\pi + 4}$$

$\Rightarrow \quad$ ratio of the areas of two parts = $\dfrac{\pi}{4}$

Example : 23

A box of constant volume C is to be twice as long as it is wide. The cost per unit area of the material on the top and four sides faces is three times the cost for bottom. What are the most economical dimensions of the box?

Solution

Let 2x be the length, x be the width and y be the height of the box.

Volume = C = $2x^2y$.

Let then cost of bottom = Rs. k per sqm.

Total cost = cost of bottom + cost of other faces

$= k(2x^2) + 3x\ (4xy + 2xy + 2x^2) = 2k$

$= 2k\ (4x^2 + 9xy)$

Eliminating y using C = $2x^2y$,

Total cost = $2k\ (4x^2 + 9C/2x)$

Total cost is to be minimised.

$$\text{Let total cost} = f(x) = 2k\left(4x^2 + \frac{9C}{2x}\right)$$

$$f'(x) = 2k\left(8x - \frac{9c}{2x^2}\right)$$

$$f'(x) = 0 \quad \Rightarrow \quad 8x - \frac{9c}{2x^2} = 0$$

$$\Rightarrow \quad x = \left(\frac{9C}{16}\right)^{1/3}$$

$f''(x) = 2k\left(8 + \frac{9C}{x^3}\right) > 0$

hence the cost is minimum for $x = \left(\frac{9C}{16}\right)^{1/3}$ and $y = \frac{C}{2x^2} = \frac{C}{2}\left(\frac{16}{9C}\right)^{2/3} = \left(\frac{32C}{81}\right)^{1/3}$

The dimensions are : $2\left(\frac{9C}{16}\right)^{1/3}, \left(\frac{9C}{16}\right)^{1/3}, \left(\frac{32C}{81}\right)^{1/3}$

Example : 34

Show that the semi-vertical angle of a cone of given total surface and maximum volume is $\sin^{-1} 1/3$.

Solution

Let r and h be the radius and height of the cone and ℓ be the slant height of the cone.

Total surface area = $S = \pi r\ell + \pi r^2$ ....(i)

Volume = $V = \pi/3 r^2 h$ is to be maximised

Using, $\ell^2 = r^2 + h^2$ and $S = \pi r\ell + \pi r^2$

$$V = \frac{\pi}{3} r^2 \sqrt{\ell^3 - r^2}$$

$$\Rightarrow \quad V = \frac{\pi}{3} r^2 \sqrt{\left(\frac{S - \pi r^2}{\pi r}\right)^2 - r^2}$$

$$\Rightarrow \quad V = \frac{\pi}{3} r^2 \sqrt{\frac{S^2}{\pi^2 r^2} - \frac{2S}{\pi}}$$

We will maximise V^2

Let $V^2 = f(r) = \frac{\pi^2}{9} r^4 \left(\frac{S^2}{\pi^2 r^2} - \frac{2S}{\pi}\right) = f(r) = \frac{S}{9}(Sr^2 - 2\pi r^4)$

$\Rightarrow \quad f'(r) = 0 \Rightarrow \quad 2Sr - \pi r^3 = 0$

$\Rightarrow \quad r = \sqrt{\frac{S}{4\pi}}$ ....(ii)

$f''(r) = \frac{S}{9}(2S - 24\pi r^2)$

$f''\left(\sqrt{\frac{S}{4\pi}}\right) = \frac{S}{9}(2S - 6S) < 0$

Hence the volume is maximum for $r = \sqrt{\frac{S}{4\pi}}$

To find the semi-vertical angle, eliminate S between (i) and (ii), to get :

$4\pi r^2 = \pi r\ell + \pi r^2$

$\Rightarrow \quad \ell = 3r$

$\sin\theta = r/\ell = 1/3$

$\Rightarrow \quad \theta = \sin^{-1}(1/3)$ for maximum volume.

Example : 25

Find the maximum surface area of a cylinder that can be inscribed in a given sphere of radius R.

Solution

Let r be the radius and h be the height of cylinder. Consider the right triangle shown in the figure.

$2r = 2R \cos\theta$ and $h = 2R \sin\theta$

Surface area of the cylinder = $2\pi rh + 2pr^2$

$\Rightarrow \quad S(\theta) = 4\pi R^2 \sin\theta \cos\theta + 2\pi R^2 \cos^2\theta$

$\Rightarrow \quad S(\theta) = 2\pi R^2 \sin 2\theta + 2\pi R^2 \cos^2\theta$

$\Rightarrow \quad S'(\theta) = 4\pi R^2 \cos 2\theta - 2\pi R^2 \sin 2\theta$

$S'(\theta) = 0 \quad \Rightarrow \quad 2\cos 2\theta - \sin 2\theta = 0$

$\Rightarrow \quad \tan 2\theta = 2 \quad \Rightarrow \quad \theta = \theta_0 = 1/2 \tan^{-1} 2$

$S''(\theta) = -8\pi R^2 \sin 2\theta - 4\pi R^2 \cos 2\theta$

$$S''(\theta_0) = -8\pi R^2 \left(\frac{2}{\sqrt{5}}\right) - 4\pi R^2 \left(\frac{1}{\sqrt{5}}\right) < 0$$

Hence surface area is maximum for $\theta = \theta_0 = 1/2 \tan^{-1} 2$

$$\Rightarrow \quad S_{max} = 2\pi R^2 \left(\frac{2}{\sqrt{5}}\right) + 2\pi R^2 \left(\frac{1 + 1/\sqrt{5}}{2}\right)$$

$$\Rightarrow \quad S_{max} = \pi R^2 \left(1 + \sqrt{5}\right)$$

Example : 26

Find the semi-vertical angle of the cone of maximum curved surface area that can be inscribed in a given sphere of radius R.

Solution

Let h be the height of come and r be the radius of the cone. Consider the right ΔOMC where O is the centre of sphere and AM is perpendicular to the base BC of cone.

$OM = h - R$, $OC = R$, $MC = r$

$R^2 = (h - R)^2 + r^2$ ....(i)

and $r^2 + h^2 = \ell^2$ ....(ii)

where ℓ is the slant height of cone.

Curved surface area = $C = \pi r \ell$

Using (i) and (ii), express C in terms of h only.

$$C = \pi r \sqrt{r^2 + h^2}$$

$$\Rightarrow \quad C = \pi \sqrt{2hR - h^2} \ \sqrt{2hR}$$

We will maximise C^2.

Let $C^2 = f(h) = 2\pi^2 hR (2hR - h^2)$

$\Rightarrow \quad f'(h) = 2\pi^2 R (4hR - 3h^2)$

$f'(h) = 0 \quad \Rightarrow \quad 4hR - 3h^2 = 0$

$\Rightarrow \quad h = 4R/3.$

$f''(h) = 2\pi^3 R (4R - 6h)$

$$f''\left(\frac{4R}{3}\right) = 2\pi R^2 (4R - 8R) < 0$$

Hence curved surface area is maximum for $h = \frac{4R}{3}$

Using (i), we get $\quad r^2 = 2hR - h^2 = \frac{8R^2}{9}$

$$\Rightarrow \quad r = \frac{2\sqrt{2}}{3} R$$

Semi-vertical angle = $\theta \tan^{-1} r/h = \tan^{-1} 1/\sqrt{2}$

Example : 27

A cone is circumscribed about a sphere of radius R. Show that the volume of the cone is minimum if its height is 4R.

Solution

Let r be the radius, h be the height, and be the slant height of cone.

If O be the centre of sphere,

$\Delta AON - \Delta ACM$

$\Rightarrow \quad \dfrac{h-R}{R} = \dfrac{\ell}{r}$ ....(i)

$\Rightarrow \quad \dfrac{h-R}{R} = \dfrac{\sqrt{r^2+h^2}}{r}$

Squaring and simplifying we get ;

$r^2 = \dfrac{hR^2}{h-2R}$ ....(ii)

Now volume of cone = $1/3\ \pi r^2\ h$

$\Rightarrow \quad V = \dfrac{1}{3}\pi\left(\dfrac{hR^2}{h-2R}\right)h$

$\Rightarrow \quad V = \dfrac{1}{3}\dfrac{\pi R^2}{\left(\dfrac{1}{h}-\dfrac{2R}{h^2}\right)}$

For volume to be minimum, the denominator should be maximum. Hence we will maximise :

$f(h) = \dfrac{1}{h} - \dfrac{2R}{h^2}$

$f'(h) = -\dfrac{1}{h^2} + \dfrac{4R}{h^3}$

$f'(h) = 0 \quad \Rightarrow \quad h = 4R$

$f''(h) = \dfrac{2}{h^3} - \dfrac{12R}{h^4} = \dfrac{2h-12R}{h^4}$

$f''(4R) = \dfrac{8R-12R}{256R^4} < 0$

Hence f(h) is maximum and volume is minimum for h = 4R.

Example : 28

The lower corner of a page in a book is folded over so as to reach the inner edge of the page. Show that the fraction of the width folded over when the area of the folded part is minimum is 2/3.

Solution

The corner A is folded to reach A_1.

The length of the folded part = AB = A_1B = x

Let total width = 1 unit

$\Rightarrow$ Length of the unfolded part = OB = 1 – x.

If CM || OA, $\quad \Delta A_1 CM \sim \Delta BA_1O$

$\Rightarrow \quad \dfrac{A_1C}{CM} = \dfrac{BA_1}{A_1O}$

$\Rightarrow \quad A_1 C = y = CM\left(\dfrac{BA_1}{A_1O}\right)$

$\Rightarrow \quad y = 1\left(\dfrac{x}{\sqrt{x^2-(1-x)^2}}\right)$ ....(i)

Area of folded part = Area (ΔA_1BC)

$A = \dfrac{1}{2}\,xy = \dfrac{1}{2} \times \dfrac{x}{\sqrt{2x-1}}$

$\Rightarrow \quad A^2 = \dfrac{x^4}{4(2x-1)} = \dfrac{1}{4\left(\dfrac{2}{x^3}-\dfrac{1}{x^4}\right)}$

For area to be minimum, denominator in R.H.S. must be maximum.

Let $f(x) = \dfrac{2}{x^3} - \dfrac{1}{x^4}$

$f'(x) = \dfrac{-6}{x^4} + \dfrac{4}{x^5}$

$f'(x) = 0 \quad \Rightarrow \quad -6x + 4 = 0 \quad \Rightarrow \quad x = 2/3$

$f''(x) = \dfrac{24}{x^5} - \dfrac{20}{x^6} = \dfrac{24x-20}{x^6}$

$f''(2/3) = \dfrac{16-20}{(2/3)^6} < 0$

Hence f(x) is maximum and area is minimum if x = 2/3

i.e. 2/3 rd of the width

Example : 29

Prove that the minimum intercept made by axes on the tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is a + b. Also find the ratio in which the point of contact divides this intercept.

Solution

Intercept made by the axes on the tangent is the length of the portion of the tangent intercepted between the axes. Consider a point P on the ellipse whose coordinates are x = a cost, y = b sin t (where t is the parameter)

The equation of the tangent is :

$$y - b\sin t = \frac{b\cos t}{-a\sin t}(x - a\cos t)$$

$\Rightarrow \quad \dfrac{x}{a}\cos t + \dfrac{y}{b}\sin t = 1$

$\Rightarrow \quad OA = \dfrac{a}{\cos t},\ OB = \dfrac{b}{\sin t}$

Length of intercept = ℓ = AB = $\sqrt{\dfrac{a^2}{\cos^2 t}+\dfrac{b^2}{\sin^2 t}}$

We will minimise ℓ^2 .

Let $\ell^2 = f(t) = a^2\sec^2 t + \text{cosec}^2 t$

$\Rightarrow \quad f(t) - 2a^2\sec^2 t\tan t - 2b^2\,\text{cosec}^2 t\cos t$

$f'(t) = 0 \quad \Rightarrow \quad a^2\sin^4 t = b^2\cos^4 t$

$\Rightarrow \quad t = \tan^{-1}\sqrt{\dfrac{b}{a}}$

$f''(t) = 2a^2 (\sec^4 t + 2 \tan^2 t \sec^2 t) + 2b^2 (\text{cosec}^4 t + 2 \text{cosec}^2 t \cot^2 t)$
which is positive

Hence f(t) is minimum for tan t = $\sqrt{b/a}$

$\Rightarrow \quad \ell_{min} = \sqrt{a^2(1+b/a)+b^2(1+a/b)}$

$\Rightarrow \quad \ell_{min} = a + b$ ....(i)

$PA^2 = \left(a\cos t - \frac{a}{\cos t}\right)^2 + b^2 \sin^2 t = \frac{a^2 \sin^4 t}{\cos^2 t} + b^2 \sin^2 t = (a^2 \tan^2 t \;\; b^2) \sin^2 t = (ab + b^2) \frac{b}{a+b} = b^2$

$\Rightarrow \quad PA = b$
Using (i), PB = a

Hence $\frac{PA}{PB} = \frac{b}{a}$

$\Rightarrow$ P divides AB in the ratio b : a

Example : 30

Find the area of the greatest isosceles triangle that can be inscribed in a given ellipse having its vertex coincident with one end of the major axis.

Solution

Let the coordinates of B be (a cos t, b sin t)
$\Rightarrow$ The coordinates of C are : (a cos t, – b sin t)
because BC is a vertical line and BM = MC
Area of triangle = 1/2 (BC) (AM)
$\Rightarrow \quad A = 1/2 (2b \sin t) (a – a \cos t)$
$\Rightarrow \quad A(t) = ab (\sin t – \sin t \cos t)$
$A'(t) = ab (\cos t – \cos 2t)$
$A'(t) = ab (\cos t – \cos 2t)$
$A'(t) = 0 \quad \Rightarrow \quad \cos t – \cos 2t = 0$
$\Rightarrow \quad \cos t + 1 – 2 \cos^2 t = 0$
$\Rightarrow \quad \cos t = 1, –1/2$
$A''(t) = ab (– \sin t + 2 \sin 2t) = ab \sin t(4 \cos t – 1)$

$A''(2\pi/3) = ab \sqrt{3}/2 \; (–2 – 1) < 0$

Hence area is maximum for $t = \frac{2\pi}{3}$

Maximum area = $A\left(\frac{2\pi}{3}\right)$

$= ab (\sin 2\pi/3 – \sin 2\pi/3 \cos 2\pi/3)$

$= ab \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\frac{1}{2}\right) = \frac{3\sqrt{3}}{4} ab$

Example : 31

Find the point on the curve $y = x^2$ which is closest to the point A (0, a)

Solution

Using the parametric representation, consider an arbitrary point P (t, t^2) on the curve.
Distance of P from A = PA

$PA = \sqrt{t^2 + (t^2 – a)^2}$

We have to find t so that this distance is minimum.
We will minimise PA^2
Let $PA^2 = f(t) = t^2 + (t^2 – a)^2$
$f'(t) = 2t + 4t (t^2 – a)$

$f'(t) = 2t\,[2t^2 - 2a + 1]$

$f'(t) = 0 \quad \Rightarrow \quad t = 0, \pm\sqrt{a - \frac{1}{2}}$

$f''(t) = 2 - 4a + 12\,t^2$

we have to consider two possibilities.

Case – I : $a < 1/2$

In this case, $t = 0$ is the only value.

$f''(0) = 2 - 4a = 4\,(1/2 - a) > 0$

Hence the closest point corresponds to $t = 0$

$\Rightarrow$ (0, 0) is the closest point

Case – II : $a > 1/2$

In this case $t = 0, \pm\sqrt{a - \frac{1}{2}}$

$f''(0) = 2 - 4a = 4\left(\frac{1}{2} - a\right) < 0$

$\Rightarrow$ local maximum at $t = 0$

$f''\left(\pm\sqrt{a - \frac{1}{2}}\right) = 2 - 4a + 12\,a - 6 = 8\left(a - \frac{1}{2}\right) > 0$

Hence the distance is minimum for $t = \pm\sqrt{a - \frac{1}{2}}$

So the closest points are $\left(\sqrt{a - \frac{1}{2}}, \frac{2a - 1}{2}\right)$ and $\left(-\sqrt{a - \frac{1}{2}}, \frac{2a - 1}{2}\right)$

Example : 32

Find the shortest distance between the line $y - x = 1$ and the curve $x = y^2$

Solution

Let P (t^2, t) be any point on the curve $x = y^2$. The distance of P from the given line is $= \frac{|-t^2 + t - 1|}{\sqrt{t^2 + 1^2}}$

$= \frac{t^2 - t + 1}{\sqrt{2}}$ because $t^2 - t + 1$ is a positive expression. We have to find minimum value of this expression.

Let $f(t) = t^2 - t + 1$

$f'(t) = 2t - 1$

$f'(t) = 0 \quad \Rightarrow \quad t = 1/2$

$f''(t) = 2 > 0$

$\Rightarrow$ distance is minimum for $t = +1/2$

Shortest distance $= \left[\frac{t^2 - t + 1}{\sqrt{2}}\right]_{t=1/2} = \frac{\frac{1}{4} - \frac{1}{2} + 1}{\sqrt{2}} = \frac{3\sqrt{2}}{8}$

Example : 33

Find the point on the curve $4x^2 + a^2y^2 = 4a^2$; $4 < a^2 < 8$ that is farthest from the point (0, –2).

Solution

The given curve is an ellipse $\frac{x^2}{a^2} \quad \frac{y^2}{4} = 1$

Consider a point (a cost, 2 sin t) lying on this ellipse.

The distance of P from (0, –2) = $\sqrt{a^2 \cos^2 t + (2 + 2\sin t)^2}$

This distance is to be maximised.

Let $f(t) = a^2 \cos^2 t + 4(1 + \sin t)^2$

$f'(t) = -2a^2 \sin t \cos t + 8(1 + \sin t)(\cos t)$

$f'(t) = (8 - 2a^2) \sin t \cos t + 8 \cos t$

$f'(t) = 0 \Rightarrow \quad \cos t = 0 \quad$ or $\quad \sin t = \frac{4}{a^2 - 4}$

$\Rightarrow \quad t = \pi/2$ or $t = \sin^{-1}\left(\frac{4}{a^2 - 4}\right)$

($t = 3\pi/2$ is rejected because it makes the distance zero)

Let us first discuss the possibility of $t = \sin^{-1}\left(\frac{4}{a^2 - 4}\right)$

We are give that $4 < a^2 < 8$

$\Rightarrow \quad 0 < a^2 - 4 < 4$

$\Rightarrow \quad 0 < 1 < \frac{4}{a^2 - 4}$

as $\frac{4}{a^2 - 4}$ is greatest than 1,

$t = \sin^{-1} \frac{4}{a^2 - 4}$ is not possible.

Hence $t = \pi/2$ is the only value.

Now, $f''(t) = (8 - 2a^2) \cos 2t - 8 \sin t$

$f''(\pi/2) = 2a^2 - 8 - 8 = 2(a^2 - 8) < 0$

$\Rightarrow$ The farthest point corresponds to $t = \pi/2$ and its

Coordinates are $\equiv (a \cos \pi/2, 2 \sin \pi/2) \equiv (0, 2)$

Example : 34

If a + b + c = 0, then show that the quadratic equation $3ax^2 + 2bx + c = 0$, has at least one root in 0 and 1.

Solution

Consider the polynomial $f(x) = ax^3 + bx^2 + cx$. We have f(0) = 0 and f(1) = a + b + c = 0 (Given)

$\Rightarrow \quad f(0) = f(1)$

Also f(x) is continuous and differentiable in [0, 1], it means Rolle's theorem is applicable.

Using the Rolle's Theorem there exists a root of $f'(x) = 0$

i.e. $\quad 3ax^2 + 2bx + c = 0$ between 0 and 1

Hence proved.

Example : 35

Let A (x_1, y_1) and B(x_2, y_2) be any two points on the parabola $y = ax^2 + bx + c$ and let C (x_3, y_3) be the point on the arc AB where the tangent is parallel to the chord AB. Show that $x_3 = (x_1 + x_2)/2$.

Solution

Clearly $f(x) = ax^2 + bx + c$ is a continuous and differentiable function for all values of $x \in [x_1, x_2]$.

On applying Langurange's Mean value theorem on f(x) in (x_1, x_2) we get

$$f'(x_3) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \qquad [\because \quad x_3 \in (x_1, x_2)]$$

On differentiating f(x), we get :

$f'(x) = 2ax + b \quad \Rightarrow \quad f'(x_3) = 2ax_3 + b$

On substituting x_1 and x_2 in the quadratic polynomial, we get

$f(x_1) = ax_1^2 + bx_1 + c \quad$ and $\quad f(x_2) = ax_2^2 + bx_2 + c$

On substituting the values of $f(x_1)$, $f(x_2)$ and $f'(x_3)$ in (i), we get :

$$2ax_3 + b = \frac{ax_2^2 + bx_2 + c - (ax_1^2 + bx_1 + c)}{x_2 - x_1}$$

$$\Rightarrow \quad ax_3 = a(x_1 + x_3)$$

$$\Rightarrow \quad x_3 = \frac{x_1 + x_2}{2}.$$ Hence Proved

Example : 36

Find the condition so that the line $ax + by = 1$ may be a normal to the curve $a^{n-1} y = x^n$.

Solution

Let (x_1, y_1) be the point of intersection of line $ax + by = 1$ and curve $a^{n-1} y = x^n$.

$\Rightarrow \quad ax_1 + by_1 = 1$ ....(i)

and $\quad a^{n-1} y_1 = x_1^n$ ....(ii)

The given curve is : $a^{n-1}y = x^n$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right]_{at(x_1, y_1)} = n\,\frac{x_1^{n-1}}{a^{n-1}} = n\,\frac{x_1^{n-1}}{x_1^{n}}\,y_1 = \frac{ny_1}{x_1} \qquad \text{[using (ii)]}$$

Equation of normal to (x_1, y_1) is :

normal is $y - y_1 = \dfrac{-x_1}{ny_1}(x - x_1)$

$\Rightarrow \quad xx_1 + ny\,y_1 = ny_1^2 + x_1^2$ .... (iii)

But the normal is the line $xa + yb = 1$ .... (iv)

Comparing (iii) and (iv), we get

$$\frac{x_1}{a} = \frac{ny_1}{b} = \frac{ny_1^2 + x_1^2}{1}$$

Let each of these quantities by K, $\quad$ i.e. $\dfrac{x_1}{a} = \dfrac{ny_1}{b} = \dfrac{ny_1^2 + x_1^2}{1} = K$

$\Rightarrow \quad x_1 = aK,\ ny_1 = bK,\ ny_1^2 + x_1^2 = K$

On substituting the values of x_1 and y_1 from first two equations into third equation, we get

$$n\,\frac{b^2K^2}{n^2} + a^2K^2 = K$$

$$\Rightarrow \quad K = \frac{n}{b^2 + na^2},\ x_1 = \frac{an}{b^2 + na^2} \text{ and } y_1 = \frac{b}{b^2 + na^2}$$

Replacing the values of x_1 and y_1 in (ii), we get :

$$a^{n-1}\,\frac{b}{b^2 + na^2} = \left(\frac{an}{b^2 + na^2}\right)^n$$ as the required condition.

Example : 37

Find the vertical angle of right circular cone of minimum curved surface that circumscribes in a given sphere.

Solution

When cone is circumscribed over a sphere

we have : $\Delta AMC \sim \Delta APO$

$\Rightarrow \quad \frac{AC}{MC} = \frac{AO}{OP} \Rightarrow \frac{\ell}{r} = \frac{r-R}{R}$ ....(i)

In cone, we can define $r^2 + h^2 = \ell^2$ ....(ii)

Eliminating ℓ in (i) and (ii), we get

$r^2 = \frac{hR^2}{h-2R}$ ....(iii)

Let curved surface area of cone = $C - \pi r\ell$

$\Rightarrow \quad C = \pi r \frac{r(h-R)}{R}$ [using (i)]

$\Rightarrow \quad C = \frac{\pi hR(h-R)}{(h-2R)}$ [using (iii)]

As C is expressed in terms on one variable only i.e. h, we can maximise C by use of derivatives

$\frac{dC}{dh} = \frac{\pi R}{(h-2R)^2}$ $[(h - 2R)(2h - R) - (h^2 - hR)] = 0$

$\Rightarrow \quad h^2 - 4Rh + 2R^2 = 0$

$\Rightarrow \quad h = (2 + \sqrt{2})R$ ....(iv)

It can be shown that $\frac{d^2C}{dh^2} > 0$ for this value of h.

Substituting $h = (2 + \sqrt{2})R$ in (iii), we get $\frac{(\sqrt{2}+1)R^2}{(\sqrt{2}+1)R^2 + 2(\sqrt{2}+1)^2 R^2}$

$r^2 = (\sqrt{2} + 1)R^2$

Let semi-vartical angle = θ

$\Rightarrow \quad \sin^2\theta = r^2/\ell^2 = \frac{r^2}{r^2+h^2}$

Using (iv) and (v), we get :

$\sin^2\theta = \qquad = \frac{1}{3+2\sqrt{2}}$

$\Rightarrow \quad \sin^2\theta = 3 - 2\sqrt{2} = (\sqrt{2} - 1)^2$

$\Rightarrow \quad \sin\theta = \sqrt{2} - 1$

Example : 38

Let $f(x) = x^3 - x^2 + x + 1$ and $g(x) = \begin{cases} \max[f(t)] & 0 \le t \le x \quad 0 \le x \le 1 \\ 3 - x & ; \quad 1 < x \le 2 \end{cases}$

Discuss the continuity and differentiability of f(x) in (0, 2)

Solution

It is given that $f(x) = x^3 - x^2 + x + 1$

$f'(x) = 3x^2 - 2x + 1$

$f'(x) > 0$ for all x

($\because$ coeff. of $x^2 > 0$ and Discriminant < 0)

Hence f(x) is always increasing function.

Consider $0 \le t \le x$

$\Rightarrow$ $f(0) \le f(t) \le f(x)$ ($\because$ f(t) is an in creasing function)

$\Rightarrow$ $1 \le f(t) \le f(x)$

$\Rightarrow$ Maximum $[f(t)] = f(x) = x^3 - x^2 + x + 1$

$$\Rightarrow \quad g(x) = \begin{cases} x^3 - x^2 + x + x + 1 & , \quad 0 \le x \le 1 \\ 3 - x & , \quad 1 < x \le 2 \end{cases}$$

As g(x) is polynomial in [0, 1] and (1, 2], it is continuous and differentiable in these intervals.

At x = 1

LHL = 2, RHL = 2 and f(1) = 2

$\Rightarrow$ g(x) is continuous at x = 1

LHD = 2 and RHD = – 1

$\Rightarrow$ g(x) is non-differentiable at x = 1

Example : 39

Two considers of width a and b meet at right angles show that the length of the longest pipe that can be passes round the corner horizontally is $(a^{2/3} + b^{2/3})^{3/2}$

Solution

Consider a segment AB touching the corner at P. AB = a cosec θ + b sec θ

Let f(θ) = a cosec θ + b sec θ ....(i)

f′(θ) = – a cosec θ cot θ + b sec θ tan θ

$$\Rightarrow \quad f'(\theta) = \frac{-a\cos\theta}{\sin^2\theta} + \frac{b\sin\theta}{\cos^2\theta} = \frac{-a\cos^3\theta + b\sin^3\theta}{\sin^2\theta\cos^2\theta}$$

f′(θ) = 0 $\Rightarrow$ $\tan^3\theta = a/b$

$\Rightarrow$ $\tan\theta = (a/b)^{1/3}$

Using first derivative test, see yourself that f(θ) possesses local minimum at $\theta = \tan^{-1}(a/b)^{1/3}$.

Using (i), the minimum length of segment AB is :

$$f_{min} = (a^{2/3} + b^{2/3})^{3/2} \qquad \text{for } \theta = \tan^{-1}\sqrt[3]{\frac{b}{a}}$$

This is the minimum length of all the line segments that can be drawn through corner P. If the pipe passes through this segment, it will not get blocked in any other position. Hence the minimum length of segment APB gives the maximum length of pipe that can be passed.

Example : 40

Find the equation of the normal to the curve $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$ at x = 0.

Solution

We have

$y = (1 + x)^y + \sin^{-1}(\sin^2 x)$

Let $A = (1 + x)^y$ and $B = \sin^{-1}\sin^2 x$

$\Rightarrow$ y = A + B ....(i)

Consider A

Taking log and differentiating, we get

$\ell n\, A = y\, \ell n\, (1 + x)$

$$\frac{1}{A}\frac{dA}{dx} = \frac{dy}{dx}\,\ell n\,(1 + x) + \left(\frac{y}{1+x}\right)$$

$$\text{or} \quad \frac{dA}{dx} = A\left[\frac{dy}{dx}\ell n(1+x) + \frac{y}{1+x}\right] = (1 + x)^y\left[\frac{dy}{dx}\ell n(1+x) + \frac{y}{1+x}\right] \qquad \text{........(ii)}$$

Consider B

$B = \sin^{-1}(\sin^2 x)$ $\Rightarrow$ $\sin B = \sin^2 x$

Differentiating wrt x, we get

$$\cos B\,\frac{dB}{dx} = 2\sin x\cos x$$

$$\frac{dB}{dx} = \frac{1}{\cos B}\ (2 \sin x \cos x) = \frac{2\sin x \cos x}{(1-\sin^2 B)^{1/2}} = \frac{2\sin x\cos x}{(1-\sin^4 x)^{1/2}}$$

Now since $y = A + B$

we have $$\frac{dy}{dx} = \frac{dA}{dx} + \frac{dB}{dx} = (1 + x)^y \left[\frac{dy}{dx}\ \ell n\,(1+x) + \frac{y}{1+x}\right] + \frac{2\sin x\cos x}{(1-\sin^2 x)^{1/2}}$$

or $$\frac{dy}{dx} = \frac{y(1+x)^{y-1} + \dfrac{2\sin x\cos x}{(1-\sin^4 x)^{1/2}}}{1-(1-x)^y\,\ell n\,(1+x)}$$

Using the equation of given curve, we can find f(0).
Put x = 0 in the given curve.
$y = (1 + 0)^y + \sin^{-1}(\sin^2 0) = 1$

$$\frac{dy}{dx} = \frac{1(1+0)^{1-1} + \dfrac{2\sin 0\cos 0}{(1-\sin^4 0)^{1/2}}}{1-(1-0)^1\ell n(1+0)} \quad\Rightarrow\quad \frac{dy}{dx} = 1$$

The slope of the normal is $m = -\dfrac{1}{(dy/dx)} = -1$

Thus, the required equation of the normal is $y - 1 = (-1)(x - 0)$
i.e. $y + x - 1 = 0$

Example : 41

Tangent at a point P_1(other than (0, 0) on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve at P_3, and so on. Show that the abscissa of P_1, P_2, P_3 ......P_n , from a GP. Also find the ratio [area ($\Delta P_1 P_2 P_3$)] / [area ($\Delta P_2 P_3 P_4$)].

Solution

Let the chosen point on the curve $y = x^3$ be P_1 (t, t^3). The slope of the tangent to the curve at (t, t^3) is

given as $\dfrac{dy}{dx} = 3x^2 = 3t^2$ ....(i)

The equation of the tangent at (t, t^3) is

$y - t^3 = 3t^2 (x - t)$

$y - 3t^2 x + 2t^3 = 0$ ....(ii)

Now to get the points where the tangent meets the curve again, solve their equations

i.e. $x^3 - 3t^2 x + 2t^3 = 0$ ....(iii)

One of the roots of this equation must be the sbscissa of P_1 i.e. t. Hence, equation (iii) can be factorised as

$(x - t)(x^2 + tx - 2t^2) = 0$

or $(x - t)(x - t)(x + 2t) = 0$

or $(x - t)(x - t)(x + 2t) = 0$

Hence, the abscissa of $P_2 = -2t$ ....(iv)

Let coordinates of point P_2 are (t_1, t_1^3)

Equation of tangent at P_2 is : $y - 3t_1^2 x + 2t_1^3 = 0$

[this is written by replacing t by t_1 in (ii)]

On solving tangent at P_2 and the given curve we get the coordinates of the point where tangent at P_2 meets the curve again i.e.

coordinates of P_3 are $(-2t_1, -t_1)$

Using (iv), abscissa of $P_3 = -2(-2)t$

$\Rightarrow$ abscissa of $P_3 = 4t$

So the abscissa of P_1, P_2 and P_3 are t, (–2) t, (–2) (–2) t respectively, that is, each differing from the preceding one by a factor of (–2).

Hence, we conclude that the abscissae of P_1, P_2, P_3 , ..., P_n from a GP with common ratio of – 2.

Now area $(\Delta P_1 P_2 P_3) = \frac{1}{2}\begin{vmatrix} t & t^3 & 1 \\ -2t & -8t^3 & 1 \\ 4t & 64t^3 & 1 \end{vmatrix} = \frac{t^4}{2}\begin{vmatrix} 1 & 1 & 1 \\ -2 & -8 & 1 \\ 4 & 64 & 1 \end{vmatrix}$

area $(\Delta P_2 P_3 P_4) = \frac{1}{2}\begin{vmatrix} -2t & -8t^3 & 1 \\ 4t & 64t^3 & 1 \\ -8t & (-2)^9 t^3 & 1 \end{vmatrix} = \frac{16t^4}{2}\begin{vmatrix} 1 & 1 & 1 \\ -2 & -8 & 1 \\ 4 & 64 & 1 \end{vmatrix}$

Hence $\frac{\text{area}(\Delta P_1 P_2 P_3)}{\text{area}(\Delta P_2 P_3 P_4)} = \frac{t^4}{16t^4} = \frac{1}{16}$

Example : 42

Let $f(x) = \begin{cases} -x^3 + \frac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)} & , \quad 0 \le x < 1 \\ 2x - 3 & , \quad 1 \le x \le 3 \end{cases}$

Find all possible real values of b such that f(x) has the smallest value at x = 1

Solution

The value of function f(x) at x = 1 is f(x) = 2x – 3 = 2 (1) – 3 = – 1

The function f(x) = 2x – 3 is an increasing function on [1, 3]. hence, f(1)= – 1 is the smallest value of f(x) at x = 1.

Now $f(x) = -x^3 + \frac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)}$

is a decreasing function on [0, 1] for fixed values of b. So its smallest value will occur at the right end of the interval.

$\Rightarrow$ Minimum [(f(x) in [0, 1]) ≥ -1

$\Rightarrow$ $f(1) \ge -1$

$-1 + \frac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)} \ge -1$

In order that this value is not less than – 1, we must have $\frac{b^3 - b^2 + b - 1}{b^2 + 3b + 2} \ge 0$

$\Rightarrow \frac{(b^2 + 1)(b - 1)}{(b + 2)(b + 1)} \ge 0 \qquad \Rightarrow \frac{(b - 1)}{(b + 2)(b + 1)} \ge 0$

The sign of b is positive for $b \in (-2, -1) \cup [1, \infty)$

Hence, the possible real values of b such that f(x) has the smallest value at x = 1 are $(-2, -1) \cup [1, \infty)$

Example : 43

Find the locus of a point that divides a chord of slope 2 of the parabola $y^2 = 4x$ internally in the ratio 1 : 2.

Solution

Let $P \equiv (t_1^2, 2t_1)$, $Q \equiv (t_2^2, 2t_2)$ be the end points of chord AB. Also let $M \equiv (x_1, y_1)$ be a point which divides AB internally in ratio 1 : 2.

It is given that slope of PQ = 2,

$\Rightarrow$ slope (PQ) = $\frac{2t_2 - 2t_1}{t_2^2 - t_1^2} = 2$

$\Rightarrow$ $t_1 + t_2 = 1$ ....(i)

As M divides PQ in 1 : 2 ratio, we get

$\Rightarrow$ $x_1 = \frac{2t_1^2 + t_2^2}{3}$ ....(ii) and $y_1 = \dfrac{2t_2 + 4t_1}{3}$ ....(iii)

We have to eliminate two variables t_1 and t_2 between (i), (ii) and (iii).
From (i), put $t_1 = 1 - t_2$ in (iii) to get :
$3y_1 = 2(t - t_2) + 4t_2 = 2(1 + t_2)$
$\Rightarrow$ $t_2 = (3y - 2)/2$ and $t_1 = -3y_1/2$
On substituting the values of t_1 and t_2 in (ii), we get : $4x_1 = 9y_1^2 - 16y_1 + 8$
Replacing x_1 by x and y_1 by y, we get the required locus as : $4x = 9y^2 - 16y + 8$

Example : 44

Determine the points of maxima and minima of the function $f(x) = 1/8\ \ell n\ x - bx + x^2 + x^2$, $x > 0$, where $b \geq 0$ is a constant.

Solution

Consider $f(x) = 1/8\ \ell n\ x - bx + x^2$
$\Rightarrow$ $f'(x) = 1/8\ x - b + 2x = 0$
$\Rightarrow$ $16x^2 - 8bx + 1 = 0$
$\Rightarrow$ $x = \dfrac{b \pm \sqrt{b^2 - 1}}{4}$
For $0 \leq b < 1$ $f'(x) > 0$ for all x
$\Rightarrow$ f(x) is an increasing function
$\Rightarrow$ No local maximum or local minimum
For $b > 1$ $f'(x) = 0$ at $x_1 = \dfrac{b - \sqrt{b^2 - 1}}{4}$ and $x_2 = \dfrac{b + \sqrt{b^2 - 1}}{4}$
Check yourself that x_1 is a point of local maximum and x_2 is a point of local minimum.
For b = 1
$f'(x) = 16x^2 - 8x^2 + 1 = (4x - 1)^2 = 0$
$\Rightarrow$ $x = 1/4$
$f''(x) = 2(4x - 1)(4)$
$\Rightarrow$ $f''(1/4) = 0$
$f'''(x) = 32$ $\Rightarrow$ $f'''(1/4) \neq 0$
$\Rightarrow$ 1/4 is a point of inflexion
i.e. no local maxima or minima
So points of local maximum and minimum are :
$0 \leq b \leq 1$: No local maximum or minimum
$b > 1$: Local maximum at $x = \dfrac{b - \sqrt{b^2 - 1}}{4}$
Local minimum at $x - \dfrac{b + \sqrt{b^2 - 1}}{4}$

Example : 45

Let $f(x) = \begin{cases} xe^{ax} & , \quad x \leq 0 \\ x + ax^2 - x^3 & , \quad x > 0 \end{cases}$

Where a is a positive constant. Find the interval in which $f'(x)$ is increasing

Solution

Consider $x \leq 0$
$f'(x) = e^{ax}(1 + xa)$
$f''(x) = a\,e^{ax}(1 + xa) + e^{ax}a$
$\Rightarrow$ $f''(x) = e^{ax}(2a + xa^2) > 0$
$\Rightarrow$ $x > -2/a$ ($\because$ e^{ax} is always + ve)
So $f'(x)$ in increasing in $-2/a < x < 0$ ....(i)
Consider $x > 0$

$f'(x) = 1 + 2ax - 3x^2$

$f''(x) = 2a - 6x > 0 \quad \Rightarrow \quad x < a/3$

$\Rightarrow \quad f'(x)$ is increasing in $0 < x < a/3$ ....(ii)

From (i) and (ii), we can conclude that :

$f'(x)$ is increasing in $x \in (-2/a, 0) \cup (0, a/3)$

Example : 46

What normal to the curve $y = x^2$ forms the shortest chord?

Solution

Let (t, t^2) be any point P on the parabola $y = x^2$

Equation of normal at P to $y = x^2$ is :

$y - t^2 = -1/2t\,(x - t)$

Now assume that normal at P meets the curve again at Q whose coordinates are (t_1, t_1^2).

$\Rightarrow$ The point $Q(t_1, t_1^2)$ should satisfy equation of the normal

$\Rightarrow \quad t_1^2 - t^2 - 1/2t\,(t_1 - t)$

$\Rightarrow \quad t_1 + t = -1/2t \quad \Rightarrow \quad t_1 = -t - 1/2t$ ....(i)

$PQ^2 = (t - t_1)^2 + (t^2 - t_1^2)^2 = (t - r_1)^2\ [1 + (t_1 + t)^2]$

On substituting the value of t_1 from (i), we get ;

$$\Rightarrow \quad PQ^2 = \left(2t + \frac{1}{2t}\right)^2 \left(1 + \frac{1}{4t^2}\right) = 4t^2 \left(1 + \frac{1}{4t^2}\right)^3$$

Let $PQ^2 = f(t)$

$$\Rightarrow \quad f'(f) = 8t\left(1 + \frac{1}{4t^2}\right)^3 + 12t^2\left(1 + \frac{1}{4t^2}\right)^2 \left(\frac{-2}{4t^3}\right)$$

$$\Rightarrow \quad f'(t) = 2\left(1 + \frac{1}{4t^2}\right)^2 \left[4t\left(1 + \frac{1}{4t^2}\right) - \frac{3}{t}\right]$$

$f'(t) = 0 \quad \Rightarrow \quad 2t - 1/t = 0$

$\Rightarrow \quad t^2 = 1/2 \quad \Rightarrow \quad t = \pm 1/\sqrt{2}$

It is easy to see $f''(t) > 0$ for $t = \pm 1/\sqrt{2}$

equation of PQ :

for $t = 1\sqrt{2} \equiv \sqrt{2}\,x + 2y - 2 = 0$ and

for $t = -1\sqrt{2} \equiv \sqrt{2}\,x - 2y + 2 = 0$

Sets, Relations & Functions

Example : 1

Let A {1, 2, 3} and B = {4, 5}. Check whether the following subsets of A × B are functions from A to B or not.

(i) f_1 = {(1, 4), (1, 5), (2, 4), (3, 5)}   (ii) f_2 = {(1, 4), (2, 4), (3, 4)}

(iii) f_3 = {1, 4), (2, 5), (3, 5)}   (iv) f_4 = {(1, 4), (2, 5)}

Solution

(i) f_1 = {(1, 4), (1, 5), (2, 4), (3, 5)}
It is not a function since an element of domain
(i.e. 1) has two image in co-domain (i.e. 4, 5)

(ii) f_2 = {(1, 4), (2, 4), (3, 4)}
It is function as every element of domain has exactly
one image f(A) = Range = {4}

(iii) f_3 = {1, 4), (2, 5), (3, 5)}
It is a function. f(A) = Range = {4, 5} = co-domain

(iv) f_4 = {(1, 4), (2, 5)}
It is not a function because one element (i.e. 3)
in domain does no have an image

Example : 2

Which of the following is a function from A to B?

(i) A = {x | x > 0 and x ∈ R}, B = {y/y ∈ R}
(A is the set of positive reals numbers and B is the set of all real numbers)
f = {(x, y) / y = √x}

(ii) A = {x/x ∈ R} B{y/y ∈ R}
f = {(x, y) / y = √x}

Solution

(i) F is a function from A to B because every element of domain (+ve reals) has a unique image (square root) in codomain

(ii) f is not a function from A to B because – ve real nos. are present in domain and they do not have any image in codomain
(∵ y = √x is meaningless for –ve reals of x}

Example : 3

Check the following functions for injective and surjective

(i) f : R → R and f(x) = x^2

(ii) f : R → R^+ and f(x) = x^2

(iii) f : R^+ → R^+ and f(x) = x^2

Solution

(i) Injective

Let $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$

⇒ it is not necessary that $x_1 = x_2$

⇒ It is not injective

Surjective

y = x^2

⇒ x = ± √y for – ve values of y in codomain, there does not exist any value of x in domain

⇒ It is not surjecitve

(ii) Injective

Let $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2 \Rightarrow$ not injective

Surjective

y = x^2 ⇒ x = ± √y

As the codomain contains only positive real numbers, there exists some x for every values of y

⇒ it is surjecitve

(iii) Injective

$f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = x_2$ because domain contains only +ve reals

⇒ it is injective

Surjective

y = x^2 ⇒ x = ± √y

for +ve values of y, there exists some x, As codomains is R^+ , it is surjecitve

Example : 4

Let A = R – {3} and B = R – {1}

Let f : A → B be defined by $f(x) = \frac{x-2}{x-3}$

Is f bijective ?

Solution

Injective

Let $f(x_1) = f(x_2)$ where $x_1, x_2 \in A$

$\Rightarrow \quad \frac{x_1-2}{x_1-3} = \frac{x_2-2}{x_2-3} \quad \Rightarrow \quad (x_1 - 2)(x_2 - 3) = (x_2 - 2)(x_1 - 3) \quad$ (because $x_1, x_2 \neq 3$)

$\Rightarrow \quad x_1 = x_2$ (on simplification)

Hence f(x) is injective

Surjective

$y = \frac{x-2}{x-3}$

$\Rightarrow \quad y(x - 3) = x + 2$

$\Rightarrow \quad x = \frac{3y-2}{y-1}$

For $y \neq 1$, there exists some value of x, As the codomain does not contain 1, we have some value of x in domain for every value of y in codomain

⇒ it is surjecitve

Hence f(x) is bijective

Inverse of f(x)

Interchanging x and y in y = f(x) we have $x = \frac{y-2}{y-3} \quad \Rightarrow \quad y = \frac{3x-2}{x-1}$

$\Rightarrow \quad f^{-1}(x) = \frac{3x-2}{x-1}$ is the inverse of f(x)

Example : 5

Is f : R → R, f(x) = cos (5x + 2) invertible ?

Solution

Injecitve

Let $f(x_1) = f(x_2)$ where $x_1, x_2 \in R$

$\Rightarrow \quad \cos(5x_1 + 2) = \cos(5x_2 + 2)$

$\Rightarrow \quad 5x_1 + 2 = 2n\pi \pm (5x_2 + 2)$

⇒ it is not necessary that $x_1 = x_2$

hence if it not injective

Surjecitve

y = cos (5x + 2)

$\Rightarrow \quad x = \frac{\cos^{-1} y - 2}{5}$

⇒ there is no value of x for $y \in (-\infty, -1) \cup (1, +\infty)$

As this interval is included in codomain, there are some values of y in codomain for which there does no exist any value of x. Hence it is not surjecitve.

As f is neither injective nor surjective, it is not invertible.

Example : 6

(i) Let f(x) = x – 1 and g(x) = x^2 + 1.
What is fog and gof?

(ii) f = {(1, 2), (3, 5), (4, 1)} and g = {(2, 3), (5, 1), (1, 3)}
write down the pairs in the mappings fog.

Solution

(i) fog = f[g(x)] = f(x^2 + 1) = x^2 + 1 – 1 = x^2
gof = g[f(x)] = g[x – 1] = $(x – 1)^2$ + 1

(ii) domain of fog is the domain of g(x) i.e. {2, 5, 1}
fog (2) = f[g(2)] = f(3) = 5
fog (5) = f[g(5)] = f(1) = 2
fog (1) = f[g(1)] = f(3) = 5

$\Rightarrow$ fog = {2, 5), (5, 2), (1, 5)}

Example : 7

If A = $\left\{x : \frac{\pi}{6} \le x \le \frac{\pi}{3}\right\}$ and f(x) = cos x – x (1 + x). Find f(A).

Solution

We have to find the range with A as domain.
As f(x) is decreasing in the given domain

$$\frac{\pi}{6} \le x \le \frac{\pi}{3} \quad \Rightarrow \quad f\left(\frac{\pi}{6}\right) \ge f(x) \ge f\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \quad f(x) \in \left[\frac{1}{2} - \frac{\pi}{3} - \frac{\pi^2}{9}, \frac{\sqrt{3}}{2} - \frac{\pi^2}{36}\right]$$

$$\Rightarrow \quad \text{the range is the interval : } \left[\frac{1}{2} - \frac{\pi}{3} - \frac{\pi^2}{9}, \frac{\sqrt{3}}{6} - \frac{\pi}{6} - \frac{\pi^2}{36}\right]$$

Integration

Example : 1

Evaluate the following integrals

Hint : Express Integrals in terms of standard results :

(1) $\int \sec^2(2-3x)\ dx = \frac{-1}{3}\tan(2-3x) + C$

(2) $\int \frac{\sin(2-3x)}{\cos^2(2-3x)}\ dx = \int \sec(2-3x)\tan(2-3x)\ dx = \frac{1}{-3}\sec(2-3x) + C$

(3) $\int e^{2x-3}dx = \frac{1}{2}e^{2x-3} + C$

(4) $\int \sec(2-3x)\ dx = \frac{1}{-3}\log|\sec(2-3x) + \tan(2-3x)| + C$

(5) $\int \frac{1}{\sqrt{4x+1}}\ dx = \frac{1}{4}\left(2\sqrt{4x+1}\right) + C$

(6) $\int \frac{dx}{(1-2x)^3} = \frac{1}{-2}\left(\frac{-1}{2(1-2x)^2}\right) + C$

Example : 2

Evaluate the following integrals

Hint : Express numerator in terms of denominator

(1) $\int \frac{x-1}{x+1}\ dx = \int \frac{x+1-2}{x+1}\ dx = \int\left(1-\frac{2}{x+1}\right) dx = \int dx - 2\int \frac{dx}{x+1} = x - 2\log|x+1| + C$

(2) $\int \frac{x^2-1}{x^2+1}\ dx = \int \frac{x^2+1}{x^2+1}\ dx - \int \frac{2dx}{x^2+1} = x - 2\tan^{-1}x + C$

(3) $\int \frac{x}{(2x+1)^2}\ dx = \frac{1}{2}\int \frac{2x+1-1}{(2x+1)^2}\ dx = \frac{1}{2}\int \frac{dx}{(2x+1)} - \frac{1}{2}\int \frac{dx}{(2x+1)^2}$

$= \frac{1}{2}\left(\frac{1}{2}\log|2x+1|\right) - \frac{1}{2}\left[\frac{1}{2}\left(\frac{-1}{2x+1}\right)\right] + C$

(4) $\int \frac{x^4+1}{x^2+1}\ dx = \int \frac{x^4-1}{x^2+1}\ dx + \int \frac{2}{x^2+1}\ dx\ \int (x^2-1)\ dx + 2\int \frac{dx}{x^2+1} = x^3/3 - x + 2\tan^{-1}x + C$

(5) $\int \frac{x^7}{x+1}\ dx = \int \frac{x^7+1}{x+1}\ dx - \int \frac{dx}{x+1} = \int \frac{(x+1)(x^6-x^5+x^4-x^3+x^2-x+1)}{x+1}\ dx - \log|x+1|$

$= \frac{x^7}{7} - \frac{x^6}{6} + \frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x - \log|x+1| + C$

(6) $\int \frac{x^3}{(x+1)^2}\,dx = \int \frac{x^3+1}{(x+1)^2}\,dx - \int \frac{dx}{(x+1)^2} = \int \frac{x^2-x+1}{(x+1)}\,dx - \int \frac{dx}{(x+1)^2}$

$= \int \frac{x^2+x}{x+1}\,dx + \int \frac{1-2x}{x+1}\,dx - \int \frac{dx}{(x+1)^2} = |x\,dx - 2| \int \frac{x+1}{x+1}\,dx + \int \frac{3dx}{x+1} - \int \frac{dx}{(x+1)^2}$

$= \frac{x^2}{2} - 2x + 3\log|x+1| + \frac{1}{x+1} + C$

(7) $\int \frac{ax+b}{cx+d}\,dx = \frac{a}{c} \int \frac{(cx+d) - \left(d - \frac{bc}{a}\right)}{cd+d}\,dx$

$= \frac{a}{c} \int dx - \frac{a}{c} \frac{\left(d - \frac{bc}{a}\right)}{cx+d}\,dx = \frac{ax}{c} - \left(\frac{ad-bc}{c^2}\right) \log|cx + d| + C$

Example : 3

Evaluate the following integrals

Hint : Use $\int [f(x)]^n\ f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C$

(1) $\int \frac{(\sin^{-1}x)^3}{\sqrt{1-x^2}}\,dx = \frac{1}{4}(\sin^{-1}x)^4 + C$

(2) $\int \sec^4 x \tan x\,dx = \int \sec^3 x(\sec x \tan x)\,dx = \frac{\sec^4 x}{4} + C$

(3) $\int \frac{\log^n x}{x}\,dx = \frac{\log^{n+1} x}{n+1} + C$

(4) $\int \frac{x}{(x^2+1)^3}\,dx = \frac{1}{2} \int \frac{1}{(x^2+1)^3}\,2x\,dx = \frac{1}{2}\,\frac{(x^2+1)^{-2}}{-2} + C$

(5) $\int \sin^5 x \cos x\,dx = \frac{\sin^6 x}{6} + C$

Example : 4

Evaluate the following integrals

Hint : Use $\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + C$

(1) $\int \frac{x^3}{1+x^4}\,dx = \frac{1}{4} \int \frac{4x^3}{1+x^4}\,dx = \frac{1}{4}\log|1+x^4| + C$

(2) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx = \log |e^x + e^{-x}| + C$

(3) $\int \frac{e^x + 1}{e^x - 1}\, dx = \int \frac{e^{x/2} + e^{-x/2}}{e^{x/2} - e^{-x/2}} = 2 \int \frac{\frac{1}{2}e^{x/2} + \frac{1}{2}e^{-x/2}}{e^{x/2} - e^{-x/2}} = 2 \log |e^{x/2} - e^{-x/2}| + C$

(4) $\int \frac{x^3}{(x^2+1)}\, dx = \int \frac{x^3 + x}{(x^2+1)^2}\, dx - \int \frac{x\, dx}{(x^2+1)^2} = \int \frac{x}{x^2+1}\, dx - \int \frac{x}{(x^2+1)^2}\, dx$

$= \frac{1}{2} \int \frac{2x}{x^2 - 1}\, dx - \frac{1}{2} \int \frac{2x}{(x^2+1)^2}\, dx = \frac{1}{2} \log |x^2 + 1| - \frac{1}{2} \left(\frac{-1}{x^2+1} \right) + C$

(5) $\int \frac{dx}{a + be^x} = \int \frac{e^{-x}}{ae^{-x} + b}\, dx - \frac{1}{a} \int \frac{-ae^{-x}}{ae^{-x} + b}\, dx = -\frac{1}{a} \log |ae^{-x} + b| + C$

(6) $\int \frac{\tan x + 1}{\tan x - 1}\, dx = \int \frac{\sin x + \cos x}{\sin x - \cos x}\, dx = \log |\sin x - \cos x| + C$

(7) $\int \frac{\sec x}{\log(\sec x + \tan x)}\, dx = \log |\log (\sec x + \tan x)| + C$

Note that $\frac{d}{dx} \log (\sec x + \tan x) = \sec x$

(8) $\int \frac{x^2 - 1}{x(x^2+1)}\, dx \int \frac{1 - \frac{1}{x^2}}{x + \frac{1}{x}}\, dx = \log \left| x + \frac{1}{x} \right| + C$

(9) $\int \frac{dx}{x + \sqrt{x}} = \int \frac{dx}{\sqrt{x}\left(\sqrt{x} + 1\right)} = 2 \int \frac{\frac{1}{2\sqrt{x}}}{\left(\sqrt{x} + 1\right)}\, dx = 2 \log |\sqrt{x} + 1| + C$

(10) $\int \frac{x}{(x^4+1) \tan^{-1} x^2}\, dx = \frac{1}{2} \int \frac{\frac{2x}{1 + x^4}}{\tan^{-1} x}\, dx = \frac{1}{2} \log |\tan^{-1} x^2| + C$

(11) $\int \frac{dx}{x \log x \log \log x} = \int \frac{\frac{1}{x \log x}}{\log \log x}\, dx = \log |\log \log x| + C$

(12) $\int \frac{\sin 2x}{1 + \sin^2 x}\, dx = \log |1 + \sin^2 x| + C$

(13) $\int \frac{e^{x-1} + x^{e-1}}{e^x + x^e}\, dx = \frac{1}{e} \int \frac{e^x + ex^{e-1}}{e^x + x^e}\, dx = \frac{1}{e} \log |e^x + x^e| + C$

Example : 5

Evaluate

(1) $\int \sin^2 x\, dx$ **(2)** $\int \sin^3 x\, dx$ **(3)** $\int \sin^4 x\, dx$ **(4)** $\int \sin^4 x \cos^4 x\, dx$

Hint : Reduce the degree of integral and to one by transforming it into multiple angles of sine and cosine.

Solution

(1) $\int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2}\, dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + C$

(2) $\int \sin^3 x\, dx = \int \frac{3\sin x - \sin 3x}{4}\, dx = \frac{1}{4}\left[\int 3\sin x\, dx - \int \sin 2x\, dx\right] dx = \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right] + C$

$= \frac{-3}{4}\cos x + \frac{1}{12}\cos 3x + C$

(3) $\int \sin^4 x\, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 - 2\cos 2x)\, dx + \frac{1}{4}\int (\cos^2 2x\, dx)$

$= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{1}{8}\int (1 + \cos 4x)\, dx$

$= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{x}{8} + \frac{\sin 4x}{32} + C$

(4) $\int \sin^4 x \cos^4 x\, dx = \frac{1}{16}\int \sin^4 2x$. Now proceed on the pattern of $\int \sin^4 x\, dx$

Example : 6

Evaluate :

(1) $\int \frac{dx}{1+\sin x}$ **(2)** $\int \frac{dx}{1+\cos x}$

Solution

(1) $\int \frac{dx}{1+\sin x} = \int \frac{1-\sin x}{\cos^2 x}\, dx = \int \sec^2 x\, dx - \int \sec x \tan x\, dx = \tan x - \sec x + C$

Alternative Method

$\int \frac{dx}{1+\sin x} = \int \frac{dx}{1+\cos\left(\frac{\pi}{2} - x\right)} = \int \frac{dx}{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)} = \frac{1}{2}\int \sec^2\left(\frac{\pi}{4} - \frac{\pi}{2}\right) dx = \frac{1}{2}\, \frac{\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)}{-1/2} + C$

$= -\tan\left(\frac{x}{4} - \frac{x}{2}\right) + C$

(2) $\int \frac{dx}{1+\cos x} = \int \frac{1-\cos x}{\sin^2 x} = \int \text{cosec}^2 x\, dx - \int \text{cosec}\, x \cot x\, dx = -\cot x + \text{cosec}\, x + C$

Alternative Method

$\int \frac{dx}{1+\cos x} = \int \frac{dx}{2\cos^2 x/2} = \frac{1}{2}\int \sec^2 \frac{x}{2}\, dx = \frac{1}{2}\, \frac{\tan x/2}{1/2} + C = \tan\frac{x}{2} + C$

Example : 7

Evaluate :

(1) $\int \sin 2x \sin 3x \, dx$ **(2)** $\int \sin 2x \sin 4x \cos 5x \, dx$ **(3)** $\int \sin^2 x \cos^2 x \, dx$

Hint : Apply trigonometric formulas to convert product form of the integrand into sum of sines and cosines of multiple angle

Solution

(1) $\int \sin 2x \sin 3x \, dx = \frac{1}{2}\int 2\sin 2x \sin 3x \, dx = \frac{1}{2}\int (\cos x - \cos 5x) \, dx = \frac{1}{2} \sin x - \frac{1}{10} \sin 5x + C$

(2) $\int \sin 2x \cos 4x \cos 5x \, dx = \frac{1}{2}\int (2\sin 2x \cos 4x) \cos 5x \, dx$

$= \frac{1}{2}\int (\sin 6x - \sin 2x) \cos 5x \, dx = \frac{1}{4}\int 2\sin 6x \cos 5x \, dx - \frac{1}{4}\int 2\sin 2x \cos 5x \, dx$

$= \frac{1}{4}\int (\sin 11x + \sin x) \, dx - \frac{1}{4}\int (\sin 7x - \sin 3x) \, dx$

$= -\frac{1}{4}\frac{\cos 11x}{11} - \frac{1}{4}\cos x + \frac{1}{4}\frac{\cos 7x}{7} - \frac{1}{4}\frac{\cos 3x}{3}$

(3) $\int \sin^2 x \cos^2 x \, dx = \frac{1}{4}\int \sin^2 2x \, dx = \frac{1}{4}\int \frac{1 - \cos 4x}{2} \, dx = \frac{1}{8}\left(x - \frac{\sin 4x}{4}\right) + C$

Example : 8

Evaluate : $\int \frac{dx}{a\sin x + b\cos x}$

Solution

$$\int \frac{dx}{a\sin x + b\cos x} = \frac{1}{\sqrt{a^2+b^2}}\int \frac{dx}{\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x}$$

$= \frac{1}{\sqrt{a^2+b^2}}\int \frac{dx}{\sin x \cos\alpha + \cos x \sin\alpha}$ where $\alpha = \tan^{-1}(b/x)$

$= \frac{1}{\sqrt{a^2+b^2}}\int \text{cosec}(x+\alpha) \, dx$

$= \frac{1}{\sqrt{a^2+b^2}} \log|\text{cosec}(x+\alpha) - \cot(x+\alpha)| + C$ where $\alpha = \tan^{-1}(b/a)$

Example : 9

Evaluate $\int e^x (x+1)\cos(xe^x) \, dx$

Solution

The given integral is in terms of the variable x, we can simplify the integral by connecting it in the terms of another variable t using substitution

Here let us put $x e^x = t$

and hence $xe^x \, dx + e^x \, dx = dt$

$\Rightarrow$ $e^x (x + 1) \, dx = dt$

The given integral = $\int \cos(xe^x)[e^x(x+1)\,dx] = \int \cos t\,dt$ = sin t + C = sin (x e^x) + C

Note that the final result of the problem must be in terms of x.

Example : 10

Evaluate :

(1) $\int \frac{x^2}{1+x^6}\,dx$ **(2)** $\frac{dx}{x+\sqrt[3]{x}}$ **(3)** $\int \frac{x^2}{\sqrt{1+x}}\,dx$

Solution

(1) Let $x^3 = t$ $\Rightarrow$ $3x^2\,dx = dt$

$$\Rightarrow \int \frac{x^2 dx}{1+x^6} = \frac{1}{3}\int \frac{3x^2 dx}{1+x^6} = \frac{1}{3}\int \frac{dt}{1+t^2} = \frac{1}{3}\tan^{-1} t + C = \frac{1}{3}\tan^{-1} x^3 + C$$

(2) $\sqrt[3]{x}$ indicates that we should try $x = t^3$

$\Rightarrow$ $dx = 3t^2\,dt$

$$\Rightarrow \int \frac{dx}{x+\sqrt[3]{x}} = \int \frac{3t^2 dt}{t^3+t} = 3\int \frac{t\,dt}{t^2+1} = \frac{3}{2}\int \frac{2t\,dt}{t^2+1} = \frac{3}{2}\log|t^2+1| + C = \frac{3}{2}\log|x^{2/3}+1| + C$$

(3) Let $1 + x = t^2$ $\Rightarrow$ $dx = 2t\,dt$

$$\Rightarrow \int \frac{x^2}{\sqrt{1+x}}\,dx = \int \frac{(t^2-1)^2}{\sqrt{t^2}}\,2t\,dt = \int \frac{t^4+1-2t^2}{t}\,2t\,dt$$

$$= 2\,\frac{t^5}{5} + 2t - \frac{4t^3}{3} + C = \frac{2}{5}(1+x)^{5/2} + 2\sqrt{1+x} - \frac{4}{3}(1+x)^{3/2} + C$$

Example : 11

(1) $\int \frac{a^{x+\tan^{-1}a^x}}{a^{2x}+1}\,dx$ **(2)** $\int \sqrt{\sin\theta}\cos^3\theta\,d\theta$ **(3)** $\int \sqrt{\frac{x}{1-x^3}}\,dx$

Solution

(1) The given integral can be written as : $\int \frac{a^{x+\tan^{-1}a^x}}{a^{2x}+1}\,dx$

Let $\tan^{-1} a^x = t$

$$\Rightarrow \frac{1}{1+a^{2x}}\,a^x \log a\,dx = dt$$

$$\Rightarrow I = \int \frac{a^{\tan^{-1}a^x} a^x \log a\,dx}{\log a\,(1+a^{2x})} = \int \frac{a^t\,dt}{\log a}$$

$$\Rightarrow I = \frac{1}{\log a}\,\frac{a^t}{\log a} + C$$

$$\Rightarrow I = \frac{a^{\tan^{-1}a^x}}{(\log a)^2} + C$$

(2) Let $\sin\theta = t^2$ $\Rightarrow$ $\cos\theta\,d\theta = 2t\,dt$

$$\Rightarrow \int \sqrt{\sin\theta}\cos^3\theta\,d\theta = \int \sqrt{\sin\theta}\,(1-\sin^2\theta)\cos\theta\,d\theta$$

$= \int \sqrt{t^2}\ (1 - t^4)\ 2t\ dt = 2 \int (t^2 - t^6)\ dt$

$= \frac{2t^3}{3} - \frac{2t^7}{7} + C = \frac{2}{3}\ (\sin \theta)^{3/2} - \frac{2}{7}\ (\sin q)^{7/2} + C$

(3) The given integral is $I = \int \frac{\sqrt{x}\ dx}{\sqrt{1-x^3}}$

$\sqrt{x}$ appears in the derivative of $x^{3/2}$

hence, let $x^{3/2} = t \quad \Rightarrow \quad 3/2\ \sqrt{x}\ dx = dt$

$\Rightarrow \quad I = \frac{2}{3} \int \frac{\frac{3}{2}\sqrt{x}\ dx}{\sqrt{1-x^3}} = \frac{2}{3} \int \frac{dt}{\sqrt{1-t^2}} = \frac{2}{3} \sin^{-1} t + C = \frac{2}{3} \sin^{-1} x^{3/2} + C$

Example : 12

Evaluate the following integrals

(1) $\int \tan^2 x\ dx = \int (\sec^2 x - 1)\ dx = \tan x - x + C$

(2) $\int \tan^3 x\ dx = \int \tan x\ (\sec^2 x - 1)\ dx = \int \tan x \sec^2 x\ dx - \int \tan x\ dx = \frac{\tan^2 x}{2} - \log|\sec x| + C$

(3) $\int \tan^4 x\ dx = \int \tan^2 x\ (\sec^2 x - 1)\ dx = \int \tan^2 x\ (\sec^2 x\ dx) - \int \tan^2 x\ dx$

$= \frac{\tan^3 x}{3} - \int \sec^2 x\ dx + \int dx = \frac{\tan^3 x}{3} - \tan x + x + C$

(4) $\int \sec^4 x\ dx = \int \sec^2 x \sec^2 x\ dx = \int (1 + \tan^2 x)\ \sec^2 x\ dx$

$= \int (1 + t^2)\ dt = t + t^3/3 + C = \tan x = \frac{\tan^3 x}{3} + C$

Example : 13

Evaluate :

(1) $\int \sin^3 x \cos^4 x\ dx$ **(2)** $\int \sin^5 x\ dx$

Solution

(1) $\int \sin^3 x \cos^4 x\ dx = \int \sin^2 x \cos^4 x\ (\sin x\ dx)$

$= -\int (1 - t^2)\ t^4\ dt$ where $t = \cos x$

$= \frac{t^7}{7} - \frac{t^5}{5} + C = \frac{\cos 7x}{7} - \frac{\cos^5 x}{5} + C$

(2) $\int \sin^5 x\,dx = \sin^4 x \sin x\,dx = -\int (1-\cos^2 x)^2\ (-\sin x\,dx)$

$= -\int (1-t^2)^2\,dt$ where t = cos x

$= -\int (1+t^4-2t^2)\,dt\ \ -t-\dfrac{t^5}{5}+\dfrac{2t^3}{3}+C$

$= -\cos x - \dfrac{\cos^5 x}{5} + \dfrac{2}{3}\cos^3 x + C$

Example : 14

Type : $\int \dfrac{dx}{ax^2+bx+c}$

(1) $\int \dfrac{dx}{x^2+x+1} = \int \dfrac{dx}{x^2+2\frac{1}{2}x+\frac{3}{4}+\frac{1}{4}} = \int \dfrac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$

$= \dfrac{1}{\sqrt{3}/2}\tan^{-1}\left(\dfrac{x+1/2}{\sqrt{3}/2}\right) + = \dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right) + C$

(2) $\int \dfrac{dx}{1-4x-2x^2} = \dfrac{1}{2}\int \dfrac{dx}{1/2-(x^2+2x)} = \dfrac{1}{2}\int \dfrac{dx}{\left(\sqrt{3/2}\right)^2-(x+1)^2}$

$= \dfrac{1}{2}\ \dfrac{1}{2\sqrt{3/2}}\log\left|\dfrac{\sqrt{3/2}+x+1}{\sqrt{3/2}-(x+1)}\right| + C$

$= \dfrac{1}{2\sqrt{6}}\log\left|\dfrac{\sqrt{3}+\sqrt{2}x+\sqrt{2}}{\sqrt{3}-\sqrt{2}x-\sqrt{2}}\right| + C$

(3) $\int \dfrac{dx}{x^2+6x+1} = \int \dfrac{dx}{x^2+6x+9-8} = \int \dfrac{dx}{(x+3)^2-\left(2\sqrt{2}\right)^2} = \dfrac{1}{2\left(2\sqrt{2}\right)}\log\left|\dfrac{x+3-2\sqrt{2}}{x+3+2\sqrt{2}}\right| + C$

Example : 15

Type : $\int \dfrac{dx}{\sqrt{ax^2+bx+c}}$

(1) Let $I = \int \dfrac{dx}{\sqrt{ax^2+bx+c}}$

Treat $1-x-x^2$ as $1-(x+x^2) = 1-(x^2+x+1/4)+1/4 = 5/4-(x+1/2)^2$

$\Rightarrow\quad I = \int \dfrac{dx}{\sqrt{\frac{5}{4}-\left(x+\frac{1}{2}\right)^2}} = \sin^{-1}\left(\dfrac{x+1/2}{\sqrt{5}/2}\right) = \sin^{-1}\left(\dfrac{2x+1}{\sqrt{5}}\right) + C$

Let $I = \int \frac{dx}{\sqrt{2x^2+6x+2}}$

Now $2x^2 + 6x + 2 = 2(x^2 + 3x + 1) = 2\left(x^2+\frac{6x}{2}+\frac{9}{4}-\frac{9}{4}+1\right) = 2\left[\left(x+\frac{3}{2}\right)^2-\frac{5}{4}\right]$

This is in the form $x^2 - a^2$.

$\Rightarrow \quad I = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(x+3/2)^2-5/4}} = \frac{1}{\sqrt{2}} \log \left| x+\frac{3}{2}+\sqrt{(x+3/2)^2-5/4} \right| + C$

Example : 16

Type : $\int \sqrt{ax^2+bx+c}\ dx$

Let $I = \int \sqrt{ax^2+5x+1}\ dx = \int \sqrt{(x+5/2)^2-21/4}\ dx$

$= \frac{x+5/2}{2} \sqrt{x^2+5x+1} - \frac{21}{8} \log \left|x+5/2+\sqrt{(x+5/2)^2-21/4}\right| + C$

$= \frac{2x+5}{4} \sqrt{x^2+5x+1} - \frac{21}{8} \log \left| x+5/2+\sqrt{x^2+5x+1} \right| + C$

Example : 17

Evaluate $\int \frac{3x+1}{\sqrt{x^2+4x+1}}\ dx$

Solution

The linear expression in the numerator can be expressed as $3x + 1 = \ell\ d/dx\ (x^2 + 4x + 1) + m$

$\Rightarrow \quad 3x + 1 = \ell\,(2x + 4) + m$

comparing the coefficients of x and x^0,

$3 = 2\ell$ and $1 = 4\ell + m$

$\Rightarrow \quad \ell = 3/2$ and $m = -5$

$\Rightarrow \quad I = \int \frac{3x+1}{\sqrt{x^2+4x+1}} = \int \frac{3/2(2x+4)-5}{\sqrt{x^2+4x+1}}\ dx = \frac{3}{2} \int \frac{2x+4}{\sqrt{x^2+4x+1}} - 5 \int \frac{dx}{\sqrt{x^2+4x+1}}$

Let $I_1 = \frac{3}{2} \int \frac{2x+4}{\sqrt{x^2+4x+1}} = \frac{3}{2} \int \frac{dt}{\sqrt{t}}$ (where $t = x^2 + 4x + 1$)

$= 3\sqrt{t} + C = 3\sqrt{x^2+4x+1} + C$

Let $I_2 = 5 \frac{dx}{\sqrt{x^2+4x+1}} = 5 \int \frac{dx}{\sqrt{(x+2)^2-3}} = 5 \log \left| x+2+\sqrt{(x+2)^2-3} \right| + C$

$\Rightarrow \quad I = I_1 - I_2 = 3\sqrt{x^2+4x+1} - 5 \log \left| x+2+\sqrt{x^2+4x+1} \right| + C$

Example : 18

Evaluate : $\int \frac{x^2 - x + 1}{2x^2 + x + 2}\, dx$

Solution

Express numerator in terms of denominator and its derivative

Let $x^2 - x + 1 = \ell\,(2x^2 + x + 2) + m\,(4x + 1) + n$

$\Rightarrow \quad 1 = 2\ell - 1 = \ell + 4m \qquad 1 = 2\ell + m + n$

$$\Rightarrow \quad 1 = \int \frac{x^2 - x + 1}{2x^2 + x + 2}\, dx = \int \frac{1/2(2x^2 + x + 2) - 3/8(4x + 1) + 3/8}{2x^2 + x + 2}\, dx$$

$$= \frac{1}{2} \int dx - \frac{3}{8} \int \frac{4x + 1}{2x^2 + x + 2}\, dx + \frac{3}{8} \int \frac{dx}{2x^2 + x + 2}$$

$$= \frac{x}{2} - \frac{3}{8} \log|2x^2 + x + 2| + \frac{3}{8}\, I_1 \qquad \text{where } I_1 = \int \frac{dx}{2x^2 + x + 2}$$

$$= \frac{1}{2} \int \frac{dx}{x^2 + 1/2x + 1/16 - 1/16 + 1} + \frac{1}{2} \int \frac{dx}{(x + 1/4)^2 + 15/16}$$

$$= \frac{1}{2}\, \frac{1}{\sqrt{15/4}}\, \tan^{-1}\left(\frac{x + 1/4}{\sqrt{15/4}}\right) + C = \frac{2}{\sqrt{15}}\, \tan^{-1}\left(\frac{4x + 1}{\sqrt{15}}\right) + C$$

$$= \frac{x}{2} - \frac{3}{8} \log|2x^2 + x + 2| + \frac{3}{4\sqrt{15}}\, \tan^{-1}\left(\frac{4x + 1}{\sqrt{15}}\right) + C$$

Example : 20

$$\int \frac{dx}{3\sin^2 x + 4\cos^2 x}$$

Solution

$$\int \frac{dx}{3\sin^2 x + 4\cos^2 x} = \int \frac{\sec^2 x}{3\tan^2 x + 4}\, dx = \int \frac{dt}{3t^2 + 4} \qquad \text{where } t = \tan x$$

$$= \frac{1}{3} \int \frac{dt}{t^2 + (2/\sqrt{3})^2} = \frac{1}{3}\, \frac{1}{2/\sqrt{3}}\, \tan^{-1}\left(\frac{t}{2/\sqrt{3}}\right) + C$$

$$= \frac{2}{2\sqrt{3}}\, \tan^{-1}\left(\frac{\sqrt{3}}{2} \tan x\right) + C$$

Example : 21

Evaluate :

(1) $\int \frac{dx}{4 + 5\sin x}$ **(2)** $\int \frac{dx}{a + b\cos x}$ where a, b > 0

Solution

(1) $I = \int \frac{dx}{4 + 5\sin x}$

Put $\tan \frac{x}{2} = t \quad \Rightarrow \quad x = 2\tan^{-1} t$

$$\Rightarrow \quad \cos x = \frac{1 - t^2}{1 + t^2}\ ; \ \sin x = \frac{2t}{1 + t^2}\ ; \ dx\ \frac{2dt}{1 + t^2}$$

$$\Rightarrow \quad I = \int \frac{\frac{2dt}{1+t^2}}{4+5\left(\frac{2t}{1+t^2}\right)} = \int \frac{2dt}{4t^2+10t+4} = \frac{1}{2}\int \frac{dt}{t^2+5/2t+1}$$

$$= \frac{1}{2}\int \frac{dt}{(t+5/4)^2-9/16} = \frac{1}{2}\;\frac{1}{2\times 3/4}\log\left|\frac{t+5/4-3/4}{t+5/4+3/4}\right| + C$$

$$= \frac{1}{3}\log\left|\frac{4t+2}{4t+8}\right| + C = \frac{1}{3}\log\left|\frac{2\tan\frac{x}{2}+1}{2\tan\frac{x}{2}+4}\right| + C$$

(2) $\int \frac{dx}{a+b\cos x}$ where a, b > 0

Let tan x/2 = t

$$\Rightarrow \quad I = \int \frac{dx}{a+b\cos x} = \int \frac{\frac{2dt}{1+t^2}}{a+b\left(\frac{1-t^2}{1+t^2}\right)} \qquad \Rightarrow \quad \int \frac{2dt}{(a+b)+(a-b)\,t^2}$$

Case – 1 Let a = b

$$\Rightarrow \quad I = \int \frac{2dt}{a+b} = \frac{2t}{a+b} = \frac{2}{a+b}\tan\frac{x}{2} + C$$

Case – 2 Let a > b

$$\Rightarrow \quad I = \frac{1}{a-b}\int \frac{2dt}{\frac{a+b}{a-b}+t^2} = \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\tan^{-1}\left(t\sqrt{\frac{a-b}{a+b}}\right) + C$$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\tan\frac{x}{2}\sqrt{\frac{a-b}{a+b}}\right) + C$$

Case – 3 Let a < b

$$\Rightarrow \quad I = \int \frac{2dt}{(a+b)-(b-a)\,t^2} = \frac{2}{b-a}\int \frac{dt}{\frac{b+a}{b-a}-t^2}$$

$$= \frac{2}{b-a}\;\frac{\sqrt{b-a}}{\sqrt{b+a}}\log\left|\frac{\sqrt{\frac{b+a}{b-a}}+t}{\sqrt{\frac{b+a}{b-a}}-t}\right| + C$$

$$= \frac{1}{\sqrt{b^2-a^2}}\log\left|\frac{\sqrt{b+a}+\sqrt{b-a}\tan\frac{x}{2}}{\sqrt{b+a}-\sqrt{b-a}\tan\frac{x}{2}}\right| + C$$

Example : 22

Evaluate : $\int \frac{2\sin x + 3\cos x}{\sin x + 4\cos x}\, dx$

Solution

Express numerator as the sum of denominator and its derivative

Let 2 sin x + 3 cos x = ℓ (sin x + 4 cos x) + m (cos x – 4 sin x)

comparing coefficients of sin x and cos x

$2 = \ell - 4m, \qquad 3 = 4\ell + m$

$\Rightarrow \quad \ell = 14/17 \qquad m = -5/17$

$\Rightarrow \quad I = \int \frac{2\sin x + 3\cos x}{\sin x + 4\cos x}\, dx$

$\Rightarrow \quad I = \frac{14}{17} \int \frac{\sin x + 4\cos x}{\sin x + 4\cos x}\, dx - \frac{5}{17} \int \frac{\cos x - 4\sin x}{\sin x + 4\cos x}\, dx$

$\Rightarrow \quad I = \frac{14}{17}\, x - \frac{5}{17} \log\ |\sin x + 4\cos x) + C$

Example : 23

Evaluate

(1) $\int \frac{x^2+1}{x^4+1}\, dx$ **(2)** $\int \frac{x^2-1}{x^4+1}\, dx$

Solution

(1) Let $I_1 = \int \frac{x^2+1}{x^4+1}\, dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\, dx = \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}\, dx = \int \frac{dt}{t^2+2}$

where $t = x - \frac{1}{x} = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right) + C$

(2) Let $I_2 = \int \frac{x^2-1}{x^4+1}\, dx = \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\, dx = \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}\, dx$

Let $I_2 = \int \frac{dt}{t^2-2}$ where $t = x + \frac{1}{x} = \frac{2}{2\sqrt{2}} \log\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| + C$

$= \frac{1}{2\sqrt{2}} \log\left|\frac{x^2 - x\sqrt{2}+1}{x^2+x\sqrt{2}+1}\right| + C$

Example : 24

Evaluate : $\int \frac{x^2-1}{(x^2+1)\sqrt{x^4+1}}\,dx$

Solution

The given integral is

$$I = \int \frac{1-\frac{1}{x^2}}{\frac{x^2+1}{x}\sqrt{\frac{x^4+1}{x^2}}}\,dx = \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)\sqrt{x^2+\frac{1}{x^2}}}\,dx = \int \frac{dt}{t\sqrt{t^2-2}} \qquad \text{where } x+\frac{1}{x} = t$$

$$\Rightarrow \qquad I = \frac{1}{\sqrt{2}}\sec^{-1}\left|\frac{t}{\sqrt{2}}\right| + C = \frac{1}{\sqrt{2}}\sec^{-1}\left|\frac{x^2+1}{x\sqrt{2}}\right| + C$$

Example : 25

Evaluate : $\int x\cos x\,dx$

Solution

$$I = \int \underset{\text{part 1}}{x}\ \underset{\text{part 2}}{\cos dx} = x\int \cos x\,dx - \int\left[\int \cos x\,dx\right]dx$$

$$\Rightarrow \qquad I = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$$

Example : 26

Study the following examples carefully

(1) $\int x\sec^2 x\,dx \quad = x\int \sec^2 x\,dx - \int \tan x\,dx = x\tan x - \log|\sec x| + C$

(2) $\int \sin^{-1} dx \quad = \sin^{-1}\int dx - \int x\frac{1}{\sqrt{1-x^2}}\,dx = x\sin^{-1}x - \frac{1}{2}\int \frac{2x}{\sqrt{1-x^2}}\,dx$

$= x\sin^{-1}x + \frac{1}{2}\int \frac{dt}{\sqrt{t}} \qquad \text{where } I - x^2 = t$

$= x\sin^{-1}x + \frac{1}{2}\,2\sqrt{t} + C = x\sin^{-1}x + \sqrt{1-x^2} + C$

(3) $\int \tan^{-1}x\,dx \quad = \tan^{-1}\int dx - \int \frac{x}{1+x^2}\,dx$

$= x\tan^{-1}x - \frac{1}{2}\log|1+x^2| + C$

(4) $\int x\,e^x\,dx \quad = x\int e^x\,dx - \int e^x\,dx = x\,e^x - e^x + C$

(5) $\int \log x\,dx \quad = \log x\int dx - \int x\,\frac{1}{x}\,dx = x\log x - x + C$

(6) $\int x^2 \sin x\, dx = x^2 \int \sin x\, dx - \int (-\cos x)\, 2x\, dx$

$= -x^2 \cos x + 2 \int x \cos x\, dx$

$= -x^2 \cos x + 2 \left[x \int \cos x\, dx - \int \sin x\, dx \right]$

$= -x^2 \cos x + 2x \sin x + 2 \cos x + C$

Example : 27

Evaluate : $\int x \sin^{-1} dx$

Solution

$$\int x \sin^{-1} dx = \sin^{-1} x \int x\, dx - \int \frac{x^2 dx}{2\sqrt{1-x^2}} = \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{1-x^2-1}{\sqrt{1-x^2}}\, dx$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \sqrt{1-x^2}\, dx - \frac{1}{2} \int \frac{dx}{\sqrt{1-x^2}}$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x \right] - \frac{1}{2} \sin^{-1} x + C$$

Example : 28

Evaluate : $e^x \sin x\, dx$

Solution

Let $I = \int e^x \sin x\, dx$

$\Rightarrow \quad I = \int \sin x\, e^x dx = \sin x \int e^x\, dx - \int e^x\, [\cos x\, dx]$

$\Rightarrow \quad I = e^x \sin x - \int \cos x\, e^x\, dx$

$\Rightarrow \quad I = e^x \sin x - [\cos x \int e^x dx - \int e^x\, (-\sin x\, dx]$

$\Rightarrow \quad I = e^x (\sin x - \cos x) - \int e^x \sin x\, dx$

$\Rightarrow \quad I = e^x (\sin x - \cos x) - I$

$\Rightarrow \quad I + I = e^x (\sin x - \cos x)$

$\Rightarrow \quad I = e^x/2\, (\sin x - \cos x) + C$

Example : 29

Evaluate : $\int \sec^3 x\, dx$

Solution

Let $I = \int \sec^3 x\, dx = \int \sec x \sec^2 x\, dx = \sec x \int \sec^2 x - \int \tan x\, (\sec x \tan x)\, dx$

$\Rightarrow \quad I = \sec x \tan x - \int \sec x\, (\sec^2 x - 1)\, dx = \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$

$\Rightarrow \quad I = \sec x \tan x - I + \log |\sec x + \tan x|$

$\Rightarrow \quad I = 1/2\, [\sec x \tan x + \log| \sec x + \tan x| + C$

Example : 30

Evaluate : **(1)** $\int e^x\left[\frac{2+\sin 2x}{1+\cos 2x}\right] dx$ **(2)** $\int \frac{xe^x}{(1+x)^2}\, dx$

(3) $\int \frac{e^x(x^2+1)}{(x+1)^2}\, dx$ **(4)** $\int \left[\log(\log x)+\frac{1}{\log x}\right] dx$

Solution

(1) $I = \int e^x\left[\frac{2+\sin 2x}{1+\cos 2x}\right] dx$

$\Rightarrow \quad I = \int e^x\left[\frac{2}{1+\cos 2x}+\frac{\sin 2x}{1+\cos 2x}\right] dx$

$\Rightarrow \quad I = e^x\left[\frac{2}{2\cos^2 x}+\frac{2\sin x\cos x}{2\cos^2 x}\right] dx = \int e^x \ [\sec^2 + \tan x]\ dx$

$\Rightarrow \quad I = \int e^x\ [\tan x+\sec^2 x]\ dx = e^x \tan x + C$

(2) $I = \int \frac{xe^x}{(1+x)^2}\, dx = \left[\frac{1+x-1}{(1+x)^2}\right] dx$

$\Rightarrow \quad I = \int e^x\left[\frac{1}{1+x}-\frac{1}{(1+x)^2}\right] dx = e^x\left(\frac{1}{1+x}\right) + C$

(3) $I = \int \frac{e^x(x^2+1)}{(x+1)^2}\, dx = \int\left[\frac{x^2-1}{(x+1)^2}+\frac{2}{(x+1)^2}\right] dx$

$\Rightarrow \quad I = \int e^x\left[\frac{x-1}{x+1}+\frac{2}{(x+1)^2}\right] dx$

We now se that $\frac{d}{dx}\left(\frac{x-1}{x+1}\right) = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$

$\Rightarrow \quad I = e^x\left[\frac{x-1}{x+1}\right] + C$

(4) $I = \int\left[\log(\log x)+\frac{1}{\log x}\right] dx$

Substitute log x = 1 $\Rightarrow$ $x = e^t$ and $dx = e^t\, dt$

$\Rightarrow \quad = \int \left(\log t + \frac{1}{t}\right) e^t\, dt = e^t \log t + C$

$\Rightarrow \quad = e^{\log x} \log\log x + C = x \log\log x + C$

Example : 31

Evaluate : $\int \frac{x^2\,dx}{(x-1)\,(2x+3)}$

Solution

Let I = $\int \frac{x^2 dx}{(x-1)(2x+3)}$

The degree of numerator is not less than the degree of denominator. Hence we divide N by D.

$$\frac{x^2}{(x-1)(2x+3)} = \text{quotient} + \frac{\text{remainder}}{(x-1)(2x+3)} = \frac{1}{2} + \frac{-\frac{1}{2}x+\frac{3}{2}}{(x-1)(2x+3)} = \frac{1}{2} + \frac{1}{2}\ \frac{3-x}{(x-1)(2x+3)}$$

We now split $\frac{3-x}{(x-1)(2x+3)}$ in two partial fractions.

Let f(x) = $\frac{3-x}{(x-1)(2x+3)} = \frac{A}{x-1} + \frac{B}{2x+3}$ where A and B are constants.

Equating the numerators on both sides :

3 – x = A (2x + 3) + B (x – 1)

Now there are two ways to calculate A and B.

1. Comparing the coefficients of like terms

2. Substituting the appropriate values of x.

Method 1 :

Comparing the coefficients of x and x^0, we get :

– 1 = 2A + B and 3 = 3A – B

On solving we have a = 2/5 B = –9/5

Method 2 :

In 3 – x = A(2x + 3) + B (x – 1), pur x = 1, – 3/2

x = 1 ⇒ 3 – 1 = 5A ⇒ A = 2/5

x = –3/2 ⇒ 3 + 3/2 = B (–3/2 – 1) ⇒ B = –9/5

Hence finally we have :

$$f(x) = \frac{\frac{2}{5}}{x-1} + \frac{-\frac{9}{5}}{2x+3}$$

$$\Rightarrow \quad I = \int \left[\frac{1}{2} + \frac{1}{2} f(x)\right] dx$$

$$\Rightarrow \quad I = \frac{x}{2} + \frac{1}{2} \int \frac{\frac{2}{5}}{x-1}\,dx + \frac{1}{2} \int \frac{-\frac{9}{5}}{2x+3}\,dx$$

$$\Rightarrow \quad \frac{x}{2} + \frac{1}{5} \log|x-1| - \frac{9}{20} \log|2x+3| + C$$

Example : 32

Evaluate : $\int \frac{(x-1)\,dx}{(2x+1)(x-2)(x-3)}$

Solution

Let $f(x) = \frac{x-1}{(2x+1)(x-2)(x-3)} = \frac{A}{2x+1} + \frac{B}{x-2} + \frac{C}{x-3}$

$\Rightarrow \quad A = \left.\frac{x-1}{(x-2)(x-3)}\right]_{x=-\frac{1}{2}} = -\frac{6}{35}$

$\Rightarrow \quad B = \left.\frac{x-1}{(2x+1)(x-3)}\right]_{x=2} = -\frac{1}{5}$

$\Rightarrow \quad C = \left.\frac{x-1}{(2x+1)(x-2)}\right]_{x=3} = \frac{2}{7}$

$\Rightarrow \quad \int f(x)\,dx = \frac{-6}{35}\int \frac{dx}{2x+1} - \frac{1}{5}\int \frac{dx}{x-2} + \frac{2}{7}\int \frac{dx}{x-3}$

$= -\frac{3}{35}\log|2x+1| - \frac{1}{5}\log|x-2| + \frac{2}{7}\log|x-3| + C$

Example : 33

Evaluate : $\int \frac{(\cos\theta+1)\sin\theta}{(\cos\theta-1)^2(\cos\theta-3)}$

Solution

Let $\cos\theta = x \quad \Rightarrow \quad -\sin\theta\,d\theta = dx$

$\Rightarrow \quad I = -\int \frac{x+1}{(x-1)^2(x-3)}\,dx$

Let $f(x) = \int \frac{x+1}{(x-1)^2(x-3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-3}$

Equating numerator on both sides,

$\Rightarrow \quad x + 1 = A(x-1)(x-3) + B(x-3) + C(x-1)^2$

By taking x = 1, we get B = – 1

By taking x = 3, we get C = 1

Comparing the coefficient of x^2 , we get,

$0 = A + C \quad \Rightarrow \quad 0 = A + 1 \quad \Rightarrow \quad A = -1$

$\Rightarrow \quad I = -\int f(x)\,dx = -\left\{\int \frac{-1}{x-1}dx + \int \frac{-1}{(x-1)^2}dx + \int \frac{1}{x-3}dx\right\}$

$\Rightarrow \quad I = \log|x-1| - \frac{1}{x-1} - \log|x-3| + C$

$\Rightarrow \quad I = \log\left|\frac{x-1}{x-3}\right| - \frac{1}{x-1} + C$

$\Rightarrow \quad I = \log\left|\frac{\cos\theta-1}{\cos\theta-3}\right| - \frac{1}{\cos\theta-1} + C$

Example : 34

Evaluate : $\int \frac{dx}{x^3+1}$

Solution

Let $\quad f(x) = \frac{1}{x^3+1} =$

$\Rightarrow \quad f(x) = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$

$\Rightarrow \quad 1 = a\,(x^2 - x + 1) + (Bx + C)\,(x + 1)$

Comparing the coefficients of x^2 , x, x^0 :

$0 = A + B, \qquad 0 = -A + B + C \quad 1 = A + C$

$\Rightarrow \quad A = 1/3 C = 2/3 \qquad B = -1/3$

$\Rightarrow \quad f(x) = \frac{\frac{1}{3}}{x+1} + \frac{-\frac{x}{3}+\frac{2}{3}}{x^2-x+1}$

Let $\quad I_1 = \frac{1}{3} \int \frac{dx}{x+1} = \frac{1}{3} \log|x+1| + C_1$

Let $\quad I_2 = \int \frac{-\frac{1}{3}x+\frac{2}{3}}{x^2-x+1}\,dx = \frac{1}{3} \int \frac{2-x}{x^2-x+1}\,dx$

Express the numerator in terms of derivative of denominator.

$\Rightarrow \quad I_2 = -\frac{1}{6} \int \frac{2x-4}{x^2-x+1}\,dx$

$\Rightarrow \quad I_2 = -\frac{1}{6} \int \frac{2x-1}{x^2-x+1}\,dx + \frac{1}{2} \int \frac{dx}{x^2-x+1}$ $\frac{1}{(x+1)(x^2-x+1)}$

$\Rightarrow \quad I_2 = -\frac{1}{6} \log|x^2 - x + 1| + \frac{1}{2} \int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}}$

$\Rightarrow \quad I_2 = -\frac{1}{6} \log|x^2 - x + 1| + \frac{2}{2\sqrt{3}} \tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C_2$

$\Rightarrow \quad I_2 = -\frac{1}{6} \log|x^2 - x + 1| + \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C_2$

$\Rightarrow \quad \int \frac{dx}{x^3+1} = \int f(x)\,dx = I_1 + I_2 = \frac{1}{3} \log|x+1| - \frac{1}{6} \log|x^2 - x + 1| + \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$

$\frac{1}{3} \log\left|\frac{x+1}{\sqrt{x^2-x+1}}\right| + \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$

Example : 35

Evaluate : $\int \frac{x^2 dx}{x^4 - 1}$

Solution

$$\int \frac{x^2 dx}{x^4 - 1} = \int \frac{x^2\ dx}{(x^2-1)(x^2+1)}$$

$$\frac{x^2}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

As the function contains terms of x^2 only, substitute $x^2 = t$ and then make partial fractions

$$\frac{t}{(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1} \quad \Rightarrow \quad t = A\ (t + 1) + B\ (t - 1)$$

Put t = ± 1 to get A = 1/2, B = 1/2

$$\Rightarrow \quad \frac{t}{(t-1)(t+1)} = \frac{\frac{1}{2}}{t-1} + \frac{\frac{1}{2}}{t+1}$$

Convert $t = x^2$ again before integrating

$$\Rightarrow \quad I = \int \frac{x^2 dx}{(x^2-1)(x^2+1)} = \int \frac{1/2}{(x^2-1)}\ dx + \int \frac{1/2}{x^2+1}\ dx$$

$$= \frac{1}{2}\,\frac{1}{2} \log \left| \frac{x-1}{x+1} \right| + \frac{1}{2} \tan^{-1} x + C$$

Example : 36

Evaluate $\int \frac{dx}{(x+1)\sqrt{x+2}}$

Solution

Let $I = \int \frac{dx}{(x+1)\sqrt{x+2}}$

Substitute : $x + 2 = t^2 \quad \Rightarrow \quad dx = 2t\ dt$

$$\Rightarrow \quad \int \frac{dx}{x+1\left(\sqrt{x+2}\right)} = \int \frac{2t\ dt}{(t^2-1)\sqrt{t^2}} = 2 \int \frac{dt}{t^2-1} = \frac{2}{2} \log \left| \frac{t-1}{t+1} \right| + C = \log \left| \frac{\sqrt{x+2}-1}{\sqrt{x+2}+1} \right| + C$$

Example : 37

Evaluate : $\int \frac{x}{(x^2-3x+2)\sqrt{x-1}}\ dx$

Solution

Let $x - 1 = t^2 \quad \Rightarrow \quad dx = 2t\ dt$

$$\Rightarrow \quad I = \int \frac{(t^2+1)}{(t^2+1)^2 - 3(t^2+1) + 2}\ \frac{2t\ dt}{\sqrt{t^2}} = 2 \int \frac{(t^2+1)\ dt}{t^4 - t^4}$$

$$\Rightarrow \quad I = 2\int \frac{t^2+1}{t^2(t^2-1)}\,dt = \int\left(\frac{2}{t^2-1} - \frac{1}{t^2}\right)dt = 4\int \frac{dt}{t^2-1} - 2\int \frac{dt}{t^2}$$

$$\Rightarrow \quad I = \frac{4}{2}\log\left|\frac{t-1}{t+1}\right| + \frac{2}{t} + C$$

$$\Rightarrow \quad I = 2\log\left|\frac{\sqrt{x-1}-1}{\sqrt{x-1}+1}\right| + \frac{2}{\sqrt{x-1}} + C$$

Example : 38

Evaluate : $\int \frac{dx}{(x^2+1)\sqrt{x^2+2}}$

Solution

Let $I = \int \frac{dx}{(x^2+1)\sqrt{x^2+2}}$

Substitute : $x = \frac{1}{t} \quad \Rightarrow \quad dx = \frac{1}{t^2}\,dt$

$$\Rightarrow \quad I = \int \frac{-\frac{1}{t^2}dt}{\left(\frac{1}{t^2}+1\right)\sqrt{\frac{1}{t^2}+2}} = \int \frac{-t\,dt}{(1+t^2)\sqrt{1+2t^2}}$$

Let $\quad 1 + 2t^2 = z^2 \quad \Rightarrow \quad 4t\,dt = 2z\,dz$

$$\Rightarrow \quad I = \frac{-1}{2}\int \frac{z\,dz}{\left(1+\frac{z^2-1}{2}\right)\sqrt{z^2}} = \int \frac{dz}{z^2+1} = -\tan^{-1} z + C$$

$$\Rightarrow \quad I = -\tan^{-1}\sqrt{1+2t^2} + C = -\tan^{-1}\sqrt{1+\frac{2}{x^2}} + C$$

Example : 39

Evaluate : $\int \frac{dx}{(x+2)\sqrt{x^2+6x+7}}$

Solution

Let $I = \int \frac{dx}{(x+2)\sqrt{x^2+6x+7}}$

Substitute : $x + 2 = \frac{1}{t} \quad \Rightarrow \quad dx = -\frac{1}{t^2}$

$$\Rightarrow \quad x^2 + 6x + 7 = \left(\frac{1}{t}-2\right)^2 + 6\left(\frac{1}{t}-2\right) + 7 = \frac{1+2t-t^2}{t^2}$$

$\Rightarrow \quad I = \int \frac{dt}{\sqrt{2-(t-1)^2}} = -\sin^{-1}\left(\frac{t-1}{\sqrt{2}}\right) + C$

$\Rightarrow \quad I = \sin^{-1}\left[\frac{x+1}{(x+2)\sqrt{2}}\right] + C$

Example : 40

Evaluate : $\int \frac{dx}{\sqrt[3]{x+1}+\sqrt{x+1}}$

Solution

Let $\quad I = \int \frac{dx}{\sqrt[3]{x+1}+\sqrt{x+1}} \quad \Rightarrow \quad I = \int \frac{dx}{(x+1)^{1/3}+(x+1)^{1/2}}$

The least common multiple of 2 and 3 is 6

So substitute x + 1 = t^6 $\Rightarrow$ dx = $6t^5$ dt

$\Rightarrow \quad I = \int \frac{6t^5 dt}{t^2+t^3} = 6\int \frac{t^3 dt}{1+t}$

$\Rightarrow \quad I = 6\int \left(t^2 - t + 1 - \frac{1}{1+t}\right) dt$

$\Rightarrow \quad I = 6\left(\frac{t^3}{3} - \frac{t^2}{2} + t - \log(t+1)\right) + C$

On substituting t = $(1 + x)^{1/6}$, we get

$I = 6\left(\frac{(1+x)^{1/2}}{3} - \frac{(1+x)^{1/3}}{2} + (1+x)^{1/6} - \log\left((x+1)^{1/6}+1\right)\right) + C$

Example : 41

Evaluate : $\int x^{13/2}(1+x^{5/2})^{1/2}\ dx$

Solution

Let $\quad I = \int x^{13/2}(1+x^{5/2})^{1/2}\ dx$

Comparing with integral of type 5.6, we can see that p = 1/2 which is not an integer.

So this integral does not belong to type 5.6 (i).

Check the sign of (m + 1)/n

$\frac{m+1}{n} = \frac{\frac{13}{2}+1}{\frac{5}{2}} = \frac{15}{5} = 3 \quad \Rightarrow$ (m + 1)/n is an integer. So this integral belongs to type 5.6 (ii)

To solve this integral, substitute 1 + $x^{5/2}$ = t^2

$\Rightarrow \quad 5/2\ x^{3/2}\ dx = 2t\ dt$

$\Rightarrow \quad I = \frac{2}{5}\int (t^2-1)^2 (t^2)^{1/2}\ 2t\ dt$

$\Rightarrow \quad I = \frac{4}{5}\int t^2 (t^2-1)^2\ dt$

$\Rightarrow \qquad I = \frac{4}{5} \int t^6 + t^2 - 2t^4 \, dt$

$\Rightarrow \qquad I = \frac{4}{5} \left(\frac{t^7}{7} + \frac{t^3}{3} - 2\frac{t^5}{5} \right) + C$

On substituting $t = (1 + x^{5/2})^{1/2}$, we get

$$I = \frac{4}{5} \left(\frac{(1+x^{5/2})^{7/2}}{7} + \frac{(1+x^{5/2})^{3/2}}{3} - \frac{2(1+x^{5/2})^{5/2}}{5} \right) + C$$

Example : 42

Evaluate : **(1)** $\int \frac{dx}{(2ax + x^2)^{3/2}}$ **(2)** $\int \frac{\sqrt{x^2+1}}{x^4} \, dx$

Solution

(1) Let $I = \int \frac{dx}{(2ax + x^2)^{3/2}} \qquad \Rightarrow \qquad I = \int \frac{dx}{[(x+a)^2 - a^2]^{3/2}}$

Put $x + a = a \sec \theta \qquad \Rightarrow \qquad dx = a \sin \theta \tan \theta \, d\theta$

On substituting in I, we get

$$I = \int \frac{a \sec \theta \tan \theta d\theta}{(a^2 \sec^2 \theta - a^2)^{3/2}} = \int \frac{2 \sec \theta \tan \theta d\theta}{a^3 \tan^3 \theta}$$

$\Rightarrow \qquad I = \frac{1}{a^2} \int \sec \theta \cot^2 \theta d\theta = \frac{1}{a^2} \int \frac{\cos \theta}{\sin^2 \theta} d\theta$

$\Rightarrow \qquad I = \frac{1}{a^2} \int \frac{d(\sin \theta)}{\sin^2 \theta} d\theta = -\frac{1}{a^2 \sin \theta} + C$

$\Rightarrow \qquad I = -\frac{1}{a^2} \frac{x+a}{\sqrt{x^2 + 2ax}} + C$

(2) Let $I = \int \frac{\sqrt{x^2+1}}{x^4} \, dx$

Put x tan θ $\qquad \Rightarrow \qquad dx = \sec^2\theta \, d\theta$

On substituting x and dx in I, we get

$$I = \int \frac{\sqrt{\tan^2 \theta + 1}}{\tan^4 \theta} \sec^2\theta \, d\theta = \int \frac{\sec^3 \theta d\theta}{\tan^4 \theta}$$

$\Rightarrow \qquad I = \int \frac{\cos \theta}{\sin^4 \theta} d\theta = \int \frac{d(\sin \theta) d\theta}{\sin^4 \theta} \qquad \Rightarrow \qquad I = -\frac{1}{3 \sin^3 \theta} + C$

On substituting value of sin θ in terms of x, we get $I = -\frac{1}{3} \frac{(1+x^2)^{3/2}}{x^3} + C$

Example : 43

Find the reduction formula for $\int \sin^n x \, dx$

Solution

Let $I_0 = \int \sin^n x \, dx = \int \sin^{n-1} x \, . \sin x \, dx$

Apply by parts taking $\sin^{n-1} x$ as first part and sin x as second part.

$\Rightarrow \quad I_n = \sin^{n-1} x \,.\, (-\cos x) + \int (n-1)\sin^{n-2} x \cos^2 x \, dx$

$= -\cos x \sin^{n-1} x + (n - 1) \int \sin^{n-2} x (1 - \sin^2 x) \, dx$

$= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$

$\Rightarrow \quad I_n = -\cos x \sin^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$

$\Rightarrow \quad nI_n = -\cos x \,.\, \sin^{n-1} x + (n-1) I_{n-2}$;

$$I_n = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}$$

Example : 44

Find the reduction formula for $\int \tan^n x \, dx$.

If $I_n = \int \tan^n x \, dx$, to prove that $(n-1)(I_n + I_{n-2}) = \tan^{n-1} x$.

Solution

Here $I_n = \int \tan^n x \, dx \ \int \tan^{n-2} x \tan^2 x \, dx$

$= \int \tan^{n-2} x (\sec^2 x - 1) \, dx$

$= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$

$= \int \tan^{n-2} x \sec^2 x - I_{n-2}$

$$\Rightarrow \quad I_n + I_{n-2} = \frac{\tan^{n-1} x}{n-1}$$

Hence $(n-1)(I_n + I_{n-2}) = \tan^{n-1} x$.

Example : 45

Find reduction formula for $\int \sec^n x \, dx$

Solution

Let $I_n = \int \sec^n x$

$\Rightarrow \quad I_n = \int \sec^{n-2} x \sec^2 x \, dx$

Apply by parts taking $\sec^{n-2} x$ as the first part and $\sec^2 x$ as the second part

$$\Rightarrow \quad I_n = \sec^{n-2} x \int \sec^3 x \, dx - \int \left[\frac{d}{dx} (\sec^{n-2} x) \int \sec^2 x \, dx \right] dx$$

$\Rightarrow \quad I_n = \sec^{n-2} x \tan x - \int (n-2) \sec^{n-3} x \sec x \tan x \tan x \, dx$

$\Rightarrow \quad I_n = \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx$

$\Rightarrow \quad I_n + (n-2) I_n = \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x \, dx$

$\Rightarrow \quad (n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

Hence $\int \sec^n x\, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx$

This is the required reduction formula for $\int \sec^n x\, dx$

Example : 46

Find the reduction formula for $\int e^{ax} \cos^n x\, dx$

Solution

Let $I_n = \int e^{ax} \cos^n x\, dx$

Apply by parts taking $\cos^n x$ as the first part and e^{ax} the second part

$\Rightarrow \quad I_n = \cos^n x \int e^{ax}\, dx - \int \left[\frac{d}{dx}(\cos^n x) \int e^{ax} dx \right] dx$

$\Rightarrow \quad I_n = \frac{e^{ax}}{a} \cos^n x - \int n \cos^{n-1} x \; x\, (-\sin x)\, \frac{e^{ax}}{a}\, dx$

$\Rightarrow \quad I_n = \frac{1}{a} e^{ax} \cos^n x + \frac{n}{a} \int (\cos^{n-1} x \sin x)\; e^{ax}\, dx$

Apply by parts again taking $\cos^{n-1} x \sin x$ as first part and e^{ax} as second part

$\Rightarrow \quad I_n = \frac{1}{a} e^{ax} \cos^n x + \frac{n}{a} (\cos^{n-1} x \sin x) \int e^{ax} - \frac{n}{a} \int \left[\frac{d}{dx}(\cos^{n-1} x \sin x) \int e^{ax}\, dx \right] dx$

$\Rightarrow \quad I_n = \frac{1}{a} e^{ax} \cos^n x + \frac{n}{a} \cos^{n-1} x \sin x \frac{e^{ax}}{a} - \frac{n}{a} \int [-(n-1) \cos^{n-2} x \sin^2 x + \cos^{n-1} x \,.\, \cos x] \frac{e^{ax}}{a}\, dx$

$\Rightarrow \quad I_n = \frac{1}{a} e^{ax} \cos^n x + \frac{n}{a^2} e^{ax} \cos^{n-1} x \sin x + \frac{n(n-1)}{a^2} \int e^{ax} \cos^{n-2} x\, (1 - \cos^2 x)\, dx - \frac{n}{a^2} \int e^{ax} \cos^n x\, dx$

$\Rightarrow \quad I_n = \frac{1}{a} e^{ax} \cos^n x + \frac{n}{a^2} e^{ax} \cos^{n-1} x \sin x + \frac{n(n-1)}{a^2} I_{n-2} - \frac{n^2}{a^2} I_n$

$\Rightarrow \quad I_n = \left(1 + \frac{n^2}{a^2}\right) I_n = \frac{1}{a^2} e^{ax} (a \cos x + n \sin x) \cos^{n-1} x + \frac{n(n-1)}{a^2} I_{n-2}$

Hence $\int e^{ax} \cos^n x\, dx = e^{ax} \left(\frac{a \cos x + n \sin x}{a^2 + n^2} \right) \cos^{n-1} x + \frac{n(n-1)}{a^2 + n^2} \int e^{ax} \cos^{n-2} x\, dx$

This is the required reduction formula.

Example : 47

Find the reduction formula for $\int \cos^m x \ \sin nx \, dx$

Solution

Let $\quad I_{m,n} = \int \cos^m x \ \sin nx \, dx$

Apply by parts taking $\cos^m x$ as the first part and sin nx as the second part.

$$\Rightarrow \quad I_{m,n} = \cos^m x \left(-\frac{\cos nx}{n}\right) - \int m \cos^{m-1} (-\sin x) \left(-\frac{\cos nx}{n}\right) dx$$

$$\Rightarrow \quad I_{m,n} = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} \int \cos^{m-1} x \ (\sin x \cos nx) \, dx$$

Now sin (n – 1) x = sin nx cos x – cos nx sin x or cos nx sin x = sin nx cos x – sin (n – 1) x

$$\Rightarrow \quad I_{m,n} = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} \int \cos^{m-1} x \ [\sin nx \cos x - \sin (n-1) x] \, dx$$

$$\Rightarrow \quad I_{m,n} = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} \int \cos^m x \sin nx \, dx + \frac{m}{n} \int \cos^{m-1} x \sin (n-1) x \, dx$$

$$\Rightarrow \quad \left[1 + \frac{m}{n}\right] I_{m,n} = -\frac{\cos^m x \cos nx}{n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow \quad I_{m,n} = \frac{m}{m+n} I_{m-1,n-1} - \frac{\cos^m x \cos nx}{m+n}$$

Limits & Derivatives

Example : 1

Find the domain and the range of the following functions

(a) $y = \sqrt{1-x^2}$ (b) $y = 2\sin x$ (c) $y = \dfrac{1}{x-2}$

Solution

(a) For domain : $1 - x^2 \geq 0$

$\Rightarrow x^2 \leq 1$

$\Rightarrow -1 \leq x \leq 1$

Hence the domain is x set [–1, 1].

For range : As $-1 \leq x \leq 1$

$\Rightarrow 0 \leq x^2 \leq 1$

$\Rightarrow 0 \leq 1 - x^2 \leq 1$

$\Rightarrow 0 \leq \sqrt{1-x^2} \leq 1$

$\Rightarrow 0 \leq y \leq 1$

Hence the range is set [0, 1]

(b) $y = 2\sin x$

For domain : $x \in R$ i.e. $x\ (-\infty, \infty)$

For range : $-1 \leq \sin x \leq 1$

$-2 \leq 2\sin x \leq 2$

$-2 \leq y \leq 2$

Hence the range is $y \in [-2, 2]$

(c) As denominator cannot be zero, x can not be equal to 2

domain is $x \in R - \{2\}$

i.e. $x \in (-\infty, 2)\ (2, \infty)$

Range : As y can never become zero, the range is $y \in R - \{0\}$

i.e. $y \in (-\infty, 0)\ (0, \infty)$

Example : 2

Find the domain of the following functions :

(a) $\sqrt{3-x} + \dfrac{1}{\log_{10} x}$ (b) $\dfrac{1}{x+|x|}$ (c) $\sqrt{1-\log_{10} x}$

Solution

(a) $\sqrt{3-x}$ is defined if $3 - x \geq 0$

$\Rightarrow x \leq 3$ ....(i)

$\dfrac{1}{\log_{10} x}$ is defined if $x > 0$ and $x \neq 1$

$\Rightarrow x > 0 - \{1\}$ ....(ii)

Combining (i) and (ii), set of domain is :

$x \in (0, 1) \cup (1, 3]$

(b) f(x) is defined if : $x + |x| \neq 0$

$\Rightarrow |x| \neq -x \Rightarrow x > 0$

Hence domain is $x \in (0, \infty)$

(c) f(x) is defined if

$1 - \log_{10}x \geq 0$ and $x > 0$

$\Rightarrow \log_{10}x \leq 1$ and $x > 0$

$\Rightarrow x \leq 10$ and $x > 0$

$\Rightarrow$ domain is $x \in (0, 10]$

Example : 3

(Using factorisation) Evaluate the following limits :

(a) $\lim\limits_{x\to 2} \dfrac{x^3 - 2x - 4}{x^2 - 3x + 2}$ (b) $\lim\limits_{x\to a} \dfrac{x^3 - a^3}{x^2 - ax}$ (c) $\lim\limits_{x\to 5} \dfrac{x^4 - 625}{x^3 - 125}$

Solution

(a) $\lim_{x\to 2} \frac{x^3-2x-4}{x^2-3x+2} = \lim_{x\to 2} \frac{(x-2)(x^2+2x+2)}{(x-2)(x-1)} = \lim_{x\to 2} \frac{x^2+2x+2}{x-1} = 10$

(b) $\lim_{x\to a} \frac{x^3-a^3}{x^2-ax} = \lim_{x\to a} \frac{(x-a)(x^2+ax+a^2)}{x(x-a)} = \lim_{x\to a} \frac{x^2+ax+a^2}{x} = 3a$

(c) $\lim_{x\to 5} \frac{x^4-625}{x^3-125} = \lim_{x\to 5} \frac{\frac{x^4-5^4}{x-5}}{\frac{x^3-5^3}{x-5}} = \frac{4.5^3}{3.5^2} = \frac{20}{3}$ [using section 2.2 (ix)]

Example : 4

(Using rationalisation) Evaluate the following limits :

(a) $\lim_{x\to 3} \frac{\sqrt{3x+7}-4}{\sqrt{x+1}-2}$ (b) $\lim_{x\to a} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$ (c) $\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{x^2-1}$

Solution

(a) $\lim_{x\to 3} \frac{\sqrt{3x+7}-4}{\sqrt{x+1}-2}$

Rationalising the numerator and denominator,

$= \lim_{x\to 3} \frac{3x+7-16}{x+1-4} \left(\frac{\sqrt{x+1}+2}{\sqrt{3x+7}+4}\right)$

$= \lim_{x\to 3} \frac{3(x-3)}{x-3} \left(\frac{\sqrt{x+1}+2}{\sqrt{3x+7}+4}\right)$

$= 3 \lim_{x\to 3} \frac{\sqrt{x+1}+2}{\sqrt{3x+7}+4} = 3\left(\frac{2+2}{4+4}\right) = \frac{3}{2}$

(b) Rationalising numerator and denominator we get,

$= \lim_{x\to a} \frac{a+2x-3x}{3a+x-4x} \left(\frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{a+2x}+\sqrt{3x}}\right)$

$= \lim_{x\to a} \frac{a-x}{3(a-x)} \left(\frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{a+2x}+\sqrt{3x}}\right) = \frac{1}{3}\left(\frac{2\sqrt{a}+2\sqrt{a}}{\sqrt{3a}+\sqrt{3a}}\right) = \frac{2}{3\sqrt{3}}$

(c) $\lim_{x\to 1} \frac{x^{\frac{1}{3}}-1}{x^2-1} = \lim_{x\to 1} \frac{\left(x^{\frac{1}{3}}-1\right)\left(x^{\frac{2}{3}}+x^{\frac{1}{3}}+1\right)}{(x^2-1)\left(x^{\frac{2}{3}}+x^{\frac{1}{3}}+1\right)}$

$\lim_{x\to 1} \frac{(x-1)}{(x-1)(x+1)\left(x^{\frac{2}{3}}+x^{\frac{1}{3}}+1\right)} = \frac{1}{6}$

Example : 5

($x \to \infty$ type problems) Evaluate the following limits :

(a) $\lim_{x\to\infty} \frac{x^3 - 2x^2 + 3x + 1}{5x^3 + 7x + 2}$ (b) $\lim_{n\to\infty} \frac{1^2 + 2^2 + 3^2 + ... + n^2}{n^3}$ (c) $\lim_{x\to\infty} \frac{\sqrt{x^2 + 3x + 1} + 5x}{1 + 4x}$

Solution

In these type of problems, divide numerator and denominator by highest power of x.

(a) Dividing numerator and denominator by x^3

$$= \lim_{x\to\infty} \frac{1 - \frac{2}{x} + \frac{3}{x^2} + \frac{1}{x^3}}{5 + \frac{7}{x^2} + \frac{2}{x^3}} = \frac{1}{5}$$

[because as $x \to \infty$, $\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}$ .... $\to 0$]

(b) $\lim_{n\to\infty} \frac{1^2 + 2^2 + 3^2 + ... + n^2}{n^3}$

$$= \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3}$$

$$= \frac{1}{6} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right)$$

$$= \frac{1}{6} (1 + 0)(2 + 0) = \frac{1}{3}$$

(c) The highest power of x is 1. Hence divide the numerator and denominator by x.

$$\lim_{x\to\infty} \frac{\sqrt{x^2 + 3x + 1} + 5x}{1 + 4x}$$

$$= \lim_{x\to\infty} \frac{\sqrt{1 + \frac{3}{x} + \frac{1}{x^2}} + 5}{\frac{1}{x} + 4} = \frac{\sqrt{1} + 5}{0 + 4} = \frac{3}{2}$$

Example : 6

($\infty - \infty$ form) Evaluate the following limits :

(a) $\lim_{x\to\frac{\pi}{2}} (\sec x - \tan x)$ (b) $\lim_{x\to\infty} \left(\sqrt{(x + 2a)(2x + a)} - x\sqrt{2}\right)$ (c) $\lim_{x\to\infty} \left(x - \sqrt{x^2 + x}\right)$

Solution

(a) $\lim_{x\to\frac{\pi}{2}} (\sec x - \tan x) = \lim_{x\to\frac{\pi}{2}} \frac{\sec^2 x - \tan^2 x}{\sec x + \tan x}$

$$\lim_{x\to\frac{\pi}{2}} \frac{1}{\sec x + \tan x} = 0$$

(b) $\lim_{x\to\infty} \left(\sqrt{(x + 2a)(2x + a)} - x\sqrt{2}\right)$

Rationalising the expression, we get

$$\lim_{x\to\infty} \frac{(x + 2a)(2x + a) - 2x^2}{\left(\sqrt{(x + 2a)(2x + a)} + x\sqrt{2}\right)}$$

$$= \lim_{x\to\infty} \frac{5ax + 2a^2}{\sqrt{2x^2 + 5ax + 2a^2} + x\sqrt{2}}$$

Dividing numerator and denominator by x, we get

$$= \lim_{x\to\infty} \frac{5a + \frac{2a^2}{x}}{\sqrt{2 + \frac{5a}{x} + \frac{2a^2}{x^2}} + \sqrt{2}} = \frac{5a}{2\sqrt{2}}$$

(c) $\lim_{x\to\infty} \left(x - \sqrt{x^2 + x}\right)$

On rationalising the expression, we get

$$= \lim_{x\to\infty} \frac{x^2 - (x^2 + x)}{x + \sqrt{x^2 + x}} = \lim_{x\to\infty} \frac{-x}{x + \sqrt{x^2 + x}}$$

Divide by the highest power of x i.e. x^1

$$= \lim_{x\to\infty} \frac{-1}{1 + \sqrt{1 + \frac{1}{x}}} = \frac{-1}{1+1} = -\frac{1}{2}$$

Example : 7

$\left(\text{using } \lim_{x\to 0} \frac{\sin x}{x} = 1\right)$ Evaluate the following limits :

(a) $\lim_{x\to\frac{\pi}{3}} \frac{\tan x - \sqrt{3}}{9x^2 - \pi^2}$

(b) $\lim_{x\to\frac{\pi}{2}} \frac{\sin(\cos x)\cos x}{\sin x - \operatorname{cosec} x}$

(c) $\lim_{x\to a} \frac{\cos x - \cos a}{x - a}$

(d) $\lim_{x\to a} \frac{a\sin x - x\sin a}{ax^2 - a^2x}$

Solution

(a) $\lim_{x\to\frac{\pi}{3}} \frac{\tan x - \sqrt{3}}{9x^2 - \pi^2} = \lim_{x\to\frac{\pi}{3}} \frac{\tan x - \frac{\pi}{3}}{9x^2 - \pi^2}$

Using tan A – tan B = $\frac{\sin(A - B)}{\cos A \cos B}$ we get,

$$\lim_{x\to\frac{\pi}{3}} \frac{\sin\left(x - \frac{\pi}{3}\right)}{\cos x \cos\frac{\pi}{3}(3x - \pi)(3x + \pi)} = \frac{1}{3} \frac{1}{\cos\frac{\pi}{3}\cos\frac{\pi}{3}(\pi + \pi)} \left(\text{using } \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right) = \frac{2}{3\pi}$$

(b) $\lim_{x\to\frac{\pi}{2}} \frac{\sin(\cos x)\cos x}{\sin x - \operatorname{cosec} x} = \lim_{x\to\frac{\pi}{2}} \frac{\sin(\cos x)}{\cos x}$

$$\lim_{x\to\frac{\pi}{2}} \frac{\cos^2 x}{\sin x - \operatorname{cosec} x} = 1 \times \lim_{x\to\frac{\pi}{2}} \frac{\cos^2 x \sin x}{\sin^2 x - 1} \qquad \left(\text{using } \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right)$$

$$= -\lim_{x\to\frac{\pi}{2}} (\sin x) = -1$$

(c) $\lim\limits_{x\to a} \dfrac{\cos x - \cos a}{x-a}$ $\lim\limits_{x\to a} \dfrac{-2\sin\left(\dfrac{x+a}{2}\right)\sin\left(\dfrac{x-a}{2}\right)}{2\times\left(\dfrac{x-a}{2}\right)}$

$$= -\lim_{x\to a} \sin\left(\frac{x+a}{2}\right) \lim_{x\to a} \frac{\sin\left(\dfrac{x-a}{2}\right)}{\left(\dfrac{x-a}{2}\right)} = -\sin a$$

(d) $\lim\limits_{x\to a} \dfrac{a\sin x - x\sin a}{ax^2 - a^2x}$

$$= \lim_{x\to a} \frac{a\sin x - x\sin x + x\sin x - x\sin a}{ax(x-a)} = \lim_{x\to a} \frac{(a-x)\sin x + x(\sin x - \sin a)}{ax(x-a)}$$

$$= \lim_{x\to a} \frac{(a-x)\sin x}{ax(x-a)} + \lim_{x\to a} \frac{\sin x - \sin a}{a(x-a)} = -\frac{\sin a}{a^2} + \lim_{x\to a} \frac{2\cos\left(\dfrac{x+a}{2}\right)}{2a}\left[\frac{\sin\dfrac{(x-a)}{2}}{\dfrac{(x-a)}{2}}\right]$$

$$= \frac{\sin a}{a^2} + \frac{\cos a}{a}$$

Example : 8

$\left(\text{using} \lim\limits_{x\to 0} \dfrac{a^x - 1}{x} = \log a\right)$ Evaluate the following limits :

(a) $\lim\limits_{x\to 1} \dfrac{2^x - 2}{x-1}$

(b) $\lim\limits_{x\to a} \dfrac{e^{\sqrt{x}} - e^{\sqrt{a}}}{x-a}$

(c) $\lim\limits_{x\to 0} \dfrac{6^x - 2^x - 3^x + 1}{\sin^2 x}$

(d) $\lim\limits_{x\to 0} \dfrac{3^x - 5^x}{x}$

Solution

(a) $\lim\limits_{x\to 1} \dfrac{2^x - 2}{x-1} = 2 \lim\limits_{x\to 1} \dfrac{2^{x-1} - 1}{x-1} = 2\log 2$

(b) $\lim\limits_{x\to a} \dfrac{e^{\sqrt{x}} - e^{\sqrt{a}}}{x-a} = \lim\limits_{x\to a} \dfrac{e^{\sqrt{a}}\left(e^{\sqrt{x}-\sqrt{a}} - 1\right)}{x-a}$

$$= e^{\sqrt{a}} \lim_{x\to a} \frac{e^{\sqrt{x}-\sqrt{a}} - 1}{\sqrt{x} - \sqrt{a}} \lim_{x\to a} \frac{\sqrt{x} - \sqrt{a}}{x-a}$$

$$= e^{\sqrt{a}}\ (1) \lim_{x\to a} \frac{(x-a)}{(x-a)\left(\sqrt{x} + \sqrt{a}\right)} = \frac{e^{\sqrt{a}}}{2\sqrt{a}}$$

(c) $\lim_{x\to 0} \frac{6^x - 2^x - 3^x + 1}{\sin^2 x}$

$= \lim_{x\to 0} \frac{(2^x - 1)(3^x - 1)}{x^2} \frac{x^2}{\sin^2 x}$

$= \lim_{x\to 0} \frac{2^x - 1}{x} \lim_{x\to 0} \frac{3^x - 1}{x} \lim_{x\to 0} \left(\frac{x}{\sin x}\right)^2 = \log_e^2 \log_e^2$

(d) $\lim_{x\to 0} \frac{3^x - 5^x}{x} = \lim_{x\to 0} \left[\frac{3^x - 1}{x} - \frac{5^x - 1}{x}\right] = \log 3 - \log 5 = \log \frac{3}{5}$

Example : 9

$\left[\text{using} \lim_{x\to 0} (1+x)^{\frac{1}{x}} = e\right]$ Evaluate the following limits :

(a) $\lim_{x\to 0} (1-2x)^{\frac{1}{x}}$ (b) $\lim_{x\to 1} x^{\cot \pi x}$

Solution

(a) $\lim_{x\to 0} (1-2x)^{\frac{1}{x}} = \lim_{x\to 0} \left((1-2x)^{\frac{-1}{2x}}\right)^{-2}$

(b) $\lim_{x\to 1} x^{\cot \pi x} = \lim_{x\to 1} \left[1 + x - 1)^{\frac{1}{x-1}}\right]^{(x-1)\cot \pi x}$

$= e^{\lim_{x\to 1}(x-1)\cot \pi x} = e^{\lim_{x\to 1} \frac{1-x}{\tan(\pi - \pi x)}} = e^{\lim_{x\to 1} \frac{1}{\pi} \frac{\pi - \pi x}{\tan(\pi - \pi x)}} = e^{\frac{1}{\pi}}$ $\left(\text{using} \lim_{\theta\to 0} \frac{\tan\theta}{\theta} = 1\right)$

Example : 10

Show that the limit of :

$f(x) = \begin{cases} 2x - 1 & ; \quad x \le 1 \\ x & ; \quad x > 1 \end{cases}$ at x = 1 exists

Solution

Left hand limit = $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (2x - 1) = 2 - 1 = 1$

(we use f(x) = 2x – 1 $\because$ while calculating limit at x = 1, we approach x = 1 from LHS i.e. x < 1)

Right hand limit = $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x) = 1$

$\Rightarrow$ L.H.L. = R.H.L. = 1. Hence limit exists

Example : 11

Find whether the following limits exist or not :

(a) $\lim_{x\to 0} \sin\frac{1}{x}$ (b) $\lim_{x\to 0} x\sin\frac{1}{x}$

Solution

(a) As $x \to 0$, $\frac{1}{x} \to \infty$.

As the angle θ approaches ∞ , sin θ oscillates by taking values between – 1 and + 1.

Hence $\lim_{x\to 0} \sin\frac{1}{x}$ is not a well defined finite number.

⇒ limit does not exist

(b) $\lim_{x\to 0} x\sin\frac{1}{x} = \lim_{x\to 0} x \lim_{x\to 0} \sin\frac{1}{x}$

= 0 × (some quantity between – 1 and + 1) = 0

It can be easily seen that $\lim_{x\to 0^+} x\sin\frac{1}{x} = \lim_{x\to 0^-} x\sin\frac{1}{x} = 0$

Hence the limit exists and is equal to zero (0)

Example : 12

Comment on the following limits :

(a) $\lim_{x\to 1} [x-3]$ (b) $\lim_{x\to 0} \frac{|x|}{x}$

Solution

(a) Right Hand limit = $\lim_{x\to 1^+} [x-3]$

$= \lim_{h\to 0} [1+h-3] = \lim_{h\to 0} [h-2]$

= – 2 (because h – 2 is between – 1 and – 2)

Left hand limit = $\lim_{x\to 1^-} [x-3]$

$= \lim_{h\to 0} [1-h-3] = \lim_{h\to 0} [-2-h]$

= – 3 (because – h – 2 is between – 2 and – 3)

Hence R.H.L. ≠ L.H.L.

⇒ limit does not exist.

(b) Left hand limit = $\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1$

Right hand limit = $\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = +1$

Hence R.H.L. ≠ L.H.L.

⇒ limit does not exist

Example : 13

Find a and b so that the function :

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x & ; \quad 0 \le x < \frac{\pi}{4} \\ 2x\cot x + b & ; \quad \frac{\pi}{4} \le x \le \frac{\pi}{2} \\ a\cos 2x - b\sin x & ; \quad \frac{\pi}{2} < x \le \pi \end{cases}$$

is continuous for $x \in [0, \pi]$

Solution

At $x = \pi/4$

Left hand limit = $\lim\limits_{x \to \frac{\pi}{4}^-} f(x) = \lim\limits_{x \to \frac{\pi}{4}^-} (x + a\sqrt{2}\sin x) = \frac{\pi}{4} + a$

Right hand limit = $\lim\limits_{x \to \frac{\pi}{4}^+} f(x) = \lim\limits_{x \to \frac{\pi}{4}^+} (2x \cot x + b) = \frac{\pi}{2} + b$

$f\left(\frac{\pi}{4}\right) = 2\left(\frac{\pi}{4}\right) \cot \frac{\pi}{4} + b = \frac{\pi}{2} + b$

for continuity, these three must be equal

$\Rightarrow \quad \frac{\pi}{4} + a = \frac{\pi}{2} + b \Rightarrow \quad a - b = \frac{\pi}{4}$ ....(i)

At $x = \pi/2$

Left hand limit = $\lim\limits_{x \to \frac{\pi}{2}^-} (2x \cot x + b) = 0 + b = b$

Right hand limit = $\lim\limits_{x \to \frac{\pi}{2}^+} (a \cos 2x - b \sin x) = -a - b$

$f\left(\frac{\pi}{2}\right) = 0 + b$

for continuity, $b = -a - b$

$\Rightarrow \quad a + 2b = 0$ ....(ii)

Solving (i) and (ii) for a and b, we get : $b = -\frac{\pi}{12}$, $a = \frac{\pi}{6}$

Example : 14

A function f(x) satisfies the following property $f(x + y) = f(x) f(y)$. Show that the function is continuous for all values of x if it is continuous at x = 1

Solution

As the function is continuous at x = 1, we have

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} f(x) = f(1)$

$\Rightarrow \quad \lim\limits_{h \to 0} f(1 - h) = \lim\limits_{h \to 0} f(1 + h) = f(1)$

using $f(x + y) = f(x) f(y)$, we get

$\Rightarrow \quad \lim\limits_{h \to 0} f(1) f(-h) = \lim\limits_{h \to 0} f(1) f(h) = f(1)$

$\Rightarrow \quad \lim\limits_{h \to 0} f(-h) = \lim\limits_{h \to 0} f(h) = 1$ ....(i)

Now consider some arbitrary point x = a

Left hand limit = $\lim\limits_{h \to 0} f(a - h) = \lim\limits_{h \to 0} f(a) f(-h)$

$= f(a) \lim\limits_{h \to 0} f(-h) = f(a)$ .... using (i)

Right hand limit = $\lim\limits_{h \to 0} f(a + h) = \lim\limits_{h \to 0} f(a) f(h)$

$= f(a) \lim\limits_{h \to 0} f(h) = f(a)$ .... using (i)

Hence at any arbitrary point (x = a)

L.H.L. = R.H.L. = f(a)

$\Rightarrow$ function is continuous for all values of x.

Example : 15

$$f(x) = \begin{cases} 1+x & ; \quad 0 \le x \le 2 \\ 3-x & ; \quad 2 < x \le 3 \end{cases}$$

Determine the form of g(x) = f(f(x)) and hence find the point of discontinuity of g, if any

Solution

$$g(x) = f(f(x)) = \begin{cases} 1+x & ; \quad 0 \le x \le 2 \\ 3-x & ; \quad 2 < x \le 3 \end{cases} = \begin{cases} f(1+x) & ; \quad 0 \le x \le 1 \\ f(1+x) & ; \quad 1 < x \le 2 \\ f(3-x) & ; \quad 2 < x \le 3 \end{cases}$$

Now $x \in [0, 1] \Rightarrow (1 + x) \in [1, 2]$
$x \in (0, 2] \Rightarrow (1 + x) \in (2, 3]$
$x \in (2, 3] \Rightarrow (3 - x) \in [0, 1)$

Hence

$$g(x) = \begin{cases} f(1+x) & \text{for} \quad 0 \le x \le 1 \Rightarrow 1 \le x+1 \le 2 \\ f(1+x) & \text{for} \quad 1 < x \le 2 \Rightarrow 2 < x+1 \le 3 \\ f(3-x) & \text{for} \quad 2 < x \le 3 \Rightarrow 0 \le 3-x < 1 \end{cases}$$

Now if $(1 + x) \in [1, 2]$, then f(1 + x) = 1 + (1 + x) = 2 + x ....(i)
[from the original definition of f(x)]

Similarly if $(1 + x) \in (2, 3)$, then
f(1 + x) = 3 – (1 + x) = 2 – 2 ....(ii)

If $(3 - x) \in (0, 1)$, then
f(3 – x) = 1 + (3 + x) = 4 – x ....(iii)

Using (i), (ii) and (iii), we get $g(x) = \begin{cases} 2+x & ; \quad 0 \le x \le 1 \\ 2-x & ; \quad 1 < x \le 2 \\ 4-x & ; \quad 2 < x \le 3 \end{cases}$

Now we will check the continuity of g(x) at x = 1, 2

At x = 1

L.H.L. = $\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} (2 + x) = 3$

R.H.L. = $\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} (2 - x) = 1$

As L.H.L., g(x) is discontinuous at x = 1

At x = 2

L.H.L. = $\lim_{x \to 2^+} g(x) = \lim_{x \to 2^-} (2 - x) = 0$

R.H.L. = $\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} (4 - x) = 2$

As L.H.L. ≠ R.H.L., g(x) is discontinuous at x = 2

Example : 16

Discuss the continuity of $f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1} & ; \quad x \ne 0 \\ 0 & ; \quad x = 0 \end{cases}$ at the point x = 0

Solution

$$\text{LHL} = \lim_{x \to 0^-} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} = \lim_{t \to -\infty} \frac{e^t - 1}{e^t + 1} = \frac{0-1}{0+1} = -1$$

$$\text{RHL} = \lim_{x\to 0^+} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} = \lim_{t\to\infty} \frac{e^t-1}{e^t+1} = \lim_{t\to\infty} \frac{1-e^{-t}}{1+e^{-t}}$$

$\Rightarrow$ R.H.L. = $\frac{1-0}{1+0}$ = 1

$\Rightarrow$ L.H.L. $\neq$ R.H.L. $\Rightarrow$ f(x) is discontinuous at x = 0

Example : 17

Discuss the continuity of the function g(x) = [x] + [–x] at integral values of x.

Solution

Let us simplify the definition of the function

(i) If x is an integer :

[x] = x and [–x] = – x $\Rightarrow$ g(x) = x – x = 0

(ii) If x is not integer :

let x = n + f where n is an integer and f $\in$ (0, 1)

$\Rightarrow$ [x] = [n + f] = n

and [–x] = [–n – f] = [(–n – 1) + (1 – f)] = – n – 1

(because 0 < f < 1 $\Rightarrow$ 0 , (1 – f) < 1)

Hence g(x) = [x] + [–x] = n + (–n – 1) = – 1

So we get : $g(x) = \begin{cases} 0 & , \text{ if } x \text{ is an integer} \\ -1 & , \text{ if } x \text{ is not an integer} \end{cases}$

Let us discuss the continuity of g(x) at a point x = a

where a $\in$ I

L.H.L. = $\lim_{x\to a^-}$ g(x) = – 1

$\because$ as $x \to a^-$, x is not an integer

R.H.L. = $\lim_{x\to a^+}$ g(x) = – 1

as $x \to a^+$, x is not an integer

but g(a) = 0 because a is an integer

Hence g(x) has a removable discontinuity at integral values of x.

Example : 18

Which of the following functions are even/odd?

(a) $f(x) = \frac{a^x-1}{a^x+1}$

(b) $f(x) = x \log\left(\frac{1+x}{1-x}\right)$

(c) f(x) = |x|

(d) $f(x) = \log\left(x+\sqrt{x^2+1}\right)$

Solution

(a) $f(-x) = \frac{a^{-x}-1}{a^{-x}+1} = \frac{1-a^x}{1+a^x}$ = – f(x) $\Rightarrow$ f(x) is odd

(b) $f(-x) = -x\log\left(\frac{1-x}{1+x}\right) = x\log\left(\frac{1-x}{1+x}\right)^{-1} = x\log\left(\frac{1+x}{1-x}\right) = f(x)$

$\Rightarrow$ f(x) is even

(c) f(–x) = |–x| = |x| = f(x)

$\Rightarrow$ f(x) is even

(d) $f(-x) = \log\left(-x+\sqrt{1+x^2}\right) = \log\left(\frac{1+x^2-x^2}{x+\sqrt{1+x^2}}\right) = -\log\left(x+\sqrt{x^2+1}\right) = -f(x)$

$\Rightarrow$ f(x) is odd.

Example : 19

Which of the following functions are periodic ? Give reasons

(a) $f(x) = x + \sin x$ (b) $\cos\sqrt{x}$

(c) $f(x) = x - [x]$ (d) $\cos^2 x$

Solution

If a function f(x) is periodic, then there should exist some positive value of constant a for which

f(x + a) = f(x) is an identity (i.e. true for all x)

The smallest value of a satisfying the above condition is known as the period of the function

(a) Assume that f(x + a) = f(x)

$\Rightarrow$ $x + a + \sin(x + a) = x + \sin x$

$\Rightarrow$ $\sin x - \sin(x + a) = a$

$\Rightarrow$ $2\cos\left(x+\frac{a}{2}\right)\sin\frac{a}{2} = -a$

$\Rightarrow$ $2\cos\left(x+\frac{a}{2}\right)\sin\frac{a}{2} = -a$

This cannot be true for all values of x.

Hence f(x) is non-periodic

(b) Assume that f(x + a) = f(x)

$\Rightarrow$ $\cos\sqrt{x+a} = \cos\sqrt{x}$

$\Rightarrow$ $\sqrt{x+a} = 2np \pm \sqrt{x}$

$\Rightarrow$ $\sqrt{x+a} \pm \sqrt{x} = 2n\pi$

$\Rightarrow$ $2x + a \pm 2\sqrt{x^2+ax} = 4n^2\pi^2$

$\Rightarrow$ $2x \pm 2\sqrt{x^2+ax} = 4n^2\pi^2 - a$

As this equation cannot be an identity, 3 f(x) is non-periodic

(c) Assume that f(x + a) = f(x)

$\Rightarrow$ $x + a - [x + a] = x - [x]$

$\Rightarrow$ $[x + a] - [x] = a$

This equation is true for all values of x if a is an integer hence f(x) is periodic

Period = smallest positive value of a = 1

(d) Let f(x + a) = f(x)

$\Rightarrow$ $\cos^2(x + a) = \cos^2 x$

$\Rightarrow$ $\cos^2(x + a) - \cos^2 x = 0$

$\Rightarrow$ $\sin(2x + a)\sin(a) = 0$

this equation is true for all values of x if a is an integral multiple of π

Hence f(x) is periodic. Period = smallest positive value of a = π

Example : 20

Find the natural number a for which $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$ where the function f satisfies the relation f(x + y) = f(x) f(y) for all natural numbers x, y and further f(1) = 2.

Solution

Since the function f satisfies the relation f(x + y) = f(x) f(y)

It must be an exponential function.

Let the base of this exponential function be a.
Thus $f(x) = a^x$
It is given that f(1) = 2. So we can make
$f(1) = a^1 = 2 \quad \Rightarrow \quad a = 2$
Hence, the function is $f(x) = 2^x$ ....(i)
[Alternatively, we have
$f(x) = f(x - 1 + 1) = f(x - 1)\, f(1) = f(x - 2 + 1)\, f(1) = f(x - 2)\, [f(1)]^2 = \ldots\ldots\ldots\ldots = [f(1)]^x = 2^x$]
Using equation (i), the given expression reduces to :

$$\sum_{k=1}^{n} 2^{a+k} = 16\,(2^n - 1)$$

$$\Rightarrow \quad \sum_{k=1}^{n} 2^a \,.\, 2^k = 16\,(2^n - 1)$$

$$\Rightarrow \quad 2^a \sum_{k=1}^{n} 2^k = 16\,(2^n - 1)$$

$$\Rightarrow \quad 2^a\,(2 + 4 + 8 + 16 + \ldots\ldots\ldots\ldots + 2^n) = 16\,(2^n - 1)$$

$$\Rightarrow \quad 2^a \left[\frac{2(2^n - 1)}{2 - 1}\right] = 16\,(2^n - 1)$$

$$\Rightarrow \quad 2^{a+1} = 16 \quad \Rightarrow \quad 2^{a+1} = 2^4$$

$$\Rightarrow \quad a + 1 = 4 \quad \Rightarrow \quad a = 3$$

Example : 21

Evaluate the following limits :

(i) $\lim\limits_{x \to 2} \dfrac{3^x + 3^{3-x} - 12}{3^{3-x} - 3^{x/2}}$ (ii) $\lim\limits_{x \to \pi/3} \dfrac{\tan^3 x - 3\tan x}{\cos\left(x + \dfrac{\pi}{6}\right)}$ (iii) $\lim\limits_{x \to -\infty} \dfrac{x^4 \sin\dfrac{1}{x} + x^2}{1 + |x^3|}$

Solution

(i) Let $L = \lim\limits_{x \to 2} \dfrac{3^x + 3^{3-x} - 12}{3^{3-x} - 3^{x/2}}$

$$\Rightarrow \quad L = \lim_{x \to 2} \frac{3^x + \dfrac{27}{3^x} - 12}{\dfrac{27}{3^x} - 3^{x/2}}$$

$$\Rightarrow \quad L = \lim_{x \to 2} \frac{3^{2x} - 12.3^x + 27}{(3^{x/2})^3 - 3^3}$$

$$\Rightarrow \quad L = \lim_{x \to 0} \frac{(3^x - 9)(3^x - 3)}{(3^{x/2} - 3)(3^x + 9 + 3.3^{x/2})}$$

$$\Rightarrow \quad L = \lim_{x \to 2} \frac{(3^{x/2} + 3)(3^x - 3)}{(3^x + 3.3^{x/2} + 9)}$$

$$\Rightarrow \quad L = \frac{6.6}{9 + 3.3 + 9} = \frac{36}{27} = \frac{4}{3}$$

(ii) Let L = L = $\lim_{x\to\pi/3} \dfrac{\tan^3 x - 3\tan x}{\cos\left(x+\frac{\pi}{6}\right)}$ and $x - \frac{\pi}{3} = t$

$\Rightarrow$ L = $\lim_{t\to 0} \dfrac{\tan^3\left(1+\frac{\pi}{3}\right) - 3\tan\left(t+\frac{\pi}{3}\right)}{\cos\left(t+\frac{\pi}{2}\right)}$

$\Rightarrow$ L = $\lim_{t\to 0} \dfrac{\tan(3t+\pi)\left[3\tan^2\left(1+\frac{\pi}{3}\right)-1\right]}{-\sin t}$

$\Rightarrow$ L = $\lim_{t\to 0} \dfrac{-\tan(3t)}{-\sin t} \cdot \lim_{t\to 0}\left[3\tan^2\left(t+\frac{\pi}{3}\right)-1\right]$

$\Rightarrow$ L = 3 $\lim_{t\to 0} \dfrac{\tan(3t)}{3t} \times \lim_{t\to 0} \dfrac{1}{\sin t} \times \lim_{t\to 0}\left[3\tan^2\left(1+\frac{\pi}{3}\right)-1\right]$

$\Rightarrow$ L = 3 × 1 × 1 × 8 = 24

(iii) Let L = $\lim_{x\to -\infty} \dfrac{x^4 \sin\frac{1}{x} + x^2}{1+|x^3|}$

Divided Numerator and Denominator by x^3 to get

L = $\lim_{x\to -\infty} \dfrac{x\sin\frac{1}{x}+x^2}{\frac{1}{x^3}+|x|^3} = \dfrac{\frac{\sin\frac{1}{x}}{1/x}+\frac{1}{x}}{\frac{1}{x^3}+\frac{(-x)^3}{x^3}}$ ($\because$ for x < 0, $|x^3| = -x^3$)

$\Rightarrow$ L = $\lim_{x\to -\infty} \dfrac{\to(1)+\to(0)}{\to(0)+(-1)} = -1$

Example : 22

Let f(x) = $\begin{cases} (1+|\sin x|)^{a/|\sin x|} & ; \frac{\pi}{6} < x < 0 \\ b & ; x = 0 \\ e^{\frac{\tan 2x}{\tan 3x}} & ; 0 < x < \frac{\pi}{6} \end{cases}$

Determine a and b such that f(x) is continuous at x = 0

Solution

Left hand limit at x = 0

L.H.L. = $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left[(1+|\sin x|)^{\frac{a}{|\sin x|}}\right]$

$\Rightarrow$ L.H.L. = $\lim_{h\to 0} f(0-h)$

$\Rightarrow$ L.H.L. = $\lim_{h\to 0}\left[(1+|\sinh|)^{\frac{a}{|\sinh|}}\right] = e^x$ $\left[\text{using} : \lim_{t\to 0}(1+t)^{\frac{1}{t}} = e\right]$

Right hand limit x = 0

R.H.L. = $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\frac{\tan 2x}{\tan 3x}}$

$\Rightarrow$ R.H.L. $= \lim_{h\to 0} f(0 + h)$

$\Rightarrow$ R.H.L. $= \lim_{h\to 0} e^{\frac{\tan 2h}{\tan 3h}}$

$\Rightarrow$ R.H.L. $= \lim_{h\to 0} e^{\frac{2}{3}\left(\frac{\tan 2h}{2h}.\frac{3h}{\tan 3h}\right)} = e^{\frac{2}{3}}$

for continuity

L.H.L. = R.H.L. = f(0)

$\Rightarrow$ $e^x = e^{\frac{2}{3}} = b$

$\Rightarrow$ $a = \frac{2}{3}$, $b = e^{2/3}$

Example : 23

Discuss the continuity of f(x) in [0, 2] where $f(x) = \lim_{n\to\infty}\left(\sin\frac{\pi x}{2}\right)^{2n}$

Solution

Since $\lim_{n\to\infty} x^{2n} = \begin{cases} 0 & ; \ |x| < 1 \\ 1 & ; \ |x| = 1 \end{cases}$

$\therefore$ $f(x) = \lim_{n\to\infty}\left(\sin\frac{\pi x}{2}\right)^{2n}$

$= \begin{cases} 0 & ; \ \left|\sin\frac{\pi x}{2}\right| < 1 \\ 1 & ; \ \left|\sin\frac{\pi x}{2}\right| = 1 \end{cases}$

Thus f(x) is continuous for all x, except for those values of x for which $\left|\sin\frac{\pi x}{2} = 1\right|$

i.e. x is an odd integer

$\Rightarrow$ x = (2n + 1) where $x \in I$

Check continuity at x = (2n + 1) :

L.H.L. $= \lim_{x\to 2n+1} f(x) = 0$ ....(i)

and f(2n + 1) = 1 ....(ii)

from (i) and (ii), we get :

L.H.L. $\neq$ f(2n + 1),

$\Rightarrow$ f(x) is discontinuous at x = 2n + 1

(i.e. at odd integers)

Hence f(x) is discontinuous at x = (2n + 1).

Example : 24

Let f (x) = $\begin{cases} \dfrac{1-\cos 4x}{x^2} & ; \quad x < 0 \\ a & ; \quad x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & ; \quad x > 0 \end{cases}$

Determine the value of a, if possible, so that the function is continuous at x = 0.

Solution

It is given that f(x) = $\begin{cases} \dfrac{1-\cos 4x}{x^2} & ; \quad x < 0 \\ a & ; \quad x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & ; \quad x > 0 \end{cases}$

is continuous at x = 0. So we can take :

$$\lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$$

Left hand limit at x = 0

L.H.L. = $\lim_{t\to 0^-} f(x) = \lim_{t\to 0^-} \dfrac{1-\cos 4x}{x^2}$

Now, L.H.L. = $\lim_{h\to 0} f(0 - h)$

$\Rightarrow$ L.H.L. = $\lim_{h\to 0} \dfrac{1-\cos 4h}{h^2} = \lim_{h\to 0} \dfrac{2\sin^2 2h}{h^2} = 8$ $\left[\text{using} : \lim_{t\to 0} \dfrac{\sin t}{t} = 1\right]$

Right hand limit at x = 0

R.H.L. = $\lim_{t\to 0^+} f(x) = \lim_{t\to 0^+} \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}-4}}$

Now, R.H.L. = $\lim_{h\to 0} f(0 + h)$

$\Rightarrow$ R.H.L. = $\lim_{h\to 0} \dfrac{\sqrt{h}}{\sqrt{16+\sqrt{h}-4}}$

Rationalising denominator to get :

$\Rightarrow$ R.H.L. = $\lim_{h\to 0} \dfrac{\sqrt{h}}{\sqrt{h}} \left(\sqrt{16+\sqrt{h}}+4\right) = 8$

For function f(x) to be continuous at x = 0,

L.H.L. = R.H.L. = f(0)

$\Rightarrow$ 8 = 8 = a

$\Rightarrow$ a = 8

Example : 25

Ler {x} and [x] denote the fractional and integral part of a real number x respectively.

Solve 4{x} = x + [x].

Solution

We can write x as :

x = integral part + fractional part

$\Rightarrow$ x = [x] + {x}

The given equation is 4{x} = x + [x]

$\Rightarrow \quad 4\{x\} = [x] + \{x\} + [x]$

$\Rightarrow \quad 3\{x\} = 2[x]$ ....(i)

$\Rightarrow \quad 3\{x\}$ is an even integer

But we have $0 \le \{x\} < 1$

$\Rightarrow \quad 0 \le 3\{x\} < 3$

$\Rightarrow \quad 3\{x\} = 0, 2$ [become 3{x} is even}

$\Rightarrow \quad \{x\} = 0, \frac{2}{3}$ and $[x] = \frac{3}{2}$ $\{x\} = 0, 1$ .... (using i)

$\Rightarrow \quad x = 0 + 0$ or $x = \frac{2}{3} + 1$

$\Rightarrow \quad x = 0$ or $x = \frac{5}{3}$

Example : 26

Discuss the continuity of f(x) in [0, 2] where $f(x) = \begin{cases} [\cos \pi x] & ; \quad x \le 1 \\ |2x-3|\,[x-2] & ; \quad x > 1 \end{cases}$

where [ ] : represents the greatest integer function.

Solution

First of all find critical points where f(x) may be discontinuous

Consider x – [0, 1] :

$f(x) = [\cos \pi x]$

$[f(x)]$ is discontinuous where $f(x) \in I$

$\Rightarrow \quad \cos \pi x = I$

In [0, 1], $\cos \pi x$ is an integer at $x = 0$, $x = \frac{1}{2}$ and $x = 1$

$\Rightarrow \quad x = 0, x = \frac{1}{2}$ and $x = 1$ are critical points ....(i)

Consider x – (1, 2] :

$f(x) = [x - 2]\,|2x - 3|$

In $x \in (1, 2)$ $[x - 2] = -1$ and

for $x = 2$; $[x - 2] = 0$

Also $|2x - 3| = 0 \quad \Rightarrow \quad x = \frac{3}{2}$

$\Rightarrow \quad x = \frac{3}{2}$ and $x = 2$ are critical points ....(i)

combining (i) and (ii), critical points are $0, \frac{1}{2}, 1, \frac{3}{2}, 2$

On dividing f(x) about the 5 critical points, we get

$$f(x) = \begin{cases} 1 & ; & x = 0 & \because & \cos(\pi 0) = 1 \\ 0 & ; & 0 < x \le \frac{1}{2} & \because & 0 \le \cos \pi x < 0 \Rightarrow [\cos \pi x] = 0 \\ -1 & ; & \frac{1}{2} < x \le 1 & \because & -1 \le \cos \pi x < 0 \Rightarrow [\cos \pi x] = -1 \\ -1(3-2x) & ; & 1 < x \le \frac{3}{2} & \because & |2x-3| = 3-2x \text{ and } [x-2] = -1 \\ -1(2x-3) & ; & \frac{3}{2} < x < 2 & \because & |2x-3| = 2x-3 \text{ and } [x-2] = -1 \\ 0 & ; & 2 & \because & [x-2] = 0 \end{cases}$$

Checking continuity at x = 0 :

R.H.L. = $\lim_{x \to 0^+}$ (0) = 0 and f(0) = 1

As, R.H.L. ≠ f(x)

⇒ f(x) is discontinuous at x = 0

Checking continuity at a + x = 1/2

L.H.L. = $\lim_{x \to \frac{1}{2}^-}$ f(x) = 0

R.H.L. = $\lim_{x \to \frac{1}{2}^-}$ f(x) = – 1

As L.H.L. ≠ R.H.L.,

f(x) is discontinuous at x = $\frac{1}{2}$.

Checking continuity at x = 1 :

L.H.L. = $\lim_{x \to 1^-}$ f(x) = – 1

R.H.L. = $\lim_{x \to 1^+}$ f(x) = – 1 = $\lim_{x \to 1^+}$ (2x – 3) = – 1

and f(1) = – 1

As L.H.L = R.H.L. = f(1)

f(x) is continuous at x = 1

Checking continuity at x = 3/2 :

L.H.L. = $\lim_{x \to \frac{3}{2}^-}$ (2x – 3) = 0

R.H.L. = $\lim_{x \to \frac{3}{2}^+}$ (3 – 2x) = 0 and $f\left(\frac{3}{2}\right)$ = 0

As L.H.L. = R.H.L. = $f\left(\frac{3}{2}\right)$,

f(x) is continuous at x = 3/2

Checking continuity at x = 2 :

L.H.L. = $\lim_{x \to 2^-}$ (3 – 2x) = – 1 and f(2) = 0

As L.H.L. ≠ f(2),

f(x) is discontinuous at x = 2

Example : 27

If f(x) = $\frac{\sin 2x + A\sin x + B\cos x}{x^3}$ is continuous at x = 0, find the values of A and B. Also find f(0).

Solution

As f(x) is continuous at x = 0

f(0) = $\lim_{x \to a}$ f(x) and both f(0) and $\lim_{x \to a}$ f(x) = are finite

⇒ f(0) = $\lim_{x \to 0} \frac{2\sin 2x + A\sin x + B\cos x}{x^3}$

As denominator → 0 as x → 0,

∴ Numerator should also → 0 as x → 0.

Which is possible only if (for f(0) to be finite)

sin 2(0) + A sin (0) + B cos 0 = 0

⇒ B = 0

$\therefore$ f(0) = $\lim_{x\to 0} \dfrac{\sin 2x + A\sin x}{x^2}$

$\Rightarrow$ f(0) = $\lim_{x\to 0} \left(\dfrac{\sin x}{x}\right)\left(\dfrac{2\cos x + A}{x^2}\right) = \lim_{x\to 0}\left(\dfrac{2\cos x + A}{x^2}\right)$

Again we can see that Denominator $\to$ 0 as x $\to$ 0

$\therefore$ Numerator should also approach 0 as x $\to$ 0 (for f(0) to be finite)

$\Rightarrow$ 2 + A = 0 $\Rightarrow$ A = – 2

$\Rightarrow$ f(0) = $\lim_{x\to 0}\left(\dfrac{2\cos x - 2}{x^2}\right) = \lim_{x\to 0}\left(\dfrac{-4\sin^2\frac{x}{2}}{x^2}\right) = \lim_{x\to 0}\left(\dfrac{-\sin^2\frac{x}{2}}{\frac{x^2}{4}}\right) = -1$

So we get A = – 2, B = 0 and f(0) = – 1

Example : 28

Evaluate $\lim_{x\to 0} \dfrac{64^x - 32^x - 16^x + 4^x + 2^x - 1}{\left(\sqrt{3+\cos x} - 2\right)\sin x}$

Solution

Let L = $\lim_{x\to 0} \dfrac{64^x - 32^x - 16^x + 4^x + 2^x - 1}{\left(\sqrt{3+\cos x} - 2\right)\sin x}$

On rationalising the denominated, we get

L = $\lim_{x\to 0} \dfrac{2^{6x} - 2^{5x} - 2^{4x} + 2^{2x} + 2^x - 1}{(\cos x - 1)\sin x}\left(\sqrt{3+\cos x} + 2\right)$

On factorising the numerator, we get

L = $\lim_{x\to 0} \dfrac{(2^x - 1)[2^{5x} - 2^{5x}(2^x - 1) + 1}{(\cos x - 1)\sin x} \times \lim_{x\to 0}\left(\sqrt{3+\cos x} + 2\right)$

$\Rightarrow$ L = $\lim_{x\to 0} \dfrac{(2^x - 1)[(2^{5x} - 2^{3x}) - (2^{2x} - 1)]}{(\cos x - 1)\sin x} \times 4$

$\Rightarrow$ L = $\lim_{x\to 0} \dfrac{(2^x - 1)(2^{2x} - 1)(2^{3x} - 1)}{(\cos x - 1)\sin x} \times 4$

$\Rightarrow$ L = $4 \lim_{x\to 0}\left(\dfrac{2^x - 1}{x}\right) \times 2 \lim_{x\to 0}\left(\dfrac{2^{2x} - 1}{2x}\right) \times 3$

$\lim_{x\to 0}\left(\dfrac{2^{3x} - 1}{3x}\right) \times \lim_{x\to 0}\left(\dfrac{x^2}{-2\sin^2(x/2)}\right) \times \lim_{x\to 0}\left(\dfrac{x}{\sin x}\right)$

$\Rightarrow$ L = 4 (ℓn 2) = 2 (ℓn 2) × 3 (ℓn 2) × (–2) $\Rightarrow$ L = – 48 (ℓn 2)^3

Example : 29

(i) If f is an even function defined on the interval (–5, 5), then find the four real values of x satisfying the equation f(x) = f $\left(\dfrac{x+1}{x+2}\right)$.

(ii) Evaluate : $\lim_{x\to 0}\left(\dfrac{1+5x^2}{1+3x^2}\right)^{\frac{1}{x^2}}$.

(iii) If $f(x) = \sin^2 x + \sin^2\left(x+\frac{\pi}{3}\right) = \cos x \cos\left(x+\frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right) = 1$, then find g [f(x)].

(iv) Let $f(x) = [x] \sin \frac{(\pi)}{[x+1]}$ where [ ] denotes the greater integer function. Find the domain of f(x) and the points of discontinuity of f(x) in the domain.

Solution

(i) It is given that $f(x) = f\left(\frac{x+1}{x+2}\right)$

$\Rightarrow \quad x = \left(\frac{x+1}{x+2}\right) \quad \Rightarrow \quad x^2 + x - 1 = 0$

$\Rightarrow \quad x = \frac{-1 \pm \sqrt{5}}{2}$ ....(i)

As f(x) is even, f(x) = f(–x)

$-x = \left(\frac{x+1}{x+2}\right) \quad \Rightarrow \quad x^2 + 3x + 1 = 0 \quad \Rightarrow \quad x = \frac{-3 \pm \sqrt{5}}{2}$ ....(ii)

One combining (i) and (ii), we get :

$x = \frac{-1 \pm \sqrt{5}}{2}$ and $x = \frac{-3 \pm \sqrt{5}}{2}$.

(ii) Let $L = \lim_{x \to 0} \left(\frac{1+5x^2}{1+3x^2}\right)^{\frac{1}{x^2}}$

$\Rightarrow \quad L = \lim_{x \to 0} \left(1 + \frac{1+5x^2}{1+3x^2} - 1\right)^{\frac{1}{x^2}}$

$\Rightarrow \quad L = \lim_{x \to 0} \left(1 + \frac{2x^2}{1+3x^2}\right)^{\frac{1}{x^2}}$

$\Rightarrow \quad L = \lim_{x \to 0} e^{\left(\frac{2}{1+3x^2}\right)} = e^2 \qquad \left[\text{using} : \lim_{t \to 0} (1+t)^{\frac{1}{t}} = e\right]$

(iii) It is given that $f(x) = 1 - \cos^2 x + \sin^2\left(x+\frac{\pi}{3}\right) + \cos x \cos\left(x+\frac{\pi}{3}\right)$

$= 1 - \left[\cos^2 x - \sin^2\left(x+\frac{\pi}{3}\right)\right] + \frac{1}{2}\left[2\cos x \cos\left(x+\frac{\pi}{3}\right)\right]$

$= 1 - \cos\left(2x+\frac{\pi}{3}\right) \cos\frac{\pi}{3} + \frac{\cos\left(2x+\frac{\pi}{3}\right)}{2} + \frac{\cos\left(\frac{\pi}{3}\right)}{2} = 1 + \frac{\cos\left(\frac{\pi}{3}\right)}{2} = \frac{5}{4}$

$\Rightarrow$ For all values of x, f(x) = $\frac{5}{4}$. (constant function)

Hence, g[f(x)] = $g\left(\frac{5}{4}\right)$

But $g\left(\frac{5}{4}\right) = 1 \Rightarrow$ g [(f(x)] = 1

Hence, g[f(x)] = 1 for all values of x

(iv) Let f(x) = [x] sin $\frac{(\pi)}{[x+1]}$

Domain of f(x) is $x \in$ R excluding the point where [x + 1] = 0
($\because$ denominator cannot be zero)
Find values of x which satisfy [x + 1] = 0
[x + 1] = 0
$\Rightarrow$ $0 \le x + 1 < 1$
$\Rightarrow$ $-1 \le x < 0$
i.e. for all $x \in [-1, 0)$, denominator is zero.
So, domain is $x \in$ R [–1, 0)
$\Rightarrow$ Domain is $x \in (-\infty, -1) \cup [0, \infty)$
Point of Discontinuity
As greatest integer function is discontinuous at integer points, f(x) is continuous for all non-integer points.
Checking continuity at x = a (where a – 1)

L.H.L. $= \lim_{h\to 0} [a-h] \sin\left(\frac{\pi}{[a+1-h]}\right)$

$\Rightarrow$ L.H.L. $= (a-1)\sin\left(\frac{\pi}{a}\right)$ ....(i)

R.H.L. $= \lim_{h\to 0} [a+h] \sin\left(\frac{\pi}{[a+1+h]}\right)$

$\Rightarrow$ L.H.L. $= a\sin\left(\frac{\pi}{a+1}\right)$ ....(ii)

From (i) and (ii), L.H.L. $\neq$ R.H.L.
$\Rightarrow$ f(x) is discontinuous at x = a (i.e. at integer values of x)
So, points of discontinuity are $x \in I \cap D$. (i.e. integers lying in the set of domain)
$\Rightarrow$ $x \in I - \{-1\}$.

Permutations & Combinations

Example : 1

How many (a) 5 – digit (b) 3–digit numbers can be formed using 1, 2, 3, 7, 9 without any repetition of digits?

Solution

(a) 5-digit numbers

Making a 5-digit number is equivalent to filling 5 places

Places :					
No. of choices :	**1**	**2**	**3**	**4**	**5**

The last place (unit’s place) can be filled in 5 ways using any of the five given digits.
The ten’s place can be filled in four ways using any of the remaining 4 digits.
The number of choices for other places can be calculated in the same way.
No. of ways to fill all five places = $5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$

$\Rightarrow$ 120 five-digit numbers can be formed

(b) 3-digit numbers

Making a three-digit number is equivalent to filling three places (unit’s, ten’s, hundred’s)

Places :			
No. of choices :	**3**	**4**	**5**

No. of ways to fill all the three places = $5 \times 4 \times 3 = 60$

$\Rightarrow$ 60 three-digit numbers can be formed

Example : 2

How many 3-letter words can be formed using a, b, c, d, e if :

(a) repetition is not allowed (b) repetition is allowed ?

Solution

(a) Repetition is not allowed :

The number of words that can be formed is equal to the number of ways to fill the three places

Places :			
No. of choices :	**5**	**4**	**3**

$\Rightarrow$ $5 \times 4 \times 3 = 60$ words can be formed

(b) Repetition is allowed:

The number of words that can be formed is equal to the number of ways to fill the three places.

Places :			
No. of choices :	**5**	**5**	**5**

First place can be filled in five ways (a, b, c, d, e)
If repetition is allowed, all the remaining places can be filled in five ways using a, b, c, d, e.
No. of words = $5 \times 5 \times 5 = 125$ words can be formed

Example : 3

How many four-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5?

Solution

For a four-digit number, we have to fill four places and - cannot appear in the first place (thousand’s place)

Places :				
No. of choices :	**5**	**5**	**4**	**3**

For the first place, there are five choices (1, 2, 3, 4, 5); Second place can then be filled in five ways (0 and remaining four-digits); Third place can be filled in four ways (remaining four-digits); Fourth place can be filled in three ways (remaining three-digit).
Total number of ways = $5 \times 5 \times 4 \times 3 = 300$

$\Rightarrow$ 300 four-digits numbers can be formed

Example : 4

In how many ways can six persons be arranged in a row?

Solution

Arranging a given set of n different objects is equivalent to filling n places

So arranging six persons along a row is equivalent to filling 6 places

Places :						
No. of choices :	**6**	**5**	**4**	**3**	**2**	**1**

No. of ways to fill all places = 6 × 5 × 4 × 3 × 2 × 1 = 6! = 720

Hence 720 arrangements are possible

Example : 5

How many nin-letter words can be formed by using the letters of the words

(a) E Q U A T I O N S (b) A L L A H A B A D

Solution

(a) All nine letters in the word EQUATIONS are different

Hence number of words = ${}^{9}P_{9}$ = 9! = 362880

(b) ALLAHABAD contains LL, AAAA, H, B, D.

No. of words = $\frac{9!}{2!\,4!} = \frac{9\times8\times7\times6\times5}{2}$ = 7560

Example : 6

(a) How many words can be made by using the letters of the word C O M B I N E all at a time?

(b) How many of these words begin and end with a vowel?

(c) In how many of these words do the vowels and the consonants occupy the same relative positions as in C O M B I N E ?

Solution

(a) The total number of words – arrangement of seven letters taken all at a time = ${}^{7}P_{7}$ = 7! = 5040

(b) The corresponding choices for all the places are as follows :

Places	**vowel**					**vowel**
No. of choices	**3**	**5**	**4**	**3**	**2**	**1**	**2**

As there are three vowels (O I E), first place can be filled in three ways and the last place can be filled in two ways. The rest of the places can be filled in 5 ! ways using five remaining letters.

No. of words = 3 × 5! × 2 = 720

(c) Vowels should be at second, fifth and seventh positions

They can be arranged in 3! ways

Consonants should be at first, third, fourth and sixth positions.

They can be arranged here in 4! ways

Total number of words = 3! × 4! = 144

Example : 7

How many words can be formed using the letters of the word TRIANGLE so that

(a) A and N are always together?

(b) T, R, I are always together?

Solution

(a) Assume (AN) as a single letter. Now there are seven letters in all : (AN), T, R, I, G, L, E

Seven letters can be arranged in 7! ways

All these 7! words will contain A and N together. A and N can now be arranged among themselves in 2! ways (AN and NA).

Hence total number of words = 7! 2! = 10080

(b) Assume (TRI) as a single letters

(i) The letters : (TRI), A, N, G, L, E can be arranged in 6! ways

(ii) TRI can be arranged among themselves in 3! ways

Total number of words = 6! 3! = 4320

Example : 8

There are 9 candidates for an examination out of which 3 are appearing in Mathematices and remaining 6 are appearing in different subjects. In how many ways can they be seated in a row so that no two Mathematices candidates are together?

Solution

Divide the work in two operations.

(i) First, arrange the remaining candidate in 6! ways

(ii) Place the three Mathematices candidate in the row of six other candidate so that no two of them are together.

X : Places available for Mathematices candidates.

O : Others

X	O	X	O	X	O	X	O	X	O	X	O	X

In any arrangement of 6 other candidates (O), there are seven places available for Mathematices candidates so that they are not together. Now 3 Mathematices candidates can be placed in these 7 places in $^7P_3$ ways.

Hence total number of arrangements = $6!\ ^7P_3 = 720 \times \frac{7!}{4!} = 151200$

Example : 9

(a) How many triangle can be formed by joining the vertices of a bexagon?

(b) How many diagonals are there in a polygon with n sides?

Solution

(a) Let $A_1, A_2, A_3, ..., A_6$ be the vertices of the bexagon. One triangle is formed by selecting a group of 3 points from 6 given vertices.

No. of triangles = No. of groups of 3 each from 6 points = $^6C_3 = \frac{6!}{3!\,3!} = 20$

(b) No. of lines that can be formed by using the given vertices of a polygon = No. of groups of 2 points each selected from the n points

$= {}^nC_2 = \frac{n!}{n!(n-2)!} = \frac{n(n-1)}{2}$

Out of $^nC_2$ lines, n are the sides of the polygon and remaining $^nC_2 - n$ are the diagonals

So number of diagonals = $\frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$

Example : 10

In how many ways can a circket team be selected from a group of 25 players containing 10 batsmen, 8 bowlers, 5 all-rounders and 2 wicketkeepers? Assume that the team of 11 players requires 5 batsmen, 3 all-rounders, 2-bowlers and 1 wicketkeeper.

Solution

Divide the selection of team into four operation.

I : Selection of bastsman can be done (5 from 10) in $^{10}C_5$ ways.

II : Selection of bowlers can be done (2 from 8) in $^8C_2$ ways

III : Selection of all-rounders can be done (3 from 5) in $^5C_3$ ways

IV : Selection of wicketkeeper can be done (1 from 2) in $^2C_1$ ways

$\Rightarrow$ the team can be selected in = $^{10}C_5 \times {}^8C_2 \times {}^5C_3 \times {}^2C_1$ ways = $\frac{10! \times 8 \times 7 \times 10 \times 2}{5!\,5!\,2!} = 141120$

Example : 11

A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be selected so that there are at least two balls of each colour?

Solution

The selection of balls from 5 red and 6 white balls will consist of any of the following possibilities

RED BALLS (out of 5)	WHITE BALLS (out of 6)
2	4
3	3
4	2

- If the selection contains 2 red and 4 white balls, then it can be done in $^5C_2\ ^6C_4$ ways
- If the selection contains 3 red and 3 white balls then it can be done in $^5C_3$ ways
- If the selection contains 4 red and 2 white balls then it can be done in $^5C_4\ ^6C_2$ ways

Any one of the above three cases can occur. Hence the total number of ways to select the balls

$= {}^5C_2\ {}^6C_4 + {}^5C_3\ {}^6C_3 + {}^5C_4\ {}^6C_2$

$= 10\ (15) + 10\ (20) + 5\ (15) = 425$

Example : 12

How many five-letter words containing 3 vowels and 2 consonants can be formed using the letters of the word E Q U A T I O N so that the two consonants occur together in every word?

Solution

There are 5 vowels and 3 consonants in EQUATION. To form the words we will do there operations.

I. Select vowels (3 from 5) in $^5C_3$ ways

II. Select consonants (2 from 3) in $^3C_2$ ways

III. Arrange the selected letters (3 vowels and 2 consonants always together) in 4! 2! ways. Hence the no. of words = $^5C_3\ ^3C_2\ 4!\ 2! = 10 \times 3 \times 24 \times 2 = 1440$

Example : 13

How many four-letter words can be formed using the letters of the word I N E F F E C T I V E ?

Solution

INEFFECTIVE contains 11 letters : EEE, FF, II, C, T, N, V.

As all letters are not different, we cannot use ^nP_r.

The four-letter words will be among any one of the following categories

(1) 3 alike letters, 1 different letter

(2) 2 alike letters, 2 alike letter

(3) 2 alike letters, 2 different letters

(4) All different letters

(1) 2 alike, 1 different :

3 alike can be selected in one way i.e. EEE

Different letters can be selected from F, I, T, N, V, C in $^6C_1$ ways

$\Rightarrow$ No. of groups = $1 \times {}^6C_1 = 6$

$\Rightarrow$ No. of words = $6 \times \dfrac{4!}{3! \times 1!} = 24$

(2) 2 alike 2alike

Two sets of 2 alike can be selected from 3 sets (EE, II, FF) in $^3C_2$ ways

$\Rightarrow$ No. of words = $^3C_2 \times \dfrac{4!}{2! \times 2!} = 18$

(3) 2 alike 2 different

$\Rightarrow$ No. of groups = $(^3C_1) \times (^6C_2) = 45$

$\Rightarrow$ No, of words = $45 = \dfrac{4!}{2!} = 540$

(4) All different

$\Rightarrow$ No. of groups = $^7C_4$ (out of E, F, I, T, N, V, C)

Hence total four-letter words = 24 + 18 + 540 + 840 = 1422

Example : 14

A man has 5 friends. In how many ways can he invite one or more of them to a party?

Solution

If he invites one person to the party No. of ways = ${}^5C_1$

If he invites two persons to the party No. of ways = ${}^5C_2$

Proceeding on the similar pattern,

Total number of ways to invite = ${}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = 5 + 10 + 10 + 5 + 1 = 31$

Alternate method :

To invite one or more friends to the party, he has to take 5 decisions - one for every friend.

Each decision can be taken in two ways - invited or not invited

Hence (the number of ways to invite one or more)

= (number of ways to make 5 decisions – 1)

$= 2 \times 2 \times 2 \times 2 - 1 = 2^5 - 1 = 5!$

Note that we have subtract 1 to exclude the case when all are not invited.

Example : 15

Find the number of ways in which one or more letters can be selected from the letters :

A A A A B B B C D E

Solution

The given letters can be divided into five following categories : (AAAA), (BBB), C, D, E

To select at least one letter, we have to take five decisions-one for every category

Selections from (AAAA) can be made in 5 ways : include no A, include one A, include AA, include AAA, include AAAA

Similarly, selections from (BBB) can be made in 4 ways, and selections from C, D, E can be made in $2 \times 2 \times 2$ ways.

$\Rightarrow$ total number of selections $= 5 \times 4 \times (2 \times 2 \times 2) - 1 - 159$

(excluding the case when no letter is selected)

Example : 16

The question paper is an examination contains three sections - A, B, C. There are 6, 4, 3 questions in sections A, V, C respectively. A student has the freedom to answer any number of questions attempting at least one from each section. In how many ways can the paper be attempted by a student?

Solution

There are three possible cases :

(i) Section A contains 6 questions. The student can select at least one from these in $2^6 - 1$ ways.

(ii) Section B contains 4 questions. The student can select at least one from these in $2^4 - 1$ ways.

(iii) Section C can similarly be attempted in $2^3 - 1$ ways

$\Rightarrow$ Hence total number of ways to attempt the paper $= (2^6 - 1)(2^4 - 1)(2^3 - 1) = 63 \times 15 \times 7 = 6615$

Example : 17

Find all number of factors (excluding 1 and the expression itself) of the product of $a^7 b^4 c^3$ d e f where a, b, c, d, e, f are all prime numbers.

Solution

A factor of expression $a^7 b^4 c^3$ d e f is simply the result of selecting one or more letters from 7 a's, 4 b's, 3a's, d, e, f. The collection of letters can be observed as a collection of 17 objects out of which 7 are alike of one kind (a's), 4 are of second kind (b's), 3 are of third kind (c, s) and 3 are different (d, e, f)

The number of selections $= (1 + 7)(1 + 4)(1 + 3)\, 2^3 = 8 \times 5 \times 4 \times 8 = 1280$.

But we have to exclude two cases :

(i) When no letter is selected

(ii) When all are selected

Hence the number of factors = 1280 – 2 = 1278

Example : 18

In how many ways can 12 books be equally distributed among 3 students ?

Solution

Each student will get 4 books

1. First student can be given 4 books from 12 in ${}^{12}C_4$ ways

2. Second student can be given 4 books from remaining 8 books in ${}^{8}C_4$ ways

3. Third student can be given 4 books from remaining 4 in ${}^{4}C_4$ ways

$\Rightarrow$ the total number of ways to distribute the books = ${}^{12}C_4 \times {}^{8}C_4 \times {}^{4}C_4$

Example : 19

How many four-letter words can be made using the letters of the word FAILURE, so that

(a) F is included in each word?

(b) F is not included in any word?

Solution

(a) To include F in every word, we will do two operators.

I. Select the remaining three letters from remaining 6 letters i.e. A, I, L, U, R, E in ${}^{7-1}C_{4-1} = {}^{6}C_3$ ways

II. Include F in each group and then arrange each group of four letters in 4! ways

No. of words = ${}^{6}C_3$ 4! = 480

(b) If F not to be included, then we have to select all the four letters from the remaining 6.

No. of words = ${}^{7-1}C_4$ 4! = ${}^{6}C_4$ 4! = 360

Example : 20

(a) In how many ways can 5 persons be arranged around a circular table ?

(b) In how many of these arrangements will two particular persons be next to each other?

Solution

(a) Let the five persons be A_1, A_2, A_3, A_4, A_5 .

Let us imagine A_1 as fixed in its position. The remaining 4 persons can be arranged among themselves in 4! ways.

Hence the number of different arrangements = (5 – 1)! = 4! = 24

(b) Let us assume that A_1 and A_2 are the two particular persons next to each other.

Treating $(A_1 A_2)$ as one person, we have 4 persons in all to arrange in a circle : $(A_1 A_2) A_3 A_4 A_5$. These can be arranged in a circle in (4 – 1)! = 3! = 6 ways.

Now A_1 and A_2 can be arranged among themselves in 2! ways.

Hence total number of arrangement = 6 × 2! = 12

Example : 21

There are 20 persons among whom are two brothers. Find the number of ways in which we can arrange them around a circle so that there is exactly one person between the two brothers.

Solution

Let B_1 and B_2 are the two brother and M be a person sitting between B_1 and B_2 .

Divide this problem into three operations.

I. Select M from 18 persons (excluding B_1 and B_2). This can be done in ${}^{18}C_1$ ways

II. Treat (B_1, M, B_2) as one person. Arrange (B_1, M, B_2) and other 17 persons around a circle. This can be done in (18 – 1) ! = 17 ! ways

III. B_1 and B_2 can be arranged among themselves in 2! ways.

So total number of ways = ${}^{18}C_1$ × 17! × 2 = 2 (18!)

Example : 22

Three tourist want to stay in five different hotels. In how many ways can they do so if :

(a) each hotel cannot accommodate more than one tourist?

(b) each hotel can accommodate any number of tourists?

Solution

(a) There tourists are to be placed in 3 different hotels out of 5. This can be done in two steps :

I. Select three hotels from five in ${}^{5}C_3$ ways.

II. Place the three tourists in 3 selected hotels in 3! ways

$\Rightarrow$ the required number of ways = ${}^{5}C_3$ 3! = 5 × 4 × 3 = 60

(b) To place the tourists we have to do following three operations

I. Place first tourist in any of the hotels in 5 ways ways

II. Place second tourist in any of the hotels in 5 ways

III. Place third tourist in any of the hotels in 5 ways

$\Rightarrow$ the required number of placements = 5 × 5 × 5 = 125

Example : 23

How many seven-letter words can be formed by using the letters of the word S U C C E S S so that :

(a) the two C are together but not two S are together?

(b) no two C and no two S are together

Solution

(a) Considering CC as single object U, CC, E can be arranged in 3! ways.

X U X C C X E X

Now the three S are to be placed in the 4 available places (X) so that CC are not separated but S are separated.

No. of ways to place SSS = (No. of ways to select 3 places) x 1 = $^4C_3$ x 1 = 4

$\Rightarrow$ No. of words = 3! x 4 = 24

(b) Let us first find the words in which no two S are together. To achieve this, we have to do following operations.

(i) Arrange the remaining letters UCCE in $\frac{4!}{2!}$ ways

(ii) Place the three SSS in any arrangement from (i)

X U X C X C X E X

There are five available places for three SSS.

No. of placements = $^5C_3$

Hence total number of words with no two S together = $\frac{4!}{2!}\ {}^5C_3 = 120$

No. of words having CC separated and SSS separated = (No. of words having SSS separated) - (No. of words having SSS separated but CC together = 120 – 24 = 96 [using result of part (a)]

Example : 24

A ten party is arranged for 16 people along two sides of a long table with 8 chairs on each side. Four men wish to sit on one particular side and two on the other side. In how many ways can they be seated?

Solution

Let $A_1, A_2, A_3, \ldots\ldots\ldots\ldots, A_{16}$ be the sixteen persons. Assume that A_1, A_2, A_3, A_4 want to sit on side 1 and A_5, A_6 wan to sit on side 2.

The persons can be made to sit if we complete the following operations.

(i) Select 4 chairs from the side 1 in $^8C_4$ ways and allot these chairs to A_1, A_2, A_3, A_4 in 4! ways

(ii) Select two chairs from side 2 in $^8C_2$ ways and allot these two chairs to A_5 , A_6 in 2! ways

(iii) Arrange the remaining 10 persons in remaining 10 chairs in 10 ! ways

$\Rightarrow$ Hence the total number of ways in which the persons can be arranged

$$= ({}^8C_4\ 4!)\ ({}^8C_2\ 2!)\ (10!) = \frac{8!}{4!\ 4!}\ 4! \times \frac{8!\ 2!}{2!\ 6!}\ 10! = \frac{8!\ 8!\ 10!}{4!\ 6!}$$

Example : 25

A mixed doubles tennis game is to be arranged from 5 married couples. In how many ways the game be arranged if no husband and wife pair is included in the same game?

Solution

To arrange the game we have to do the following operating

(i) Select two men from from 5 men in $^5C_2$ ways.

(ii) Select two women from 5 women excluding the wives of the men already selected. This can be done in $^3C_2$ ways.

(iii) Arrange the 4 selected persons in two teams. If the selected men are M_1 and M_2 and the selected women are W_1 and W_2 , this can be done in 2 ways :

$M_1\ W_1$ play against $M_2\ W_2$

$M_2\ W_1$ play against $M_1\ W_2$

Hence the number of ways to arrange the game = $^5C_2\ {}^3C_2\ (2) = 10 \times 3 \times 2 = 60$

Example : 26

A man has 7 relatives, 4 of them ladies and 3 gentlemen; his wife has 7 relatives, 3 of them are ladies and 4 gentlemen. In how many ways can they invite a dinner party of 3 ladies and 3 gentlemen so that there are 3 of man's relatives and 3 of wife's relatives?

Solution

The possible ways of selecting 3 ladies and 3 gentleman for the party can be analysed with the help of the following table.

Man's relative		**Wife's relative**		**Number of ways**
Ladies (4)	Gentlemen (3)	Ladies (3)	Gentlemen (4)	
3	0	0	3	${}^4C_3\ {}^3C_0\ {}^3C_0\ {}^4C_3 = 16$
2	1	1	2	${}^4C_2\ {}^3C_1\ {}^3C_1\ {}^4C_2 = 324$
1	2	2	1	${}^4C_1\ {}^3C_2\ {}^3C_2\ {}^4C_1 = 144$
0	3	3	0	${}^4C_0\ {}^3C_3\ {}^3C_3\ {}^4C_0 = 1$

Total number of ways in invite = 16 + 324 + 144 + 1 = 485

Example : 27

In how many ways can 7 plus (+) signs and 5 minus (–) signs be arranged in a row so that no two minus (–) signs are together?

Solution

(i) The plus signs can be arranged in one way (because all are identical)

	+		+		+		+		+		+		+

A blank box shows available spaces for the minus signs.

(ii) The 5 minus (–1) signs are now to be placed in the 8 available spaces so that no two of them are together

(i) Select 5 places for minus signs in ${}^8C_5$ ways.

(ii) Arrange the minus signs in the selected places in 1 way (all signs being identical).

Hence number of possible arrangements = $1 \times {}^8C_5 \times 1 = 56$

Example : 28

There are p points in a plane, no three of which are in the same straight line with the exception of q, which are all in the same straight line. Find the number of

(a) straight lines,

(b) triangles

which can be formed by joining them.

Solution

(a) If no three of the p points were collinear, the number of straight lines = number of groups of two that can be formed from p points = ${}^pC_2$.

Due to the q points being collinear, there is a loss of ${}^qC_2$ lines that could be formed from these points.

But these points are giving exactly one straight line passing through all of them.

Hence the number of straight line = ${}^pC_2 - {}^qC_2 + 1$

(b) If no three points were collinear, the number of triangles = ${}^pC_3$

But there is a loss of ${}^qC_2$ triangles that could be formed from the group of q collinear points.

Hence the number of triangles formed = ${}^pC_3 - {}^qC_3$

Example : 29

(a) How many six-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5?

(b) How many of these are even?

(c) How many of these are divisible by 4?

(d) How many of these are divisible by 25?

Solution

(a) To make six digit number we have to fill six places. The corresponding choices are as follows:

Places						
No. of choices	5	5	4	3	2	1

$\Rightarrow$ $5 \times 5! = 600$ numbers.

(b) To calculate even numbers, we have to count in two parts :

(i) even number ending in 0

Places						0
No. of choices	5	4	3	2	1	1

$\Rightarrow$ $5! = 120$ numbers can be formed

(ii) even numbers ending in 2, 4

Places						2, 4
No. of choices	4	4	3	2	1	2

There are two choices (2, 4) for the last place and four choices (non-zero digit from remaining) for the first places.

$\Rightarrow$ $4 \times 4! \times 2 = 192$ numbers can be formed. Hence total even numbers that can be formed $= 120 + 192 = 312$

(c) The multiples of 4 can be divided into following groups

(i) ending with (04)

Places					0	4
No. of choices	4	3	2	1	1	1

$\Rightarrow$ $4! = 24$ multiples of 4 ending in (04) are possible

(ii) ending with (24)

Places					2	4
No. of choices	3	3	2	1	1	1

There are 3 choices (1, 3, 5) for the first place. Remaining three places can be filled in 3! ways using any of the remaining three digits

$\Rightarrow$ $3 \times 3! = 18$ numbers are possible

(iii) ending with 0

Places					2, 4	0
No. of choices	4	3	2	1	2	1

Note that there are two choices (2, 4) for the ten's place.

$\Rightarrow$ $4! \times 2 = 48$ numbers are possible

(iv) ending with 2

Places					1, 3, 5	2
No. of choices	3	3	2	1	3	1

Note that there are three choices (1, 3, 5) for the ten's place

$\Rightarrow$ $3 \times 3! \times 3 \times 1 = 54$ numbers are possible

Hence the total number of multiples of 4 = 24 + 48 + 72 = 144

(d) numbers divisible by 25 must end with 25 or 50

(i) ending with 2,5

Places					2	5
No. of choices	3	3	2	1	1	1

$\Rightarrow$ $3 \times 3! = 18$ numbers are possible

(ii) ending with 5, 0

Places					5	0
No. of choices	4	3	2	1	1	1

$\Rightarrow$ $4! = 24$ numbers are possible

Hence total numbers of multiples of 25 = 18 + 24 = 42

Example : 30

Find the sum of all five-digit numbers that can be formed using digits 1, 2, 3, 4, 5 if repetition is not allowed?

Solution

There are 5! = 120 five digit numbers and there are 5 digits. Hence by symmetry or otherwise we can see that each digit will appear in any place (unit's or ten's or ....) $\frac{5!}{5}$ times

$\Rightarrow$ X = sum of digits in any place

$$X = \frac{5!}{5} \times 5 + \frac{5!}{5} \times 4 + \frac{5!}{5} \times 3 + \frac{5!}{5} \times 2 + \frac{5!}{5} \times 1$$

$$X = \frac{5!}{5} \times (5 + 4 + 3 + 2 + 1) = \frac{5!}{5} (15)$$

$\Rightarrow$ the sum of all numbers = X + 10 X + 100X + 1000X + 10000X

= X(1 + 10 + 100 + 1000 + 10000)

$$= \frac{5!}{5} (15) (1 + 10 + 100 + 1000 + 10000)$$

= 24 (15) (11111) = 3999960

Example : 31

Find the number of ways of distributing 5 identical balls three boxes so that no box is empty and each box being large enough to accommodate all the balls.

Solution

Let x_1, x_2 and x_3 be the number of balls places in Box – 1, Box - 2 and Box - 3 respectively.

The number of ways of distributing 5 balls into Boxes 1, 2 and 3 is the number of integral solutions of the equation $x_1 + x_2 + x_3 = 5$ subjected to the following conditions on x_1, x_2, x_3 ....(i)

Conditions on x_1, x_2 and x_3

According to the condition that the boxes should contain atleast one ball, we can find the range of x_1, x_2 and x_3 i.e.

Min. $(x_i) = 1$ and Max $(x_i) = 3$ for $i = 1, 2, 3$

[using : Max $(x_i) = 5 –$ Min $(x_2) –$ Min (x_3)]

or $1 \le x_i \le 3$ for $i = 1, 2, 3$

So, number of ways of distributing balls

= number of integral solutions of (i)

= coeff. of x^5 in the expansion of $(x + x^2 + x^3)^3$

= coeff. of x^2 in $(1 – x^3) (1 – x)^{–3}$

= coeff. of x^2 in $(1 – x)^{–3}$

= ${}^{3+2–1}C_2 = 6$

Alternate solution

The number of ways of dividing n identical objected into r groups so that no group remains empty

$= {}^{n-1}C_{r-1}$ [using result 6.3(a)]

$= {}^{5+1}C_{3-1} = {}^{4}C_{2} = 6$

Example : 32

Find the number of ways of distributing 10 identical balls in 3 boxes so that no box contains more than four balls and less than 2 balls

Solution

Let x_1, x_2 and x_3 be the number of balls placed in Boxes 1, 2 and 3 respectively

Number of ways of distributing 10 balls in 3 boxes = Number of integral solutions of the equation $x_1 + x_2 + x_3 = 10$ ....(i)

Conditions on x_1, x_2 and x_3

As the boxes should contain atmost 4 ball, we can make Max$(x_i) = 4$ and Min $(x_i) = 2$ for i = 1, 2, 3

[using : Min $(x_1) = 10 -$ Max $(x_2) -$ Max (x_3)]

or $2 \le x_i \le 4$ for i = 1,2 , 3

So the number of ways of distributing balls in boxes = number of integral solutions of equation (i)

= coeff. of x^{10} in the expansion of $(x^2 + x^3 + x^4)^3$

= coeff. of x^{10} in $x^6 (1 - x^3)^3 (1 - x)^{-3}$

= coeff. of x^4 in $(1 - x^3)^3 (1 - x)^{-3}$

= coeff. of x^4 in $(1 - {}^3C_1 x^3 + {}^3C_2 x^6 + ...) (1 - x)^{-3}$

= coeff. of x^4 in $(1 - x)^{-3}$ – coeff. of x in ${}^3C_1 (1 - x)^{-3}$

$= {}^{4+3-1}C_4 - 3 \times {}^{3+1-1}C_1 = {}^6C_4 - 3 \times {}^3C_1 = 15 - 9 = 6$

Note : Instead of taking minimum value $x_i = 2$

(for i = 1, 2, 3), we can also consider it 0 i.e. we can take $0 \le x_i \le 4$

Example : 33

In a box there are 10 balls, 4 are red, 3 black, 2 white and 1 yellow. In how many ways can a child select 4 balls out of these 10 balls? (Assume that the balls of the same colour are identical)

Solution

Let x_1, x_2 x_3 and x_4 be the number of red, black, white, yellow balls selected respectively

Number of ways to select 4 balls = Number of integral solution of the equation $x_1 + x_2 + x_3 + x_4 = 4$

Conditions on x_1, x_2, x_3 and x_4

The total number of red, black, white and yellows balls in the box are 4, 3, 2 and 1 respectively.

So we can take :

Max $(x_1) = 4$, Max $(x_2) = 3$, Max $(x_3) = 2$, Max $(x_1) = 1$

There is no condition on minimum number of red, black, white and yellow balls selected, so take :

Min $(x_i) = 0$ for 1 = 1, 2, 3, 4

Number of ways to select 4 balls = coeff. of x^4 in

$(1 + x + x^2 + x^3 + x^4) \times (1 + x + x^2 + x^3) \times (1 + x + x^2) \times (1 + x)$

= coeff. of x^4 in $(1 - x^5) (1 - x^4) (1 - x^3) (1 - x^2) (1 - x)^{-4}$

= coeff. of x^4 in $(1 - x)^{-4}$ – coeff. of x^2 in $(1 - x)^{-4}$ – coeff. of x^1 in $(1 - x)^{-4}$ – coeff. off x^0 in $(1 - x)^{-4}$

$$= {}^7C_4 - {}^5C_2 - {}^4C_1 - {}^3C_0 = \frac{7 \times 6 \times 5}{3!} - 10 - 4 - 1 = 35 - 15 = 20$$

Thus, number of ways of selecting 4 balls from the box subjected to the given conditions is 20.

Alternate Solutions :

The 10 balls are RRRR BBB WW Y (where R, B, W, Y represent red, black, white and yellow balls respectively).

The work of selection of the balls from the box can be divided into following categories

Case – 1 All alike

Number of ways of selecting all alike balls = ${}^1C_1 = 1$

Case – 2 3 alike and 1 different

Number of ways of selecting 3 alike and 1 different balls = ${}^2C_1 \times {}^3C_1 = 6$

Case – 3 2 alike and 2 alike

Number fo ways of selecting 2 alike and 2 alike balls = ${}^3C_2 = 3$

Case – 4 2 alike and 2 different

Number of ways of selecting 2 alike and 2 different balls = ${}^3C_1 \times {}^3C_1 = 9$

Case – 5 All different

Number of ways of selecting all different balls = ${}^4C_4 = 1$

Total number of ways to select 4 balls = 1 + 6 + 3 + 9 + 1 = 20

Example : 34

A person writes letters to 4 friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that :

(i) atleast two of them are in the wrong envelopes

(ii) all the letters are in the wrong envelopes

Solution

(i) Number of ways to place 4 letters in 4 envelopes without any condition = 4!

Number of ways to place all letters correctly into the corresponding envelopes = 1

Number of ways to place one letter is the wrong envelop and other 3 letters in the write envelope = 0

(Because it is not possible that only one letter goes in the wrong envelop)

Number of ways to place atleast two letters in the wrong envelopes

= Total number of way to place letters

– Number of ways to place all letters correctly

– Number of ways to place on letter correctly = 4! – 1 – 0 = 23

(ii) Number of ways to put 2 letters in 2 addressed envelopes so that all are in the wrong envelopes = 1.

Number fo ways to put 3 letters in 3 addressed envelopes so that all are in the wrong envelopes = Number of ways without restriction – Number of ways in which all letters are in the correct envelopes – Number of ways in which 1 letter is in the correct envelope = $3! - 1 - 1 \times {}^3C_1 = 2$.

(${}^3C_1$ means that select one envelop to put the letter correctly)

Number of ways to put 4 letters in 4 addressed envelopes so that all are in the wrong envelopes

= Number of ways without restriction

– Number of ways in which all letters are in the correct envelopes

– Number of ways in which 1 letter is in the correct envelopes

(i.e. 3 are in the wrong envelopes)

– Number of ways in which 2 letters are in the correct envelopes

(i.e. 2 are in the wrong envelopes)

$= 4! - 1 - 4C_1 \times 2 - 4C_2 \times 1 = 24 - 1 - 8 - 6 = 9$

Alternate Solution :

Use result 6.4 (e)

The required number of ways to place all 4 letters in the wrong envelopes

$$= 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 24\left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right) = 9$$

Example : 35

Find the number of ways of distributing 5 different balls in there boxes of different sizes so that no box is empty and each box being large enough to accommodate all the five balls.

Solution

Method – 1

The five balls can be distributed in 3 non-identical boxes in the following 2 ways :

Boxes	**Box 1**	**Box 2**	**Box 3**
Number of balls	**3**	**2**	**1**
No. of choices	**2**	**2**	**1**

Case – 1 : 3 in one Box, 1 in another and 1 in third Box (3, 1, 1) ....(i)

Number of ways to divide balls corresponding to (1) = $\frac{5!}{3!1!1!}\ \frac{1}{2!}$ = 10

But corresponding to each division there are 3! ways of distributing the balls into 3 boxes.
So number of ways of distributing balls corresponding to

(i) = (No. of ways to divide balls) × 3! = 10 × 3! = 60

Case – 2 : 2 in one Box, 2 in another and 1 in third Box (2, 2, 1)

Number of ways to divide balls corresponding to (2) = $\frac{5!}{2!\,2!\,1!}\ \frac{1}{2!}$ = 15

But corresponding to each division there are 3! ways of distributing of balls into 3 boxes.
So number of ways of distributing balls corresponding to

(2) = (No. of ways to divide balls) × 3! = 15 × 3! = 90

Hence required number of ways = 60 + 90 = 150
Method – 2
Let us name to Boxes as A, b and C. Then there are following possibilities of placing the balls :

Box A	**Box B**	**Box C**	**Number of ways**
1	**2**	**2**	$^5C_1 \times {}^4C_2 \times {}^2C_2 = 30$
1	**1**	**3**	$^5C_1 \times {}^4C_1 \times {}^3C_3 = 20$
1	**3**	**1**	$^5C_1 \times {}^4C_3 \times {}^1C_1 = 20$
2	**1**	**2**	$^5C_2 \times {}^3C_1 \times {}^2C_2 = 30$
2	**2**	**1**	$^5C_2 \times {}^3C_2 \times {}^1C_1 = 30$
2	**1**	**1**	$^5C_3 \times {}^2C_1 \times {}^1C_1 = 20$

Therefore required number of ways of placing the balls = 30 + 20 + 20 + 30 + 30 + 20 = 150
Method – 3
Using Result 6.2 (a)
Number of ways of distributing 5 balls in 3 Boxes so that no Box is empty = $3^5 - 3 \times 2^5 + 3 \times 1^5 = 150$.

Example : 36

If n distinct objects are arranged in a circle, show that the number of ways of selecting three of these things o that no two of them are next to each other is $\frac{n}{6}$ (n – 4) (n – 5).

Solution

Let a_1, a_2, a_3 ...., be the n distinct objects
Number of ways to select three objects so that no two of them are consecutive = Total number of ways to select three objects – Number of ways to select three consecutive objects – Number of ways to select three objects in which two are consecutive and one is separated ....(i)
Total number of ways to select 3 objects from n distinct objects = $^nC_3$ ....(ii)
Select three consecutive objects
The three consecutive objects can be selected in the following manner
Select from :
$a_1\, a_2\, a_3$, $a_2\, a_3\, a_4$, $a_3\, a_4\, a_5$, ..., $a_{n-1}\, a_n\, a_1$, $a_n\, a_1\, a_2$
So number of ways in which 3 consecutive objects can be selected from n objects arranged in a circle in n. ....(ii)
Select two consecutive (together) and 1 separated
The three objects so that 2 are consecutive and 1 is separated can be selected in the following manner :
Take $a_1\, a_2$ ad select third object from a_4, a_5 , ..., a_{n-1}
i.e. take $a_1\, a_2$ and select third object in (n – 4) ways or in general we can say that select one pair from n available pairs i.e. a_1, a_2, a_2, a_3 , ..., $a_n\, a_1$ and third object in (n – 4) ways

So number of ways to select 3 objects so that 2 are consecutive and 1 is separated = n (n – 4) ....(iv)

Using (i), (ii), (iii) and (iv), we get

Number of ways to select 3 objects so that all are separated

$$= {}^nC_3 - n - n(n-4) = \frac{n(n-1)(n-2)}{6} - n - n(n-4) = n\left[\frac{n^2 - 3n + 2 - 6(n-3)}{6}\right]$$

$$= \frac{n}{6}(n^2 - 9n + 20) = \frac{n}{6}(n-4)(n-5)$$

Example : 37

Find the number of integral solutions of the equation 2x + y + z = 20 where x, y, z ≥ 0

Solution

2x = y + z = 20 ....(i)

Condition on x

x is maximum when y and z are minimum

⇒ 2 Max (x) = 20 – Min (y) – Min (x)

$$\Rightarrow \quad \text{Max}(x) = \frac{20-0-0}{2} = 10$$

Let x = k where 0 ≤ k ≤ 10

Put x = k in (i) to get, y + z = 20 – 2k ....(ii)

Number of non-negative integral solutions of (ii) = 20 – 2k + 1 = 2k – 2k

As k is varies from 0 to 10, the total number of non-negative integral solutions of (1)

$$= \sum_{k=0}^{10}(21-2k) = \sum_{k=0}^{10} 21 = 2\sum_{k=0}^{10} k = 231 - 110 = 121$$

$$\left(\text{using} \sum n = \frac{n(n+1)}{2}\right)$$

Hence, total number of non-negative integral solutions of (i) is 121

Example : 38

These are 12 seats in the first row of a theater of which 4 are to be occupied. Find the number of ways of arranging 4 persons so that :

(i) no two persons sit side by side

(ii) there should be atleast 2 empty seats between any two persons

(iii) each person has exactly one neighbour

Solution

(i) We have to select 4 sets for 4 persons so that no two persons are together. It means that there should be atleast one empty seat vacent between any two persons.

To place 4 persons we have to select 4 seats between the remaining 8 empty seats so tat all persons should be separated.

Between 8 empty seats 9 seats are available for 4 person to sit.

Select 4 seats in ${}^9C_4$ ways

But we can arrange 4 persons on these 4 seats in 4! ways. So total number of ways to give seats to 4 persons so that no two of them are together = ${}^9C_4$ = 4! = ${}^9P_4$

(ii) Let x_0 denotes the empty seats to the left of the first person, x_i (i = 1, 2, 3) be the number of empty seats between i th and (i + 1) st person and x_4 be the number of empty seats to the right of 4th person.

Total number of seats are 12. So we can make this equation : $x_0 + x_1 + x_2 + x_3 + x_4 = 8$ ......(i)

Number of ways to give seats to 4 persons so that there should be two empty seats between any two persons is same as the number of integral solutions of the equation (i) subjected to the following conditions.

Conditions on x_1, x_2, x_3, x_4

According to the given condition, these should be two empty seats between any two persons i.e.

Min $(x_i) = 2$ for i = 1, 2, 3

Min $(x_0) = 0$ and Min $(x_0) = 0$

Max $(x_0) = 8 – \text{Min } (x_1 + x_2 + x_3 + x_4) = 8 – (2 + 2 + 2 – 0) = 2$

Max $(x_4) = 8 – \text{Min } (x_0 + x_1 + x_2 + x_3) = 8 – (2 + 2 + 2 – 0) = 2$

Similarly,

Max (xi) = 4 for i = 1, 2, 3

No. of integral solutions of the equation (i) subjected to the above conditions = coeff. of x^8 in the expansion of $(1 + x + x^2)^2 (x^2 + x^3 + x^4)^3$ = coeff. of x^8 in $x^6 (1 + x + x^2)^5$ = coeff of x^2 in $(1 – x^3)^5 (1 – x)^{–5}$ = coeff of x^2 in $(1 – x)^{–5} = {}^{5+2–1}C_2 = {}^6C_2 = 15$

Number of ways to select 4 seats so that there should be atleast two empty seats between any two persons = 15.

But 4 persons can be arranged in 4 seats in 4! ways.

So total number of ways to arrange 4 persons in 12 seats according to the given condition = 15 × 4! = 360

(iii) As every person should have exactly one neighbour, divide 4 persons into groups consisting two persons in each group.

Let G_1 and G_2 be the two groups in which 4 persons are divided.

According to the given condition G_1 and G_2 should be separated from each other.

Number of ways to select seats so that G_1 and G_2 are separated ${}^{8+1}C_2 = {}^9C_2$

But 4 persons can be arranged in 4 seats in 4! ways. So total number of ways to arrange 4 persons so that every person has exactly one neighbout = ${}^9C_2 \times 4! = 864$

Example : 39

The number of non-negative integral solutions of $x_1 + x_2 + x_3 + x_4 \leq n$ where n is a positive integer

Solution

It is given that : $x_1 + x_2 + x_3 + x_4 \leq 4$ ....(i)

Let $x_5 \geq 0$

Add x_5 on LHS of (i) to get $x_1 + x_2 + x_3 + x_4 + x_5$ ....(ii)

Number of non-negative integral solutions of the inequation (i) = Number of non-negative integral solutions of the equation (ii)

= coeff. of x^n in $(1 + x + x^2 + x^3 + x^4 + ... + x^n)^5$

= coeff. of x^n in $(1 – x^{n+1})^5 (1 – x)^{–5}$

= coeff. of x^n in $(1 – x)^{–5} = {}^{n+5+1}C_n = {}^{n+4}C_n = {}^{n+4}C_n$.

Example : 40

If all the letters of the word RANDOM are written in all possible orders and these words are written out as in a dictionary, then find the rank of the word RANDOM in the dictionary.

Solution

In a dictionary the words at each stage are arranged in alphabetical order. In the given problem we must therefore consider the words beginning with A, D, M, N, O, R in order. A will occur in the first place as often as there are ways of arranging the remaining 5 letters all at a line i.e. A will occur 5! times. D, M, N, O will occur in the first place the same number of times.

Number of words starting with A = 5! = 120

Number of words starting with D = 5! = 120

Number of words starting with M = 5! = 120

Number of words starting with N = 5! = 120

Number of words starting with O = 5! = 120

After this words beginning with RA must follow

Number of words beginning with RAD or RAM = 3!

Now the words beginning with RAN must follow
First one is RANDMO and the next one is RANDOM.
∴ Rank of RANDOM = 5(5!) + 2 (3!) + 2 = 614

Example : 41

What is the largest integer n such that 33! divisible by 2^n?

Solution

We know that : 33! = 1 × 2 × 3 × 4 .... × 32 × 33
⇒ 33! = (2 × 4 × 6 .... × 32) (1 × 3 × 5........ × 33)
⇒ 33! = 2^{16} (1 × 2 × 3 × 4 ...... × 15 × 16) (1 × 3 × 5 ...... × 33)
⇒ 33! = 2^{16} (2 × 4 .... × 16) (1 × 3 ..... × 15) (1 × 3 ....×33)
⇒ 33! = 2^{16} (1 × 2 × .... × 8) (1 × 3 × .... × 15) (1 × 3 ...... × 33)
⇒ 33! = 2^{24} (2 × 4 × 6 × 8) (1 × 3 × 5 × 7) (1 × 3 × ..... × 15) *1 × 3 × ....× 33)
⇒ 33! = 2^{24} . 2^4 (1 × 2 × 3 × 4) (1 × 3 × 5 × 7) (1 × 3 .... × 15) (1 × 3 .... × 33)
⇒ 33! = 2^{28} (2 × 4) (1 × 3) (1 × 3 × 5 × 7) (1 × 3 × .... × 15) (1 × 3 × ...... × 33)
⇒ 33! = 2^{31} (1 × 3) (1 × 3 × 5 × 7) (1 × 3 × .... × 15) (1 × 3 × .... × 33)
Thus the maximum value of n for which 33! is divisible by 2^n is 31

Example : 42

Find the sum of all four digit numbers formed by using the digits 0, 1, 2, 3, 4, no digits being repeated in any number.

Solution

Required sum of number = (sum of four digit number using 0, 1, 2, 3, 4, allowing 0 in first place) – (Sum of three digit numbers using 1, 2, 3, 4)

$$= \frac{5!}{5} (0 + 1 + 2 + 3 + 4) (1 + 10 + 10^2 + 10^3) – \frac{4!}{4}$$

(i.e. 3 are in the wrong envelopes) (1 + 2 + 3 + 4) (1 + 10 + 10^2)
= 24 × 10 × 1111 – 6 × 10 × 111 = 259980

Example : 43

In how many ways three girls and nine boys can be seated in two vans, each having numbered seats, 3 in the front and 4 at the back? How many seating arrangements are possible if 3 girls should sit together in a back row on adjacent seats?

Solution

(i) Out of 14 seats (7 in each Van), we have to select 12 seats for 3 girls and 8 boys
12 seats from 14 available seats can be selected in $^{14}C_{12}$ ways
Now on these 12 seats we can arrange 3 girls and 9 boys in 12! ways
So total number fo ways = $^{14}C_{12}$ × 12! = 91

(ii) One Van out of two available can be selected in $^2C_1$ways
Out of two possible arrangements (see figure) of adjacent seats, select one in $^2C_1$ ways
Out of remaining 11 seats, select 9 for 9 boys in $^{11}C_9$ ways
Arrange 3 girls on 3 seats in 3! ways and 9 boys on 9 seats in 9! ways
So possible arrangement of sitting (for 3 girls and 9 boys in 2 Vans)
= $^2C_1$ × $^2C_1$ × $^{11}C_9$ × 3! × 9! = 12! ways

Example : 44

Show that the number of ways of selecting n things out of 2n things of which na re of one kind and alike and n are of a second kind and alike and the rest are unlike is (n + 2) $2^{n–1}$.

Solution

Let group G_1 contains first n similar things, group G_2 contains next n similar things let D_1, D_2, D_3, ..., D_n be the n unlike things.
Let x_1 be the number of things selected from group G_1, x_2 be the number of things selected fro group G_2 and p_1, p_2, p_3 ,.........., p_n be the number of things selected from D_1, D_2, D_3 , ..., D_n respectively.
As we have to select n things in all, we can make $x_1 + x_2 + p_1 + p_2 + ... + p_n = n$ ....(i)
Number of ways to select n things = Number of integral solutions of the equation (i) subjected to following conditions
Conditions on the variables

There is no condition on the number of items selected from group G_1 and G_2 . So we can take :
Min (x_1) = Min (x_2) = 0 and Max (x_1) = Max (x_2) = n

For items D_1 to D_n , we can make selection in two ways. That is either we take the item or we reject the item. So we can make :

Min $(P_i) = 0$ for $i = 1, 2, 3 \ldots, n$ and

Max $(p_i) = 1$ for $i = 1, 2, 3 \ldots, n$

Find solutions

Number of integral solutions of (1)

= coeff. of x^n in $(x^0 + x^1 + \ldots + x^n)^2 (1 + x) (1 + x) \ldots$ n times

= coeff. of x^n in $(1 - x^{n+1})^2 (1 - x)^{-2} (1 + x)^n$

= coeff. of x^n in $(1 - x)^{-2} [2 - (1 - x)]^n$

= coeff. of x^n in $(1 - x)^2 + \ldots + {}^nC_n\, 2^0 (1 - x)^n$ = coeff. of x^n in $[{}^nC_0\, 2^n (1 - x)^{-2} - {}^nC_1\, 2^{n-1} (1 - x)^{-1}]$

(because other terms can not product x^n)

$= 2^n \times {}^{n+2-1}C_n - n\, 2^{n-1} \times {}^{2+1-1}C_n = (n + 1)\, 2^n - n\, 2^{n-1} = (n + 2)\, 2^{n-1}$

Probability

Example : 1

Two candidates A and B appear for an interview. Their chances of getting selected are 1/3 and 1/5 respectively. Assuming that their selections are independent of each other, find :

(a) the probability that both are selected
(b) the probability that exactly one of them is selected
(c) at least one of them is selected.

Solution

Let us denote the following events

A : A is selected
B : B is selected

$\Rightarrow$ P(A) = 1/3 and P(B) = 1/5

(a) P(both selected) $= P(A \cap B) = P(A)\,P(B)$ (As A and B are independent)
$= (1/3)\,(1/5) = 1/15$

(b) P (exactly one is selected = P(only A is selected or only B is selected)

Now $A \cap \overline{B}$ represents the event that only A is selected. is the event that only B is selected

$\Rightarrow$ P(exactly one is selected) $= P(A \cap \quad \cup \quad \cap B)$

$= P(A \cap \quad) + P(\quad \cap B)$

$= P(A)\,P(\quad) + P(\quad)\,P(B)$

$$= \left(1 - \frac{1}{5}\right) + \left(1 - \frac{1}{3}\right)\frac{1}{5} = \frac{6}{15} = \frac{2}{5}$$

(c) P (at least one is selected) $= P(A \cup B) = P(A) + P(B) - P(A \cap B) = 1/3 + 1/5 - 1/3 \,.\, 1/5 = 7/15$
alternatively, we can say that
P(at least one is selected) = 1 – P (none is selected)

$= 1 - P(\overline{A} \cap \quad) = 1 - P(\quad)\,P(\quad) = 1 - (1 - 1/3)\,(1 - 1/5) = 7/15$

Example : 2

Five cards are drawn from a pack of well-shuffled pack of 52 cards. The cards are drawn one by one without replacement. $\frac{{}^{4}C_3 \times {}^{48}C_2}{3 \quad {}^{53}C_5}$

(a) What is the probability of getting 3 aces?
(b) What is the probability of obtaining aces on the first three cards only?
(c) What is the probability of getting exactly three consecutive aces?

Solution

(a) total number of ways to select 5 cards = ${}^{52}C_5$
number of ways to select three aces and two non-aces = ${}^{4}C_3\ {}^{48}C_2$

P(3 aces out of 5 cards) =

$$= \frac{4 \times 48 \times 47 \times 5!}{2 \times 52 \times 51 \times 50 \times 49 \times 48} = \frac{94}{13 \times 17 \times 5 \times 49} = \frac{94}{54145}$$

(b) Let A_i represent the vent that ace is drawn on ith card and N_i be the event that non-zero is drawn on ith card.

P (first three aces) $= P(A_1 \cap A_2 \cap A_3 \cap N_4 \cap N_5) = P(A_1)\,P(A_2/A_1)\,P(A_3/A_1 \cap A_2)$ ....

$$= \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{48}{49} \times \frac{47}{48} = \frac{47}{270725}$$

(c) P(three consecutive aces) = P(forst three aces and fourth non ace) + P(first non ace, 3 aces, last non ace) + P (first two non ace, last three aces) + P (first ace, second non ace and last three aces)

$$= \frac{5}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{48}{49} \times \frac{48}{48} + \frac{48}{52} \times \frac{4}{51} \times \frac{3}{50} \times \frac{2}{49} \times \frac{47}{48}$$

$$+ \frac{48}{52} \times \frac{47}{51} \times \frac{4}{50} \times \frac{3}{49} \times \frac{2}{48} + \frac{5}{52} \times \frac{48}{51} \times \frac{3}{50} \times \frac{2}{49} \times \frac{1}{48}$$

$$= \frac{143}{270725}$$

Example : 3

Three identical dice are thrown

(a) What is the probability of getting a total of 15?

(b) What is the probability that an odd number is obtained on each dice given that the sum obtained is 15?

Solution

(a) Let A be the event that sum is 15.

The favourable outcomes are : A = {366, 456, 465, 546, 555, 564, 636, 645, 654, 663}

and $n(S) = 6^3 = 216$

$\Rightarrow$ P(A) = 10/216

(b) Let B be the vent that odd number appears on each die.

$$\Rightarrow \quad P(B/A) = \frac{P(B \cap A)}{P(A)}$$

$\Rightarrow$ We have $B \cap A = (555)$

$$\Rightarrow \quad P(B/A) = \frac{1/216}{10/216} = \frac{1}{10}$$

Note : The number of ways to obtains a sum of 15 on three dice can also be contained by calculating the coefficient of x^{15} in the expansion of $(x + x^2 + x^3 + x^4 + x^5 + x^6)^3$

Example : 4

In a game, two persons A and B each draw a card from a pack of 52 cards one by one until an ace is obtained. The first one to drawn an ace wins the game. If A starts and the cards are replaced after each drawing, find the probability of A winning the game.

Solution

Let A_1 (or B_1) denote the event that A(or B draws $\bar{A}$ ace in ith attempt

P(A wins) = P(A wins in Ist attempt) + P(A wins in IInd attempt) + ...

$\Rightarrow$ P(A wins) = $P(A_1) + P(\bar{A}_1 \cap \bar{B}_1 \cap A_2) + \ldots\ldots$

$$= \frac{4}{52} + \frac{48}{52} \times \frac{48}{52} \times \frac{4}{52} + \ldots\ldots\ldots\ldots$$

$$= \frac{4/52}{-(48/52)^2} = \frac{4 \times 52}{52^2 - 48^2} = \frac{13}{25}$$

Example : 5

Two cards are drawn one by one without replacement from a pack of 52 cards.

(a) What is the probability of getting both aces?

(b) What is the probability that second cards is an ace?

Solution

A : first is ace

B : second is ace

(a) P(both aces) = $P(A \cap B)$ = P(A) P(B/A) = 4/52 × 3/51 = 1/221

(b) P(second is ace) = P(B) = $P(A \cap B \cup \bar{A} \cap B)$

= P(A) P(B/A) + P( ) P(B/) = 4/52 × 3/51 + 48/52 × 4/51 = 1/13

Example : 6

Cards are drawn one by one without replacement from a pack of 52 cards till all the aces are drawn out. What is the probability that only two cards are left unturned when all aces are out?

Solution

P(two cards are left) = P(52th card drawn is last ace)

Let A : 50th cards is last ace

A_1 : 3 aces are drawn in first 49 cards

A_2 : 50th card is ace

$\Rightarrow$ $A = A_1 \cap A_2$

$\Rightarrow$

$\Rightarrow$ $P(A) = P(A_1 \cap A_2) = P(A_1)\ P(A_2/A_1)$

$\Rightarrow$ $P(A_1)$ = P(3 aces and 46 non-aces in first 49 cards) =

$\Rightarrow$ $P(A_2/A_1)$ = P (50th card is ace given that 3 aces and 46 non-aces have been drawn out)

$= \frac{1}{3}$ (i.e. 1 ace out of 3 remaining cards)

$\Rightarrow$ $P(A) = \frac{{}^4C_3\,{}^{48}C_{46}}{{}^{52}C_{49}} \times \frac{1}{3} = \frac{1128}{5525}$

Example : 7

A candidate has to appear in an examination in three subjects : English, Mathematics and Physics. His chances of passing in these subjects are 0.5, 0.7 and 0.9 respectively. Find the probability that :

(a) he passes in at least one of the subjects

(b) he passes exactly in two subjects.

Solution

A : he passes in english

B : he passes in Mathematics $\frac{{}^4C_3\,{}^{48}C_{46}}{{}^{52}C_{49}}$

C : he passes in Physics

(a) P (he passes in at least one subject) = $P(A \cup B \cup C)$

To calculate $P(A \cup B \cup C)$, use :

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$= 0.5 + 0.7 + 0.9 - (0.35 + 0.63 + 0.45) + 0.315 = 0.985$

Alternatively,

it is easy to calculate $P(A \cup B \cup C)$ by :

$P(A \cup B \cup C) = 1 - P(\overline{A} \cap \quad \cap \quad) = 1 - (\overline{A}) \times P(\quad) \times P(\quad)$

$= 1 - (1 - 0.5)\ (1 - 0.7)\ (1 - 0.9) = 0.985$

(b) P(He passes exactly in two subjects) = $P(A \cap B \cap \overline{C}) + P(A \cap \overline{B} \cap C) + P(\quad \cap B \cap C)$

$= 0.5 \times 0.7\ (1 - 0.9) + 0.5 \times (1 - 0.7) \times 0.9 + (1 - 0.5) \times 0.7 \times 0.9 = 0.485$

Example : 8

A man takes a step forward with probability 0.4 and backwards with probability 0.6. Find the probability that at the end of eleven steps he is one step away from the starting point.

Solution

(1) If the man is one step forward after eleven steps, he has six forward and five backward steps.

(2) If the man is one step backward after eleven steps, he has taken five forward and six backward steps.

These are the only two probability

Let success : forwards step

failure : backward step

$\Rightarrow$ p = 0.4 and q = 0.6

P(one step away) = P(6 successes) + P(5 successes)

P(one step away) = ${}^{11}C_6\,(0.4)^6\,(0.6)^5 + {}^{11}C_5\,(0.4)^5\,(0.6)^6 = {}^{11}C_6\,(0.4)^5\,(0.6)^5$

Example : 9

(i) A coin is tossed 5 times. What is the probability of obtaining at most 3 heads?

(ii) A coin is biased in such a manner that the chances of getting head is twice the chances of getting tail. Find the probability that heads will appear an odd number of times in n tosses of the coin.

Solution

(i) success : getting head

$\Rightarrow$ p = 1/2 and q = 1/2

P(at most 3 heads) $= P(r \le 3)$

$= P(r = 0) + P(r = 1) + P(r = 2) + P(r = 3)$

$= 1 - P(r = 4) - P(r = 5)$

$= 1 - {}^5C_4\, p^4 q - {}^5C_3\, p^5 = 1 - \quad - \frac{1}{2^5} = \frac{13}{16}$

(ii) success : getting head

failure : getting tail

$\Rightarrow$ p + q = 1 and p = 2q

$\Rightarrow$ p = 2/3 and q = 1/3

P(odd number of heads) $= {}^nC_1\, p\, q^{n-1} + {}^nC_3\, p^3\, q^{n-3} + \ldots\ldots\ldots$

$$= \frac{(q+p)^n - (q-p)^n}{2} = \frac{1 - \left(\frac{1}{3} - \frac{2}{3}\right)^n}{2} = \frac{3^n - (-1)^n}{2.3^n}$$

Example : 10

Suppose the probability for A to win a game against B is 0.4. If A an option of playing either a 'best of 3 games' or 'beast of 5 games' match against B, which option should he choose so first the probability of his winning the match is higher? (no game ends in a draw and all the games of the match the played).

Solution

Success : A wins a game

$\Rightarrow$ p = 0.4 and q = 0.6

$\frac{5}{2^5}$

P(A wins 'best of 3 games' match) = P(r = 2) + P(r = 3)

$= {}^3C_2\, p^2 q + {}^3C_3\, p^3$

$= 3(0.4)^2\,(0.6) + (0.4)^2$

$= \frac{36}{125} + \frac{8}{125} = \frac{44}{125}$

P(A wins 'best of 5 games' match) = P(r = 3) + P(r = 4) + P(r = 5)

$= {}^5C_3\, p^3\, q^2 + {}^5C_4\, p^4 q + {}^5C_5\, p^5$

$= 10\left(\frac{72}{3125}\right) + 5\left(\frac{48}{3125}\right) + \frac{32}{3125} = \frac{992}{3125} = \frac{39.68}{125}$

$\Rightarrow$ P(A wins 'beast of 3 games' match) > P(A wins 'best of 5 games' match)

$\Rightarrow$ A should choose a 'best of 3 games' match

Example : 11

Six persons try to swim across a wide river. It is known that on an average, only three persons out of ten are successful in crossing the river. What is the probability that at most four of the six persons will cross river safely?

Solution

Let success : crossing the river

$\Rightarrow$ p = 3/10 and q = 7/10

number of trials = n = 6

P(at most 4 success in 6 trials) $= \sum_{r=0}^{4} P(r) = 1 - \sum_{r=5}^{6} P(r) = 1 - {}^6C_5\, p^5\, q - {}^6C_6\, p^6$

$= 1 - 6\left(\frac{3}{10}\right)^5 \left(\frac{7}{10}\right) - \left(\frac{3}{10}\right)^6 = 1 - \left(\frac{3}{10}\right)^6 (13) = 0.990523$

Example : 12

An unbiased dice is thrown. If a multiple of 3 appears, two balls are drawn from box A. If a multiple of 3 does not appear, two balls are drawn from box B. The balls drawn are found to be of different colours. Box. A contains 3 white, 2 black balls and Box B contains 4 white and 1 black balls. Find the probability that the balls were drawn from box B if the balls are drawn with replacement.

Solution

A_1 : event that balls were drawn from box A

A_2 : event that balls were drawn from box B

E : balls are of different colours

$P(A_1)$ = P(balls from A)
= P(3, 6 on dice)
= 2/6 = 1/3

$P(A_2)$ = P(balls from B)
= P(1, 2, 4, 5 on dice)
= 4/6 = 2/3

$P(E/A_1)$ = P(1 W, 1 B from box A)
= P(1 success in two trials) (taking W balls as success)
= ${}^2C_1\, pq$
= 2(3/5) (2/5) = 12/25

$P(E/A_2)$ = P(1 W, 1 B from box B)
= P(1 success in two trials) (taking W balls as success)
= ${}^2C_1\, pq$
= 2(4/5) (1/5) = 8/25

P(E) = $P(A_1) \cdot P(E/A_1) + P(A_2) \cdot P(E/A_2)$
= 1/3 (12/25) + 2/3 (8/25)
= 25/75

Required probability is

$$P(A_2/E) = \frac{P(A_2)P(E/A_2)}{P(E)} = \frac{\frac{2}{3}\left(\frac{8}{25}\right)}{\frac{28}{75}} = \frac{4}{7}$$

Example : 13

Box I contain 4 red, 5 white balls and box II contains 3 red, 2 white balls. Two balls are drawn from box I and are transferred to box II. One ball is then drawn from box II. Find the probability that :

(a) ball drawn from box Ii is white

(b) the transferred balls were both red given that the balls drawn from box II is white.

Solution

A_1 : transferred balls were both red

A_2 : transferred balls were both white

A_3 : transferred balls were one red and one white

E : ball drawn from box II is white

(a) $P(E) = P(A_1) \cdot P(E/A_1) + P(A_2) \cdot P(E/A_2) + P(A_2) \cdot P(E/A_2)$

$$= \left(\frac{{}^4C_2}{{}^9C_2}\right)\frac{2}{7} + \left(\frac{{}^5C_2}{{}^9C_2}\right)\frac{4}{7} + \left(\frac{{}^4C_1\,{}^5C_1}{{}^9C_2}\right)\frac{3}{7} = \frac{1}{21} + \frac{10}{63} + \frac{5}{21} = \frac{28}{63}$$

(b) $$P(A_1/E) = \frac{P(A_1)P(E/A_1)}{P(E)} = \frac{\frac{1}{21}}{\frac{28}{63}} = \frac{3}{28}$$

Example : 14

A person drawn two cards successively with out replacement from a pack of 52 cards. He tells that both cards are aces. What is the probability that both are aces if there are 60% chances that he speaks truth?

Solution

A_1 : both are aces

A_2 : both are not aces

E : the person tells that both are aces

$$P(E) = P(A_1 \cap E) + P(A_2 \cap E)$$
$$= P(A_1)\,P(E/A_1) + P(A_2)\,P(E/A_2)$$
$$= P(\text{both aces}) \times P(\text{speaking truth}) + P(\text{both not aces})\; P(\text{not speaking truth})$$
$$= \left(\frac{4}{52}\times\frac{3}{51}\right)\left(\frac{60}{100}\right) + \left(1-\frac{4}{52}\times\frac{3}{51}\right)\left(\frac{40}{100}\right)$$
$$= \frac{3}{1105} + \frac{440}{1105} = \frac{443}{1105}$$

Required probability is $P(A_1/E) = \dfrac{P(A_1)P(E/A_1)}{P(E)} = \dfrac{\frac{3}{1105}}{\frac{443}{1105}} = \dfrac{3}{443}$

Example : 15

A letter is known to have come either from TATANAGAR or CALCUTTA . On the envelop just two consecutive letters TA are visible. What is the probability that the letter came from Tata Nagar?

Solution

Let A_1 denotes the event that the letter come from TATANAGAR and A_2 denote the event that the letter come from CALCUTTA. Let E denotes the event that the two visible letters on the envelope be TA.

As Events A_1 and A_2 are equally likely, we can take : $P(A_1) = \dfrac{1}{2}$ and $P(A_2) = \dfrac{1}{2}$

If the letter has come from TATANAGAR, then the number of ways in which two consecutive letters choosen be TA is 2. The total number of ways to choose two consecutive letters is 8.

$\Rightarrow$ P (two consecutive letters are TA/letter has come from TATANAGAR) = $P(E/A_1) = \dfrac{2}{8} = \dfrac{1}{4}$

If the letter has come from CALCUTTA, then the number of ways in which two consecutive letters choosen be TA is 1. The total number of ways to choose two consecutive letters is 7.

$\Rightarrow$ P(two consecutive letters are TA/letter has come from CALCUTTA) = $P(E/A_1) = \dfrac{1}{7}$

Using the Baye's theorem, we get

$$P(A_1/E) = \frac{P(A_1)P(E/A_1)}{P(A_1)P(E/A_1) + P(A_2)P(E/A_2)}$$

$$\Rightarrow \quad P(A_1/E) = \frac{(1/2)(1/4)}{(1/2)(1/4) + (1/2)(1/7)} = \frac{7}{11}$$

Example : 16

A factory A produces 10% defective valves and another factory B produces 20% defective valves. A bag contains 4 values of factory A and 5 valves of factory B. If two valves are drawn at random from the bag, find the probability that atleast one valve is defective. Give your answer upto two places of decimals.

Solution

Probability of producing defective valves by factory A = $\dfrac{10}{100} = \dfrac{1}{10}$

Probability of producing defective valves by factory B = $\frac{20}{100} = \frac{1}{5}$

Bag A contains 9 valves, 4 of factory A and 5 of factory B. Two valves are to be drawn at random.
P(at least one defective) = 1 – P(both are non defective)
P(both are non defective) = P(both valves of factory A) = P(both are non defective) + P(both valves of factory B) × P(both are non defective) + P(one valve of factory A and other of factory B) × P (both are non defective)

$$= \frac{{}^4C_2}{{}^9C_2}\left(\frac{9}{10}\right)^2 + \frac{{}^5C_2}{{}^9C_2}\left(\frac{4}{5}\right)^2 + \frac{{}^4C_1 \cdot {}^5C_1}{{}^9C_2} \cdot \frac{9}{10} \cdot \frac{4}{5}$$

$$= \frac{1}{6}\left(\frac{9}{10}\right)^2 + \frac{10}{36} \cdot \frac{16}{25} + \frac{4.5.9.4}{3.6.10.5} = \frac{27}{200} + \frac{8}{45} + \frac{2}{5} = \frac{1283}{1800}$$

∴ p(at least one defective) = $1 - \frac{1283}{1800}$ = 0.2872 = 0.29 (approx.)

Example : 17

If four squares are chosen at random on a chess board, find the chance that they should be in a diagonal line.

Solution

Three are 64 squares on the chess board.
Consider the number of ways in which the squares selected at random are in a diagonal line parallel to AC.
Consider the triangle ACB. Number of ways in which 4 selected squares are along the lines, A_4C_4, A_3C_3, A_2C_2, A_1C_1 and AC are ${}^4C_4$, ${}^5C_4$, ${}^6C_4$, ${}^7C_4$ and ${}^8C_4$ respectively.
Similarly in triangle ACD there are equal number of ways of selecting 4 squares in a diagonal line parallel to AC.
∴ The total number of ways in which the 4 selected squares are in a diagonal line parallel to AC are:
$= 2({}^4C_4 + {}^5C_4 + {}^6C_4 + {}^7C_4) + {}^8C_4$
Since there is an equal number of ways in which 4 selected squares are in a diagonal line parallel to BD.
∴ the required number of ways of favourable cases is given by
$2[({}^4C_4 + {}^5C_4 + {}^6C_4 + {}^7C_4)]$
Since 4 squares can be selected out of 64 in ${}^{64}C_4$ ways, the required probability

$$= \frac{2[2({}^4C_4 + {}^5C_4 + {}^6C_4 + {}^7C_4) + {}^8C_4]}{{}^{64}C_4} = \frac{[4(1+5+15+35)+140]\times 4\times 3\times 2}{64\times 63\times 62\times 61} = \frac{91}{158844}$$

Example : 18

Of these independent events, the chance that only the first occurs is a, the chance that only the second occurs is b and the chance of only third is c. Show that the chances of three events are respectively a/(a+x), b/(b+x) c/(c+x), where x is a root of the equation $(a + x)(b + x)(c + x) = x^2$

Solution

Let A, V, C be the three independent events having probabilities p, q and r respectively.
Then according to the hypothesis, we have :
P(only that first occurs) = p (1 – q) (1 – r) = a, ...(i)
P(only the second occurs) = (p – 1) q (1 – r) = b and
P(only the third occurs) = (1 – p) (1 – q) r = c.
∴ pqr $[1 – p) (1 – q) (1 – r)]^2$ = abc

Let $\frac{abc}{pqr} = [(1-p)(1-q)(1-r)]^2 = x^2$ (say) ....(ii)

Using (i) and (ii), $\frac{a}{x} = \frac{p}{1-p}$

⇒ a – ap = px
⇒ p = a/(a + x)
Similarly q = b/(b + x) and r = c/(c + x)

Replacing the values of p, q and r in (ii), we get
$(a + x)(b + x)(c + x) = x^2$
i.e. x is a root of the equation : $(a + x)(b + x)(c + x) = x^2$

Example : 19

A and B bet on the outcomes of the successive toss of a coin. On each toss, if the coin shows a head, A gets one rupee from B, whereas if the coin shows a tail, A pays one rupee to B. They continue to do this unit one of them runs out of money. If it is assumed that the successive tosses of the coin are independent, find the probability that A ends up with all the money if A starts with five rupees and B starts with seven rupees.

Solution

Let P_i denotes the probability that A ends with all the money when A has i rupees and B has (12 – i) rupees.
Let E denote the event that A ends up with all the money.
Consider the situation of the game when A has i rupees and B has (12 – i) rupees.
$P(E) = P_i$ = P(coin shows bead) × P_{i-1} + P(coin shows tail) × P_{i-1}

$P_i = \frac{1}{2} \times P_{i+1} + \frac{1}{2} \times P_{i-1}$

$\Rightarrow \quad 2P_i = P_{i+1} + P_{i-1}$
$\Rightarrow \quad P_{i-1}\ P_i$ and P_{i+1} are in A.P. ....(i)
Also : P_0 = prob that A ends up with all the money when he has noting to begin with = 0 ....(ii)
and : Pn = prob that A ends up with all the money B has nothing to being with = 1 ....(iii)
From (i), (ii) and (iii), we get :
$P_0\ , P_1\ , P_2\ ,$ .... P_n are in A.P. with $P_0 = 0$, $P_n = 1$

common difference of the A.P. = $d = \frac{P_n - P_0}{n} = \frac{1}{n}$

The (i + 1) th term of the A.P. = $P_1 = P_0 + id = i/n$
So probability A ends up with all the money starting i rupees = $P_i = i/n$
Here A starts with 5 rupees and B with 7 rupees so i = 5, n = 12
$\therefore \quad P_5 = 5/12$

Example : 20

Two points are taken at random on the given straight line AB of length 'a' Prove that the probability of their distance exceeding a given length c(<a) is equal to $\left(1 - \frac{c}{a}\right)^2$

Solution

In this question we can not use the classical definition of probability because there can be infinite outcomes of this experiment. So we will use geometrical method to calculate probability in this question.
Let P, Q be any two points chosen randomly on the line AB of length 'a'.
Let AP = x and AQ = y
Note that : $0 \le x \le a$ and $0 \le y \le a$ ....(i)
The probability that the distance between P and Q exceeds c = $P(|x - y| > c)$
So we need to find $P(|x - y| > c)$ in this question.
Take x along X-axis and y along Y-axis
Using (i), Total Area in which the possible outcomes lie = $a.a = a^2$
Now we have to find the area in which favourable outcomes lie.
Plot the line $x - y = c$ and $y - x = c$
From figure, total area where the outcomes favourable to event $|x - y| > c$ lie = is given by :

Favourable Area $= \Delta ABC + \Delta PQR = \frac{1}{2}$ AB . BC + $\frac{1}{2}$ PQ. QR

$= \frac{1}{2}(a - c)(a - c) + \frac{1}{2}(a - c)(a - c) = (a - c)^2$

$P\{|x - y| > c\} = \frac{(a-c)^2}{a^2} = \left(1 - \frac{c}{a}\right)^2$

Example : 21

Two players A and B want respectively m and n points of winning a set of games. Their chances of winning a single game are p and q respectively where p + q = 1. The stake is to belong to the player who first makes up his set. Find the probabilities in favour of each player.

Solution

Suppose A wins in exactly m + r games. To do so he must win the last game and m – 1 out of the preceding m = r – 1 games. The chance of this is ${}^{m+r-1}C_{m-1}\ p^{m-1}\ q^{r}\ p$

$\Rightarrow$ P(A wins in m + r games) = ${}^{m+r-1}C_{m-1}\ p^{m}\ q^{r}$ ....(i)

Now the set will definitely be decided in m + n – 1 games. To win the set, A has to win m games. This be can do either in exactly m games or m + 1 games of m + 2 games, ..., or m + n – 1 games.

Hence we shall obtain the chance of A's winning the set by putting r the values 0, 1, 2, ..., n – 1 in equation (i)

Hence A's probability to win the set is :

$$P(A\text{ wins}) = \sum_{r=0}^{n-1} {}^{m+r-1}C_{m-1}\ p^{m}\ q^{r} = p^{m}\left[1 + mq + \frac{m(m+1)}{1.2}q^{2} + ... + \frac{(m+n-2)!}{(m-1)!(n-1)!}q^{n-1}\right]$$

Similarly B's probability to win the set is :

$$P(B\text{ wins}) = q^{n}\left[1 + np + \frac{n(n+1)}{1.2}p^{2} + ... + \frac{(m+n-2)!}{(m-1)!(n-1)!}p^{m-1}\right]$$

Example : 22

An unbiased coin is tossed. If the result is a head a pair of unbiased dice is rolled and sum of the numbers on top faces are noted. If the result is a tail, a chard from a well shuffled pack of eleven cards numbered 2, 3, 4, ..., 12 is picked and the number on the card is noted. What is the probability that the noted number is either 7 or 8?

Solution

Let A_1 be the event of getting head, A_2 be the event of getting tail and let E be the event that noted number is 7 or 8.

Then, $P(A_1) = \frac{1}{2}$; $P(A_2) = \frac{1}{2}$

$P(E/A_1)$ = P(getting either 7 or 8 when pair of unbaised dice is thrown) = $\frac{11}{36}$

$P(E/A_2)$ = P(getting either 7 or 8 when a card is picked from the pack of 11 cards = $\frac{2}{11}$

Using the result, $P(E) = P(A_1)\ P(E/A_1) + P(A_2)\ P(E/A_2)$, we get :

$$P(E) = \frac{1}{2} \cdot \frac{11}{36} + \frac{1}{2} \cdot \frac{2}{11}$$

$$\Rightarrow \quad P(E) = \frac{11}{72} + \frac{1}{11} = \frac{122}{792} = \frac{61}{396}$$

Example : 23

An urn contains four tickets with numbers 112, 121, 211, 222 and one ticket is drawn. Let A_i (i = 1, 2,3) be the vent that the ith digit of the number on ticket drawn is 1. Discuss the independence of the events $A_1\ A_2\ A_3$.

Solution

According to the question,

$P(A_1)$ = the probability of the event that the first digit of the selected number is 1 = $\frac{2}{4} = \frac{1}{2}$

($\because$ there are two numbers having 1 at the first place out of four)

Similarly, $P(A_2) = P(A_2) = \frac{1}{2}$

$A_1 \cap A_2$ is the and so
$P(A_1 \cap A_2)$ = the probability of the event that the first two digits in the number drawn are each equal to $1 = \frac{1}{4}$

$\Rightarrow \quad P(A_1 \cap A_2) = \frac{1}{2}, \frac{1}{2} = P(A_1)\, P(A_2)$

$\Rightarrow$ A and B are independent events

Similarly

$P(A_2 \cap A_2) = P(A_2)\, P(A_3)$

and $P(A_3 \cap A_1) = P(A_3)\, P(A_1)$

This the events A_1, A_2 and A_3 are equal to 1 and since there is no such number, we have

$P(A_1 \cap A_2 \cap A_3) = 0$

$\Rightarrow \quad P(A_1 \cap A_2 \cap A_3) \neq P(A_1)\, P(A_2)\, P(A_2)$

Hence the events A_1, A_2, A_3 are not mutually independent althrough they are pairwise independent

Example : 24

A coin is tossed (m + n) times (m > n) ; show that the probability of at least m consecutive heads is $\frac{n+2}{2^m+1}$

Solution

We denote by H the appearance of head and T the appearance of tail.

Let X denote the appearance of head or tail

Then,

$$P(H) = P(T) = \frac{1}{2} \text{ and } P(X) = 1$$

If the sequence of m consecutive heads starts from the first throw, we have

(HHH ..... m times) (XXX ...... n times)

$\therefore$ The chance of this event $= \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ ......m times $= \frac{1}{2^m}$

[Note that last n throws may be head or tail since we are considering at least m consecutive heads]

If the sequence of m consecutive heads starts from the second throw, the first must be a tail and we have

T(HHH ...... m times) (XXX ..... $\overline{n-1}$ times)

The chance of this event = $\cdot \frac{1}{2^m} = \frac{1}{2^{m+1}}$

If the sequence of heads starts with the (r + 1)th throw, then the first (r – 1) throws may be head or tail but rth throw must be tail and we have (XX .... $\overline{r-1}$ times) T (HHH ..... m times) (XX .... times)

The probability of this event = $\cdot \frac{1}{2^m} = \frac{1}{2^{m+1}}$

It can be easily observed that probability of occurrence of atleast m consecutive heads in same for all cases.

Since all the above cases are mutually exclusive, the required probability is :

$$P(\text{atleast m consecutive heads}) = \frac{1}{2^m} + \left(\frac{1}{2^{m+1}} + \frac{1}{2^{m+1}} + \ldots\ldots + \text{to n times}\right) = \frac{1}{2^m} + \frac{n}{2^{m+1}} = \frac{2+n}{2^{m+1}}$$

Example : 25

Out of 3n consecutive integers, three are selected at random. Find the chance that their sun is divisible by 3.

Solution

Let the sequence of numbers start with the integer m so that the 3n consecutive integers are
m, m + 1, m + 2, ..., m + 3n – 1
Now they can be classified as
m, m + 3, m + 6, ... m + 3n – 3
m + 1, m + 4, m + 7 ...... m + 3n – 2
m + 2, m + 5, m + 8 ...... m + 3n – 1
The sum of the three numbers shall be divisible by 3 if either all the three numbers are from the game row or all the three numbers are form different rows.
The number of ways that the three numbers are from the game row is $3\,{}^{n}C_3$ and
The number of ways that the numbers are from different rows in $n \times n \times n = n^3$ since a number can be selected from each row in n ways.
Hence the favourable no. of ways = $3 \,.\, {}^{n}C_3 + n^3$
The total number of ways = ${}^{3n}C_3$

$$\therefore \quad \text{The required probability} = \frac{\text{favourable ways}}{\text{total ways}} = \frac{3\,.\,{}^{n}C_3 + n^3}{{}^{3n}C_3} = \frac{3n^2 - 3n + 2}{(3n-1)(3n-2)}$$

Example : 26

out of (2n + 1) tickets consecutively numbered, three are drawn at random. Find the chance that the numbers on them are in A.P.

Solution

Let us consider first (2n + 1) natural numbers as (2n + 1) consecutive numbers.
The groups of three numbers in A.P. with common difference 1 :
(1, 2, 3); (2, 3, 4) ; (3, 4, 5); .... ; (2n – 1, 2n, 2n + 1) $\Rightarrow$ (2n – 1) groups with numbers in A.P.
The groups of three numbers in A.P. with common difference 2 :
(1, 3, 5); (2, 4, 6) ; (3, 5, 7); .... ; (2n – 3, 2n – 1, 2n + 1) $\Rightarrow$ (2n – 3) groups with numbers in A.P.
The groups of three numbers in A.P. with common difference 3 :
(1, 4, 7); (2, 5, 8) ; (3, 6, 9) ; .... ; (2n – 5, 2n – 2, 2n + 1) $\Rightarrow$ (2n – 5) groups with numbers in A.P.
....
....
The groups of three numbers in A.P. with common difference n :
(1, n + 1, 2n + 1) $\Rightarrow$ 1 groups with numbers in A.P.

$$\Rightarrow \quad \text{The total number of groups with numbers in A.P.} = \sum_{r=1}^{n} (2r-1) = 2\,\frac{n(n+1)}{2} - n = n^2$$

The total number of ways to select three numbers from (2n + 1) numbers = ${}^{2n+1}C_3$

$$\Rightarrow \quad \text{P(three selected numbers are in A.P.)} = \frac{\text{favorable ways}}{\text{total ways}} = \frac{n^2}{{}^{2n+1}C_3} = \frac{3n}{4n^2 - 1}$$

Example : 27

Two friends Ashok and Baldev have equal number of sons. There are 3 tickets for a circket match which are to be distributed among the sons. The probability that 2 tickets go to the sons of the one and one ticket goes to the sons of the other is 6/7. Find how many sons each of the two friends have

Solution

Let each of them have n sons each.
Hence we have to distribute 3 tickets amongst the sons of Ashok and Baldev, in such a manner that one ticket goes to the sons of one and two tickets to the sons of other.
We can make two cases.
Case 1 : 1 to Ashok's sons and 2 to Baldev'son
Case 2 : 2 to Ashok's sons and 1 to Baldev's son
$\Rightarrow$ Total number of ways of distribution the tickets = ${}^{n}C_1 \,.\, {}^{n}C_2 + {}^{n}C_2\, {}^{n}C_1 = 2.\, {}^{n}C_1\, {}^{n}C_2$

But in all 3 tickets are to be distributed amongst 2n sons of both.
Hence total number of ways to distribute tickets = ${}^{2n}C_3$

Hence, P(2 tickets go to the sons of one and 1 ticket goes to the sons of other) = $\frac{2 \cdot {}^nC_1 \cdot {}^nC_2}{{}^{2n}C_3}$ (given)

$\Rightarrow \quad 7 \cdot \frac{n(n-1)}{2} \cdot n = 3 \cdot \frac{2n(2n-1)(2n-2)}{6}$

$\Rightarrow \quad 7n = 4(2n - 1) \quad \Rightarrow \quad n = 4$

Hence both Ashok and Baldev have four sons each.

Example : 28

Sixteen players S_1, S_2 ...., S_{16} play in a tournament. They are divided into eight pairs at random. From each pair a winner is decided on the basis of a game played between the two players of the pair. Assume that all the players are of equal strength

(a) Find the probability that the players S_1 is among the eight winners

(b) Find the probability that exactly one of the two players S_1 and S_2 is among the eight winners.

Solution

According to the problem, S_1, S_2 , ... S_{16} players are divided onto eight groups and then from each from each group one winner emerges. So in all out of 16 players, 8 are winners.

(a) In this part, we have to find the probability of the event that S_1 should is there in the group of eight winners which is selected from 16 players.

So,

P(S_1 is among the winners) $= \frac{\text{No. of ways to select 8 winners such that } S_1 \text{ is always included}}{\text{No. of ways to select 8 winners}}$

$$= \frac{{}^{15}C_7}{{}^{16}C_8} = \frac{1}{2}$$

(b) In this part, we have to find the probability that exactly one of S_1 , S_2 should be among the 8 winners selected

P(exactly one of S_1, S_2) = 1 – P(both S_1 and S_2 are among the winners) – P(none of S_1 , S_2 is among the winners)

$\Rightarrow \quad$ P(exactly one of S_1 , S_2) $= 1 - \frac{{}^{14}C_6}{{}^{16}C_8} = \frac{{}^{14}C_8}{{}^{16}C_8}$

$\Rightarrow \quad$ P(exactly one of S_1, S_2) $= \frac{8}{15}$

Example : 29

If p and q are choosen randomly from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, with replacement, determine the probability that the roots of the equation $x^2 + qx + q = 0$ are real.

Solution

For the roots of the quadratic equation $x^2 + px + q = 0$ to be real, $D \geq 0$.

$\Rightarrow \quad p^2 - 4q \geq 0$ ....(i)

According to the question, coefficients p and q of the quadratic equation are choosen from the following set

{1, 2, 3, 4, 5, 6, 7, 8, 9, 10} ....(ii)

Total number of ways in which q can be choosen from the set = 10 ways

Total number of ways to choose q from the game set = 10 ways

So the total number of ways in which both p and q can be choosen = 10 × 10 ways = 100 ways

Out of these 100 ways, we have to find the favouruable ways such that p and q satisfy (i)

If p takes the values from the set (ii), then p^2 takes the values from the following set.

$p^2 \in$ {1, 4, 9, 16, 25, 36, 49, 64, 81, 100}

and similarly

$4q \in \{4, 8, 12, 16, 20, 24, 28, 32, 36, 40\}$

If we select 1 from the set of p^2 , then no element from the set of 4q can make (i) true.

If we select 4 from the set of p^2, then the favourable selection from the set of 4q is 4.

If we select 9 from the set of p^2 , then the favoruable selection from the set of 4q are 4 and 8.

Similarly find other combinations of p^2 and 4q such that (i) is true.

in all there are 62 combinations pf p^2 and 4q such that $p^2 - 4q \geq 0$

$$\Rightarrow \quad \text{P(roots are real)} = P(b^2 - 4ac \geq 0) = \frac{\text{favoruable selections of p and q}}{\text{possible selections of p and q}} = \frac{62}{100} = \frac{31}{50}$$

Progressions

Example : 1

If a, b, c are the x th, y th and z th terms of an A.P., show that :

(a) $a(y-z) + b(z-x) + c(x-y) = 0$

(b) $x(b-c) + y(c-a) + z(a-b) = 0$

Solution

Let A be the first term and D be the common difference,

$\Rightarrow$ $T_x = A + (x-1)D = a$ ....(i)

$T_y = A + (y-1)D = b$ ....(ii)

$T_x = A + (z-1)D = c$ ....(iii)

operating (2 – 3), (3 – 1) and (1 – 2) we get :

$b - c = (y-z)D$ $\quad c - a = (z-x)D$ $\quad a - b = (x-y)D$

$y - z = \dfrac{b-c}{D}$ $\quad z - x = \dfrac{c-a}{D}$ $\quad x - y = \dfrac{a-b}{D}$

Now substituting the values of (y – z), (z – x) and (x – y) in L.H.S. of the expression (a) to be proved.

$$\Rightarrow \quad \text{LHS} = \frac{a(b-c)}{D} + \frac{b(c-a)}{D} + \frac{c(a-b)}{D}$$

$$\Rightarrow \quad \text{LHS} = \frac{ab - ac + bc - ab + ca - cb}{D} = 0 = \text{RHS}$$

Now substituting the values of (b – c), (c – a) and (a – b) in LHS of the expression (b) to be proved

$\Rightarrow$ $\text{LHS} = x(y-z)D + y(z-x)D + z(x-y)D$

$= \{xy - xz + yz - xz + zx - zy\} D = 0 = \text{RHS}$

Example : 2

The sum of n terms of two series in A.P. are in the ratio 5n + 4 : 9n + 6. Find the ratio of their 13th terms.

Solution

Let a_1, a_2 be the first terms of two A.P.s and d_1, d_2 are their respective common differences.

$$\Rightarrow \quad \frac{\frac{n}{2}[2a_1 + (n-1)d_1]}{\frac{n}{2}[2a_2 + (n-1)d_2]} = \frac{5n+4}{9n+6}$$

$$\Rightarrow \quad \frac{a_1 + \frac{(n-1)}{2}d_1}{a_2 + \frac{(n-1)}{2}d_2} = \frac{5n+4}{9n+6} \quad \text{............(i)}$$

Now the ratio of 13th terms = $\dfrac{a_1 + 12d_1}{a_2 + 12d_2}$

$\Rightarrow$ put $\dfrac{(n-1)}{2} = 12$ i.e. $n = 25$ in equation (i)

$$\Rightarrow \quad \frac{a_1 + 12d_1}{a_2 + 12d_2} = \frac{5(25)+4}{9(25)+6} = \frac{129}{231}$$

Example : 3

If the sum of n terms of a series is $S_n = n(5n-3)$, find the nth term and pth term.

Solution

$S_n = T_1 + T_2 + T_3 + T_4 + \text{............} + T_{n-1} + T_n$

$S_n = \{\text{sum of } (n-1) \text{ terms}\} + T_n$

$\Rightarrow$ $T_n = S_n - S_{n-1}$

Now in the given problem :

$S_n = n(5n-1)$ and $S_{n-1}\,[5(n-1)-3]$

$\Rightarrow$ $T_n = S_n - S_{n-1} = 10n - 8$

$\Rightarrow$ $T_p = 10p - 8$

Example : 4

If a, b, c, d are in G.P., show that

(a) $(a - d)^2 = (b - c)^2 + (c - a)^2 + (d - b)^2$

(b) $a^2 - b^2$, $b^2 - c^2$ and $c^2 - d^2$ are also in G.P.

Solution

(a) a, b, c, d are in G.P.

$\Rightarrow$ $b^2 = ac$, $c^2 = bd$, $bc = ad$ ....(i)

Now expanding the RHS, we get

RHS $= (b^2 + c^2 - 2bc) + (c^2 + d^2 - 2ac) + (d^2 + b^2 - 2bd)$

$= 2(b^2 - ac) + 2(c^2 - bd) + a^2 + d^2 - 2bc$

$= 2(0) + 2(0) + a^2 + d^2 - 2ad$ (from i)

$= (a - d)^2 = $ LHS

(b) Now we have to prove that $a^2 - b^2$, $b^2 - c^2$ and $c^2 - d^2$ are in G.P. i.e.

$(b^2 - c^2)^2 = (a^2 - b^2)(c^2 - d^2)$

Consider the RHS

$(a^2 - b^2)(c^2 - d^2) = a^2c^2 - b^2c^2 - a^2d^2 + b^2d^2$

$= b^4 - b^2c^2 - b^2c^2 + c^4$

$= (b^2 - c^2) = $ LHS

Example : 5

If a, b, c are in A.P. and x, y, z are in G.P., prove that $x^{b-c}\, y^{c-a}\, z^{a-b} = 1$.

Solution

a, b, c are in A.P. $\Rightarrow$ $2b = a + c$ or $a - b = b - c$

x, y, z are in G.P. $\Rightarrow$ $y^2 = xz$

proceeding from LHS

$= x^{b-c}\, z^{b-c}\, y^{c-a}$ {as $b - c = a - b$}

$= (xz)^{b-c}\, y^{c-a} = y^{2(b-c)}\, y^{c-a}$ {as $xz = y^2$}

$= y^{\{2(b-c) + (c-a)\}}$

$= y^{2b-a-c} = y^0 = 1 = $ RHS {as $2b = a + c$}

Example : 6

The sum of three numbers in H.P. is 26 and the sum of their reciprocals is 3/8. Find the numbers.

Solution

Three numbers in H.P. are taken as :

$$\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d} \quad \Rightarrow \quad \frac{1}{a-d} + \frac{1}{a} + \frac{1}{a+d} = 26 \quad \text{........ (i)}$$

also $(a - d) + a + (a + d) = 3/8$ .... (ii)

from (i) and (ii) $a = \frac{1}{8}$ and $d = \pm \frac{1}{24}$

$\Rightarrow$ the number are 12, 8, 6 or 6, 8, 12

Example : 7

If pth term of an A.P. is 1/q and the qth term is 1/p. Find the sum of p q terms.

Solution

Let A and D be the first term and the common difference of the A.P.

$$\Rightarrow \quad \frac{1}{q} = A + (p - 1) D, \ \frac{1}{p} = A + (q - 1) D$$

solving these two equations to get A and D in terms of p and q

$$\Rightarrow \quad A = \frac{1}{pq} \text{ and } D = \frac{1}{pq}$$

sum of pq terms $= \frac{pq}{2}[2A + (pq - 1) D] = \frac{pq}{2}\left(\frac{2}{pq} + \frac{pq-1}{pq}\right)$

$\Rightarrow \quad \text{sum} = \dfrac{pq+1}{2}$

Example : 8

If the continued product of three numbers in G.P. is 216 and the sum of the products taken in pairs is 156, find the numbers.

Solution

Let $\dfrac{b}{r}$, b, br be the three numbers.

$\Rightarrow \quad \dfrac{b}{r}$, b, br = 216 $\quad \Rightarrow \quad$ b = 6

also $\dfrac{b}{r}$ (b) + b (br) + $\dfrac{b}{r}$ (br) = 156

$\Rightarrow \quad b^2 \left(\dfrac{1}{r}+r+1\right) = 156 \quad \Rightarrow \quad 6^2 (r^2 + r + 1) = 156r$

$\Rightarrow \quad 3r^2 + 3r + 3 = 13r$

$\Rightarrow \quad r = 3, \dfrac{1}{3}$

Hence the numbers are 2, 6, 18 or 18,, 6, 2

Example : 9

In a HP, the pth term is q r and qth term is r p. Show that the rth term is p q.

Solution

Let A, D be the first term and the common difference of the A.P. formed by the reciprocals of given H.P.

pth term of A.P. is $\dfrac{1}{qr}$ and qth term of A.P. is $\dfrac{1}{rp}$

$\Rightarrow \quad \dfrac{1}{qr} = A + (p-1) D$ and $\dfrac{1}{rp} = A + (q-1) D$

We will solve these two equation to get A and D

subtracting, we get $\dfrac{p-q}{pqr} = (p-q) D \quad \Rightarrow \quad D = \dfrac{1}{pqr}$

Hence $\dfrac{1}{qr} = A + \dfrac{p-1}{pqr} \quad \Rightarrow \quad A = \dfrac{1}{pqr}$

Now the rth term of A.P. = $T_r = A + (r-1) D$

$\Rightarrow \quad T_r = \dfrac{1}{pqr} + \dfrac{r-1}{pqr} = \dfrac{1}{pq}$

Hence rth term of the given H.P. is pq.

Example : 10

The sum of first p, q, r terms of an A.P. are a, b, c respectively. Show that :

$$\frac{a}{p}(q-r) + \frac{b}{q}(r-p) + \frac{c}{r}(p-q) = 0.$$

Solution

Let A be the first term and D be common difference of the A.P.

$\Rightarrow \quad a = \dfrac{p}{2} [2A + (p-1) D]$

We can write $\frac{a}{p}$ (q – r) + $\frac{b}{q}$ (r – p) + $\frac{c}{r}$ (p – q) = $\sum \frac{a}{p}(p-r)$

LHS $= \sum \frac{a}{p}(q-r)$

$= \sum \frac{1}{2}(q-r)\,[2A + (p-q)\,d]$

$= \frac{1}{2} \sum 2A(q-r) + \frac{1}{2} \sum (q-r)\,D\,(p-1)$

$= A \sum (q-r) + \frac{D}{2}\sum [p\,(q-r)] - \frac{D}{2}\sum (q-r)$ = 0 + 0 – 0 = 0 = RHS

Example : 11

If a, b, c are in A.P., then show that a^2 (b + c), b^2 (c + a), c^2 (a + b) are in A.P., if bc + ca + ab ≠ 0.

Solution

We have to prove that

a^2 (b + c), b^2 (c + a), c^2 (a + b) are in A.P.

i.e. a(ab + ca), b (bc + ab), c(ca + bc) are in A.P.

a(ab + bc + ca) – abc, b (bc + ab + ac) – abc, c (ca + bc + ab) – abc are in A.P.

a(ab + bc + ca), b(bc + ab + ca), c (ca + bc + ab) are in A.P.

⇒ a, b, c are in A.P., which is given.

Hence a^2 (b + c), b^2 (c + a), c^2 (a + b) are in A.P.

Alternate method :

As a, b, c are in A.P., we get : a – b = b – c ....(i)

Consider a^2 (b + c) – b^2 (c + a) = (a^2b – b^2a) + (a^2c – b^2c) = (a – b) (ab + ac + bc) ....(ii)

Also b^2 (c + a) – c^2 (a + b) = (b^2c – c^2b) + (b^2a – c^2a) = (b – c) (bc + ba + ca) ....(iii)

From (i), (ii), (iii), we get,

a^2 (b + c) – b^2 (c + a) = b^2 (c + a) – c^2 (a + b)

⇒ a^2 (b + c), b^2 (c + a), c^2 (a + b) are in A.P.

Example : 12

If $(b-c)^2$, (c – a), $(a-b)^2$ are in A.P., then prove that : $\frac{1}{b-c}$, $\frac{1}{c-a}$, $\frac{1}{a-b}$ are also in A.P.

Solution

$(b-c)^2$, $(c-a)^2$, $(a-b)^2$ are in A.P.

⇒ $(b-c)^2 - (c-a)^2 = (c-a)^2 - (a-b)^2$

⇒ (b – c) (b + a – 2c) = (c – b) (b + c – 2a)

⇒ (b – a) [(b – c) + (a – c)] = (c – a) [(b – a) + (c – a)]

Divide by (a – b) (b – c) (c – a)

⇒ $\frac{1}{b-c} - \frac{1}{c-a} = -\frac{1}{a-b} + \frac{1}{c-a}$

⇒ $\frac{1}{b-c} - \frac{1}{c-a} = \frac{1}{c-a} - \frac{1}{a-b}$

⇒ $\frac{1}{b-c}$, $\frac{1}{c-a}$, $\frac{1}{a-b}$ are in A.P.

Example : 13

If a, b, c are in G.P., prove that : $\frac{a^2 + ab + b^2}{bc + ca + ab} = \frac{b+a}{c+b}$

Solution

As a, b, c are in G.P., let us consider b = ar, and c = ar^2

$$\text{LHS} = \frac{a^2 + ab + b^2}{b + ca + ab} = \frac{a^2 + a^2r + a^2r^2}{a^2r^3 + a^2r^2 + a^2r} = \frac{a^2(1 + r + r^2)}{a^2r(r^2 + r + 1)} = \frac{1}{r}$$

$$\text{RHS} = \frac{b + a}{c + b} = \frac{ar + a}{ar^2 + ar} = \frac{a(r + 1)}{ar(r + 1)} = \frac{1}{r}$$

Hence LHS = RHS

Example : 14

If a, b, c are respectively pth, qth, rth terms of H.P., prove that bc (q – c) + ca (r – p) + ab (p – q) = 0.

Solution

Let A and D be the first term and common difference of the A.P. formed by the reciprocals of the given H.P.

$$\Rightarrow \quad \frac{1}{a} = A + (p - 1)\,D \qquad \text{............(i)}$$

$$\frac{1}{b} = A + (q - 1)\,D \qquad \text{............(ii)}$$

$$\frac{1}{c} = A + (r - 1)\,D \qquad \text{............(iii)}$$

Subtracting II and III we get $\frac{c - b}{bc} = (q - r)\,D$

$$\Rightarrow \quad bc\,(q - r) = -\frac{(b - c)}{D}$$

$$\text{LHS} = \sum bc\,(q - r)$$

$$= -\sum \frac{bc}{D} = -\frac{1}{D}\sum (b - c)$$

$$= -\frac{1}{D}\,[b - c + c - a + a - b] = 0 \text{ RHS}$$

Example : 15

If $a^x = b^y = c^x$ and x, y, z are in G.P., prove that $\log_b a = \log_c b$.

Solution

Consider $a^x = b^y = c^x$. Taking log

$$x \log a = y \log b = z \log c$$

$$\Rightarrow \quad \frac{x}{y} = \frac{\log b}{\log a} \text{ and } \frac{y}{x} = \frac{\log c}{\log b}$$

as x, y, z are in GP $\Rightarrow$ $\frac{x}{y} = \frac{y}{z}$

$$\Rightarrow \quad \frac{\log b}{\log a} = \frac{\log c}{\log b}$$

$$\Rightarrow \quad \frac{\log a}{\log b} = \frac{\log b}{\log c}$$

$$\Rightarrow \quad \log_b a = \log_c b$$

Example : 16

If $\sqrt[x]{a} = \sqrt[y]{b} = \sqrt[z]{c}$ and if a, b, c are in G.P., then prove that x, y, z are in A.P.

Solution

Let $a^{\frac{1}{z}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$

$$\Rightarrow \quad \frac{\log a}{x} = \frac{\log b}{y} = \frac{\log c}{z} = k$$

$\Rightarrow$ log a = kx, log b = ky, log c = kz

As $b^2 = ac$ $\Rightarrow$ 2 log b = log a + log c

We have 2 ky = kx + kz

$\Rightarrow$ 2y = x + z

$\Rightarrow$ x, y, z are in A.P.

Example : 17

If one GM G and two AM's p and q be inserted between two quantities, show that $G^2 = (2p - q)(2p - q)$.

Solution

Let a, b be two quantities

$\Rightarrow$ $G^2 = ab$ and a, p, q, b are in A.P.

$$\Rightarrow \quad p = a + \frac{(b-a)}{3} = \frac{b+2a}{3}, \qquad q = a + 2\,\frac{b+a}{3} = \frac{2b+a}{3}$$

RHS = (2p – q) (2q – p)

$$= \left(\frac{2}{3}(b+2a) - \frac{2b+a}{3}\right)\left(\frac{2(2b+a)}{3} - \frac{b+2a}{3}\right)$$

$$= \frac{1}{9}\,(2b + 4a - 2b - a)(4b + 2a - b - 2a)$$

$$= \frac{1}{9}\,(3a)(4b) = ab = G^2 = \text{RHS}$$

Example : 18

If S_n is the sum of first n terms of a G.P. (a_n) and S′n that of another G.P. $(1/a_n)$ then show that :
$S_n = S'n = a_1\, a_n$.

Solution

$S_n = a_1 + a_2 + a_3 + ... + a_n$

$$S'n = \frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}$$

For the first G.P. (a_n), $a_n = a_1 r^{n-1}$

$$S_n = \frac{a_1(1-r^n)}{1-r}, \text{ where r is the common ratio}$$

For the second G.P. $\left(\frac{1}{a_n}\right)$, common ration = $\frac{1}{r}$

$$S'_n = \frac{1}{a_1}\,\frac{\left(1-\frac{1}{r_n}\right)}{\left(1-\frac{1}{r}\right)} = \frac{(r^n-1)}{a_1(r-1)\,r^{n-1}} = \frac{r^n-1}{a_n(r-1)}$$

$$\Rightarrow \quad S'_n = \frac{1}{a_1 a_n} + \frac{a_1(r^n-1)}{r-1}$$

$$\Rightarrow \quad S'n = \frac{1}{a_1 a_n}\, S_n$$

$\Rightarrow$ $S_n = S'_n\, a_1\, a_n$

Example : 19

At what values of parameter 'a' are there values of n such that the numbers :
$5^{1+x} + 5^{1+x}$, a/2, $25^{x} + 25^{-x}$ form an A.P.?

Solution

For the given numbers to be in A.P.

$$2\left(\frac{a}{2}\right) \quad 5^{1+x} + 5^{1-x} + 25^{x} + 25^{-x}$$

Let $5^{x} = k$

$$\Rightarrow \qquad a = 5k + \frac{5}{k} + k^{2} + \frac{1}{k^{2}}$$

$$\Rightarrow \qquad a = 5\left(k + \frac{1}{k}\right) + \left(k^{2} + \frac{1}{k^{2}}\right)$$

As the sum of positive number and its reciprocal is always greater than or equal to 2,

$$k + \frac{1}{k} \geq 2 \qquad \text{and} \qquad k^{2} + \frac{1}{k^{2}} \geq 2$$

Hence $a \geq 5\,(2) + 2 \qquad \Rightarrow \qquad a \geq 12$

Example : 20

The series of natural numbers is divided into groups : (1) : (2, 3, 4); (5, 6, 7, 8, 9) and so on.
Show that the sum of the numbers in the nth group is $(n - 1)^{3} + n^{3}$

Solution

Note that the last term of each group is the square of a natural number. Hence first term in the nth group is $\quad = (n - 1)^{2} + 1 = n^{2} - 2n + 2$
There is 1 term in Ist group, 3 in IInd, 5 in IIIrd, 7 in IVth, ...
No. of terms in the nth group = nth term of (1, 3, 5, 7 .....) = 2n – 1
Common difference in the nth group = 1

$$\text{Sum} \quad = \frac{2n-1}{2}\,[2(n^{2} - 2n + 2) + (2n - 2)\,1]$$

$$= \frac{2n-1}{2}\,[2n^{2} - 2n + 2] = (2n - 1)\,(n^{2} - n + 1)$$

$$= 2n^{3} - 3n^{2} + 3n - 1 = n^{3} + (n - 1)^{3}$$

Example : 21

If $\frac{1}{a} + \frac{1}{c} + \frac{1}{a-b} + \frac{1}{c-b} = 0$, prove that a, b, c are in H.P., unless b = a + c

Solution

$$\frac{1}{a} + \frac{1}{c} + \frac{1}{a-b} + \frac{1}{c-b} = 0$$

$$\Rightarrow \qquad \frac{a+c}{ac} + \frac{a+c-2b}{ac - b(a+c) + b^{2}} = 0$$

Let a + c = t

$$\Rightarrow \qquad \frac{1}{ac} + \frac{t-2b}{ac - bt + b^{2}} = 0$$

$\Rightarrow \qquad act - bt^{2} + b^{2}t + act - 2abc = 0$
$\Rightarrow \qquad bt^{2} - b^{2}t - 2act + 2abc = 0$
$\Rightarrow \qquad bt\,(t - b) - 2ac\,(t - b) = 0$
$\Rightarrow \qquad (t - b)\,(bt - 2ac) = 0$
$\Rightarrow \qquad t = b \qquad \text{or} \qquad bt = 2ac$

$\Rightarrow \quad a + c = b \quad$ or $\quad b(a + c) = 2ac$

$\Rightarrow \quad a + c = b \quad$ or $\quad b = \dfrac{2ac}{a+c}$

$\Rightarrow \quad$ a, b, c are in H.P. or a + c = b

Example : 22

If $a_1, a_2, a_3, \ldots\ldots, a_n$ are in HP, prove that : $a_1 a_2 + a_2 a_3 + \ldots\ldots\ldots\ldots + a_{n-1} a_n = (n-1) a_1 a_n$.

Solution

Let D be the common difference of the A.P. corresponding to the given H.P.

$$\Rightarrow \quad \frac{1}{a_n} = \frac{1}{a_1} + (n-1)D \quad \ldots\ldots\ldots\ldots(i)$$

Now $\dfrac{1}{a_1}, \dfrac{1}{a_2}, \dfrac{1}{a_3}$ .... are in A.p.

$$\Rightarrow \quad \frac{1}{a_2} - \frac{1}{a_1} = D$$

$$\Rightarrow \quad a_1 a_2 = \frac{a_1 - a_2}{D} \quad \text{and} \quad a_2 a_3 = \frac{a_2 - a_3}{D} \text{ and so on.}$$

$$\Rightarrow \quad a_{n-1} a_n = \frac{a_{n-1} - a_n}{D}$$

Adding all such expressions we get

$$\Rightarrow \quad a_1a_2 + a_2a_3 + a_3a_4 + \ldots\ldots\ldots a_{n-1} a_n = \frac{a_1 - a_n}{D}$$

$$\Rightarrow \quad a_1a_2 + a_2a_3 + \ldots\ldots\ldots\ldots + a_{n-1} a_n = \frac{a_1a_n}{D}\left(\frac{1}{a_n} - \frac{1}{a_1}\right)$$

$$\Rightarrow \quad a_1 a_2 + a_2a_3 + \ldots\ldots\ldots + a_{n-1} a_n = \frac{a_1a_n}{D}[(n-1)D] \quad \ldots\ldots\ldots\ldots \text{ using (i)}$$

Hence $a_1 a_2 + a_2 a_3 + \ldots\ldots\ldots\ldots\ldots + a_{n-1} a_n = (n-1) a_1a_n$

Example : 23

If p be the first of n AM's between two numbers ; q be the first of n HM's between the same numbers, prove that the value of q cannot lie between p and $\left(\dfrac{n+1}{n-1}\right)^2$ p.

Solution

Let the two numbers be a and b. If p is first of n AM's then :

$$p = a + \frac{b-a}{n+1} = \frac{b+an}{n+1} \quad \ldots\ldots\ldots\ldots(i)$$

If q is first of n HM s then :

$$\frac{1}{q} = \frac{1}{a} + \frac{\frac{1}{b} - \frac{1}{a}}{n+1} \quad \Rightarrow \quad q = \frac{ab(n+1)}{bn+a} \quad \ldots\ldots\ldots\ldots(ii)$$

Dividing (ii) by (i) we get $\dfrac{q}{p} = \dfrac{ab\,(n+1)^2}{(bn+a)\,(an+b)}$

$\Rightarrow \quad \frac{p}{q} = \frac{(n+1)^2}{n^2+1+n\left(\frac{a}{b}+\frac{b}{a}\right)}$

As the sum of a number and its reciprocal cannot lie between – 2 and +2

$\Rightarrow \quad 2 \le \frac{a}{b} + \frac{b}{a} \le -2$

$\Rightarrow \quad (n + 1)^2 \le n \left(\frac{a}{b}+\frac{b}{a}\right) + n^2 + 1 \le (n - 1)^2$

$\Rightarrow \quad \frac{1}{(n+1)^2} \ge \frac{q}{p(n+1)} \ge \frac{1}{(n-1)^2}$

$\Rightarrow \quad p \ge q \ge p \left(\frac{n+1}{n-1}\right)^2$

$\Rightarrow$ q cannot lie between p and p $\left(\frac{n+1}{n-1}\right)^2$

Example : 24

If a, b, c are in A.P., α, β, γ are in A.P. and $a\alpha$, $b\beta$, $c\gamma$ are in G.P., prove that a : b : c = $\frac{1}{\gamma} : \frac{1}{\beta} : \frac{1}{\alpha}$.

Solution

a, b, c are in A.P. ....(i)

$\Rightarrow \quad 2b = a + c$

α, β, γ are in H.P.

$\Rightarrow \quad \beta = \frac{2\alpha\gamma}{\alpha+\gamma}$ ....(ii)

$a\alpha$, $b\beta$, $c\gamma$ are in GP

$\Rightarrow \quad b^2\beta^2 = a\alpha\, c\gamma$ ....(iii)

Using (i), (ii) and (iii), $\left(\frac{a+c}{2}\right)^2 \left(\frac{2\alpha\gamma}{\alpha+\gamma}\right)^2 = a\alpha\, c\gamma$

$\frac{(a+c)^2}{ac} = \frac{(\alpha+\gamma)^2}{\alpha\gamma}$

$\Rightarrow \quad \frac{a}{c} + \frac{c}{a} = \frac{\alpha}{\gamma} + \frac{\gamma}{\alpha}$

Multiplying by $\frac{\alpha}{\gamma}$ we get,

$\frac{\alpha^2}{\gamma^2} - \frac{\alpha}{\gamma}\left(\frac{a}{c}+\frac{c}{a}\right) + 1 = 0$

$\Rightarrow \quad \frac{\alpha}{\gamma}\left(\frac{\alpha}{\gamma}-\frac{a}{c}\right) - \frac{c}{a}\left(\frac{\alpha}{\gamma}-\frac{a}{c}\right) = 0$

$\Rightarrow \quad \left(\frac{\alpha}{\gamma}-\frac{c}{a}\right)\left(\frac{\alpha}{\gamma}-\frac{a}{c}\right) = 0$

$\Rightarrow \quad a\alpha = c\gamma \quad$ or $\quad c\alpha = a\gamma$

$a\alpha = c\gamma$ is not possible an $a\alpha$, $b\beta$, $c\gamma$ are in GP {obviously with common ratio $\neq 1$)

Hence we have only $c\alpha = a\gamma$

Using this in (iii)

$b^2\beta^2 = a2\gamma^2 \quad \Rightarrow \quad b\gamma = a\gamma$

$$\Rightarrow \quad \frac{1}{1/\gamma} = \frac{b}{1/\beta} = \frac{c}{1/\alpha}$$

$$\Rightarrow \quad a : b : c = \frac{1}{\gamma} : \frac{1}{\beta} : \frac{1}{\alpha}$$

Example : 25

Find three numbers a, b, c between 2 and 18 such that :

(i) their sum is 25,

(ii) the numbers 2, a, b are consecutive terms of an A.P.

(iii) the numbers b, c, 18 are consecutive terms of a G.P.

Solution

According to the given condition, we have

$a + b + c = 25$ ....(i)

$2a = b + 2$ ....(ii)

$c^2 = 18b$ ....(iii)

eliminating b and a, using (ii) and (iii) we get

$$\frac{1}{2}\left(\frac{c^2}{18}+2\right) + \frac{c^2}{18} + c = 25$$

$\Rightarrow \quad c^2 + 36 + 2c^2 + 36c = 25\ (36)$

$\Rightarrow \quad (c + 24)\ (c - 12) = 0$

$\Rightarrow \quad c = 12, -24$

As a, b, c are between 2 and 18, c = 12 is the only solution

Using (iii), $b = c^2/18 = \delta$

Using (ii), $a = \dfrac{b+2}{2} = 5$

Hence a = 5, b = 8, c = 12

Example : 26

If a, b, c are in G.P., and the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root then show that d/a, e/b, f/c are in A.P.

Solution

a, b, c are in G.P. $\quad \Rightarrow \quad b^2 = ac$

Hence the first equation has real roots because its discriminant $= 4b^2 - 4ac = 0$

the roots are $x = \dfrac{-2b}{2a} = -\dfrac{b}{a}$

As the two equations have a common roots, –b/a is root of the second equation also.

$$\Rightarrow \quad d\left(-\frac{b}{a}\right)^2 + 2e\left(-\frac{b}{a}\right) + f = 0$$

$\Rightarrow \quad db^2 - 2abc + a^2f = 0$

dividing by $ab^2 \quad \Rightarrow \quad \dfrac{d}{a} - \dfrac{2e}{b} + \dfrac{a^2f}{ab^2} = 0$

$$\Rightarrow \quad \frac{d}{a} - \frac{2e}{b} + \frac{a^2f}{a(ac)} = 0 \quad \Rightarrow \quad \frac{d}{a} - \frac{2e}{b} + \frac{f}{c} = 0$$

$\Rightarrow \quad \frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P.

Example : 27

The sum of the squares of three distinct real numbers, which are in G.P. is S^2 . If their sum is αS, show that : $\alpha^2 \in \left(\frac{1}{3}, 1\right) \cup (1, 3)$

Solution

Let the numbers be b, br and br^2

$b^2 + b^2 r^2 + b^2 r^4 = S^2$

$b + br + br^2 = \alpha S$

eliminating S, we get

$$\frac{b^2(1+r^2+r^4)}{b^2(1+r+r^2)^2} = \frac{S^2}{\alpha^2 S^2}$$

$$\Rightarrow \quad \alpha^2 = \frac{(1+r+r^2)^2}{1+r^2+r^4} = \frac{(1+r+r^2)^2}{(1+r^2)^2 - r^2}$$

$$\Rightarrow \quad \alpha^2 = \frac{(1+r+r^2)^2}{(1+r+r^2)(1+r^2-r)} = \frac{1+r+r^2}{1-r+r^2}$$

$\Rightarrow \quad r^2(\alpha^2 - 1) - r(\alpha^2 + 1) + \alpha^2 - 1 = 0$

as r is real, this quadratic must have non-negative discriminant

$\Rightarrow \quad (\alpha^2 + 1)^2 - 4(\alpha^2 - 1)(\alpha^2 - 1) \ge 0$

$\Rightarrow \quad [\alpha^2 + 1 + 2(\alpha^2 - 1)]\,[\alpha^2 + 1 - 2(\alpha^2 - 1)] \ge 0$

$\Rightarrow \quad (3\alpha^2 - 1)(3 - \alpha^2) \ge 0$

$\Rightarrow \quad (\alpha^2 - 1/3)(\alpha^2 - 3) \le 0$

As the numbers in G.P. are distinct, the following cases should be excluded.

$\alpha^2 = 3 \quad \Rightarrow \quad r = 1$

$\alpha^2 = 1/3 \quad \Rightarrow \quad r = 1$

$\alpha^2 = 1 \quad \Rightarrow \quad r = 0$

Hence α^2 is between 1/3 and 3, but not equal to 1.

$$\Rightarrow \quad \alpha^2 \in \left(\frac{1}{3}, 1\right) \cup (1, 3)$$

Example : 28

If the first and the (2n – 1) st term of an A.P., a G.P. and a H.P. are equal and their nth terms are a, b and c respectively, then :

(A) a = b = c (B) $a \ge b \ge c$ (C) a + c = b (D) $ac - b^2 = 0$

Solution

Let the first term = A

The last term [(2n – 1) st term] = L

No. of terms 2n – 1 i.e. odd

Middle term = $\frac{(2n-1)+1}{2}$ = n^{th} term

$\Rightarrow$ T_n is the middle term for all the three progressions. In an A.P. the middle term is the arithmetic mean of first and the last terms.

$$\Rightarrow \quad a = \frac{A+L}{2}$$

In a G.P. the middle term is the geometric mean of first and last terms.

$\Rightarrow \quad b = \sqrt{AL}$

In an H.P. the middle term is the harmonic mean of first and last terms.

$\Rightarrow \quad c = \dfrac{2AL}{A+L}$

Hence a, b, c are AM, GM and HM between the numbers A and L.

As $(GM)^2 = (AM)\,(HM)$

We have $b^2 = ac$

$\Rightarrow$ (B) and (D) are the correct choices.

Example : 29

Sum of the series : $1 + 3x + 5x^2 + 7x^2 + \ldots\ldots\ldots$

Solution

Note that the given series is an Arithmetico-Geometric series.

1, 3, 5, ... are in A.P. $\Rightarrow \quad T_n = 2n - 1$

1, x, x^2 , ... are in G.P. $\Rightarrow \quad T_n = x^{n-1}$

(a) This means that n^{th} term of A – G series = $(2n - 1)\, x^{n-1}$

$S = 1 + 3x + 5x^2 + \ldots\ldots\ldots + (2n - 3)\, x^{n-2} + (2n - 1)\, x^{n-1}$ ....(i)

$xS = x + 3x^2 + 5x^3 + \ldots\ldots\ldots\ldots + (2n - 3)\, x^{n-1} + (2n - 1)x^n$ ....(ii)

$\Rightarrow \quad (1 - x)\, S = 1 + 2x + 2x^2 + \ldots\ldots\ldots + 2x^{n-1} - (2n - 1)\, x^n$

$\Rightarrow \quad (1 - x)\, S = 1 + \dfrac{2x(1-x^{n-1})}{1-x} - (2n - 1)\, x^n$

$\Rightarrow \quad S = \dfrac{1}{1-x} + \dfrac{2x(1-x^{n-1})}{(1-x^2)} - \dfrac{(2n-1)x^n}{1-x}$

(b) $S_n = 1 + 3x + 5x^2 + \ldots\ldots\ldots\ldots\ldots$ to ∞

$xS_n = x + 3x^2 + 5x^3 + \ldots\ldots\ldots\ldots$ to ∞

$\Rightarrow \quad (1 - x)\, S_\infty = 1 + 2x + 2x^2 + \ldots\ldots\ldots\ldots$ to ∞

$\Rightarrow \quad (1 - x)\, S_\infty = 1 + 2x\,(1 + x + x^2 + \ldots\ldots\ldots\ldots$ to $\infty)$

$\Rightarrow \quad (1 - x)\, S_\infty = 1 + 2x \left(\dfrac{1}{1-x}\right) = \dfrac{1+x}{1-x}$

$\Rightarrow \quad S_\infty = \dfrac{1+x}{(1-x)^2}$

Example : 30

Sum the series : 1.2.3 + 2.3.4 + 3.4.5. + ... + to n terms.

Solution

Here $T_n = n(n + 1)\,(n + 2)$

$\Rightarrow \quad T_n = n^3 + 3n^2 + 2n$

$\Rightarrow \quad T_n = n^3 + 3n^2 + 2n$

$\Rightarrow \quad S_n = \sum T_n = \sum n^3 + 3 \sum n^2 + 2 \sum n$

$\Rightarrow \quad \dfrac{n^2(n+1)^2}{4} + \dfrac{2n(n+1)(2n+1)}{6} + \dfrac{2n(n+1)}{2}$

$\Rightarrow \quad \dfrac{n(n+1)}{4}\,[n(n + 1) + 2\,(2n + 1) + 4]$

$\Rightarrow \quad \dfrac{n(n+1)}{4}\,[n^2 + 5n + 6] = \dfrac{n(n+1)(n+2)(n+3)}{4}$

Example : 31

Sum the series : $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + ...$ to n terms.

Solution

First determine the nth term.

$\Rightarrow \quad T_n = (1^2 + 2^2 + 3^2 + ... + n^2)$

$$\Rightarrow \quad T_n = \sum n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\Rightarrow \quad T_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

$$\text{Now} \quad S_n = \sum T_n = \frac{1}{3}\sum n^3 + \frac{1}{2}\sum n^2 + \frac{1}{6}\sum n$$

$$S_n = \frac{1}{3}\frac{n^2(n+1)^2}{4} + \frac{1}{2}\frac{n(n+1)(2n+1)}{6} + \frac{1}{6}\frac{n(n+1)}{2}$$

Simplify to get $S_n = \dfrac{n(n+1)^2(n+2)}{12}$

Example : 32

Find the sum of n terms of series : $(x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + ...$

Solution

Let $S_n = (x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^2) + ...$

$$S_n = \frac{x^2-y^2}{x-y} + \frac{x^3-y^3}{x-y} + \frac{x^4-y^4}{x-y} + ...$$

$$= \frac{1}{x-y}(x^2 + x^3 + x^4 + ...) - \frac{1}{x-y}(y^2 + y^3 + y^4 + ...)$$

$$= \frac{1}{x-y}\left(\frac{x^2(1-x^n)}{1-x}\right) - \frac{1}{x-y}\left(\frac{y^2(1-y^n)}{1-y}\right)$$

Note : The following results can be very useful

(i) $\dfrac{x^n-y^n}{x-y} = x^{n-1} + x^{n-2} + x^{n-3} + ... + xy^{n-2} + y^{n-1}$ (n is an integer)

(ii) $\dfrac{x^n+y^n}{x+y} = x^{n-1} - x^{n-2} + x^{n-3}y^2 - x^{n-4}y^3 + ... + y^{n-1}$ (n is odd)

Example : 33

Sum the series : $1 + \dfrac{4}{5} + \dfrac{7}{5^2} + \dfrac{10}{5^3} + ...$ to n terms and to ∞.

Solution

The given series is arithmetico-geometric series

$$\text{Let } S = 1 + \frac{4}{5} + \frac{7}{5^2} + ... + \frac{3n-2}{5^{n-1}}$$

$$\frac{1}{5}S = \frac{1}{5} + \frac{4}{5^2} + ... + \frac{3n-5}{5^{n-1}} + \frac{3n-2}{5^n}$$

$$\Rightarrow \quad \frac{4}{5}S = 1 + \left(\frac{3}{5} + \frac{3}{5^2} + ... + \frac{3}{5^{n-1}}\right) - \left(\frac{3n-2}{5^n}\right)$$

$$\Rightarrow \quad S = \frac{5}{4} + \frac{5}{4} \times \frac{3}{5}\left(\frac{1-\frac{1}{5^{n-1}}}{1-\frac{1}{5}}\right) - \left(\frac{3n-2}{5^n} \times \frac{5}{4}\right)$$

$$\Rightarrow \quad S = \frac{5}{4} + \frac{3}{4}\left(\frac{5^{n-1}-1}{4}\right)\frac{1}{5^{n-2}} - \frac{3n-2}{4.5^{n-1}}$$

$$\Rightarrow \quad S = \frac{5}{4} + \frac{15}{16} - \frac{3}{16.5^{n-2}} - \frac{3n-2}{20.5^{n-2}}$$

$$\Rightarrow \quad S = \frac{35}{16} - \left(\frac{12n+7}{80(5^{n-2})}\right)$$

Now $S_n = 1 + \frac{4}{5} = \frac{7}{5^2} + ... \infty$

$\frac{1}{5} S_\infty = \frac{1}{5} + \frac{4}{5^2} + ... \infty$

$$\Rightarrow \quad \frac{4}{5} S_\infty = 1 + \frac{3}{5} + \frac{3}{5^2} + ... \infty$$

$$\Rightarrow \quad \frac{4}{-} S_\infty = 1 + \frac{3/5}{1-1/5}$$

$$\Rightarrow \quad S_\infty = \frac{5}{4}\left(1+\frac{3}{4}\right) = \frac{35}{16}$$

$$\frac{(n+1)^2}{4}$$

Example : 34

Sum the series :

$$\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + ... \text{ to n terms.}$$

Solution

$$T_n = \frac{1^3+2^3+.....+n^2}{1+3+5+....+n}$$

$$T_n = \frac{\sum n^3}{\frac{n}{2}[2+(n-1)2]} =$$

$$\Rightarrow \quad T_n = \frac{n^2+2n+1}{4}$$

$$S_n = \sum T_n = \frac{1}{4}\left[\sum n^2 + 2\sum n + \sum 1\right] = \frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6} + n(n-1) + n\right] = \frac{n}{24}\,[2n^2 + 3n + 1 + 6n + 6 + 6]$$

$$\Rightarrow \quad S_n = \frac{n}{24}\,[2n^2 + 9n + 12]$$

Example : 35

Find the sum of the products of every pair of the first n natural numbers.

Solution

The required sum is given as follows.

S = 1.2 + 1.3 + 1.4 + ... + 2.3 + 2.4 + ... + 3.4 + 3.5 + ... + ... + (n – 1)n

Using : $S = \dfrac{\left(\sum n\right)^2 - \sum n^2}{2}$, we get :

$$S = \frac{1}{2}\left[\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}\right] = \frac{n(n+1)}{24}\ [3n\ (n + 1) - 2\ (2n + 1)]$$

$$= \frac{n(n+1)}{24}\ [3n^2 - n - 2]$$

$\Rightarrow$ $S_n = \dfrac{n(n+1)(3n+2)(n-1)}{24}$

Example : 36

Find the sum of first n terms of the series : 3 + 7 + 13 + 2I + 3I + ...

Solution

The given series is neither an A.P. nor a G.P. but the difference of the successive terms are in A.P.

Series :	3	7	13	21	31	....
Differences :	4	6	8	10		....

In such cases, we find the nth term as follows :

Let S be the sum of the first n terms.

$S = 3 + 7 + 13 + 2I + 3I + ... + T_n$

$S = \quad 3 + 7 + 13 + 2I + 3I + ... + T_{n-1} + T_n$

On subtracting, we get :

$0 = 3 + \{4 + 6 + 8 + 10 + ...\} - T_n$

$\Rightarrow$ $T_n = 3 + \{4 + 6 + 8 + 10 + ...(n - 1) \text{ terms}\}$

$\Rightarrow$ $T_n = 3 + \dfrac{n-1}{2}\ [2(4) + (n - 2)\ 2]$

$\Rightarrow$ $T_n = n^2 + n + 1$

$\Rightarrow$ $S = \sum\limits_{k=1}^{n} T_k = \sum k^2 + \sum k + \sum 1$

$\Rightarrow$ $\dfrac{n(n+1)(2n+1)}{6} + \dfrac{n(n+1)}{2} + n = \dfrac{n}{3}\ (n^2 + 3n + 5)$

Example : 37

Sum the series : 1 + 4 + 10 + 22 + 46 + ... to n terms.

Solution

The differences of successive terms are in G.P. :

Series :	1	4	10	22	46	....
Differences :	3	6	12	24		....

Let S = sum of first n terms.

$\Rightarrow$ $S = 1 + 4 + 10 + 22 + 46 + ... + T_n$

$\Rightarrow$ $S = 1 + 4 + 10 + 22 + 46 + ... + T_{n-1} + T_n$

On subtracting, we get

$0 = 1 + \{3 + 6 + 12 + 24 + ...\} - T_n$

$T_n = 1 + \{3 + 6 + 12 + 24 + ...(n - 1) \text{ terms}\}$

$\Rightarrow$ $T_n = 1 + \dfrac{3(2^{n-1} - 1)}{2 - 1}$

$\Rightarrow \quad T_n = 3.2^{n-1} - 2$

$\Rightarrow \quad S = \sum_{k=1}^{n} T_k = \frac{3}{2} \sum 2^k - \sum 2 = \frac{3}{2} (2 + 4 + 8 + ...+ 2n) - 2n$

$= \frac{3}{2} \frac{2(2^n - 1)}{2 - 1} - 2n = 3.2^n - 2^n - 3$

Example : 38

Find the sum of the series : $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + ...$ to n terms

Solution

Let $\quad S = \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + ... + \frac{1}{(3n-2)(3n+1)}$

$\Rightarrow \quad 3S = \frac{3}{1.4} + \frac{3}{4.7} + \frac{3}{7.10} + ... + \frac{3}{(3n-2)(3n+1)}$

$\Rightarrow \quad 3S = \frac{4-1}{1.4} + \frac{7-4}{4.7} + \frac{10-7}{7.10} + ... + \frac{(3n+1)(3n-2)}{(3n-2)(3n+1)}$

$\Rightarrow \quad 3S = \left(\frac{1}{1} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{10}\right) + ... + \left(\frac{1}{(3n-2)} - \frac{1}{3n+1}\right)$

$\Rightarrow \quad S = \frac{1}{1} - \frac{1}{3n+1}$

$\Rightarrow \quad S = \frac{n}{3n+1}$

Note : The above method works in the case when nth term of a series can be expressed as the difference of the two quantities of the type :

$T_n = f(n) - f(n-1)$

or

$T_n = f(n) - f(n+1)$

In the above example, $T_n = \frac{1}{(3n-2)(3n+1)} = \frac{1}{3}\left(\frac{1}{3n-2} - \frac{1}{3n+1}\right)$

It si the form f(n) – f (n + 1)

Example : 39

Find the sum fo first n terms of the series : $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \frac{1}{4.5.6} + ...$

Solution

Let $\quad S = \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + ... + \frac{1}{n(n+1)(n+2)}$

$2S = \frac{3-1}{1.2.3} + \frac{4-2}{2.3.4} + \frac{5-3}{3.4.5} + ... + \frac{(n+2)-n}{n(n+1)(n+2)}$

$2S = \left(\frac{1}{1.2} - \frac{1}{2.3}\right) + \left(\frac{1}{2.3} - \frac{1}{3.4}\right) + ... + \left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)$

$$2S = \frac{1}{1.2} - \frac{1}{(n+1)(n+2)}$$

$$\Rightarrow \quad S = \frac{1}{4} - \frac{1}{2(n-1)(n+2)}$$

Note : You should observe that here,

$$T_n = \frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)$$

It is in the form f(n) – f(n+1)

Example : 40

Find the sum of first n terms of the series : 1(1)! + 2(2) ! + 3(3)! + 4(4)! + ...

Solution

The nth term = T_n = n(n) !

T_n can be written as

$T_n = (n + 1 – 1) (n)!$

$\Rightarrow$ $T_n = (n + 1) ! – (n) !$ ....(i)

This is in the form f(n) – f(n – 1)

$S = T_1 + T_2 + T_3 + T_4 + ... + T_n$

S = (2! – 1!) + (3! – 2!) + (4! – 3!) + ... + {(n + 1)! – n!}

$\Rightarrow$ S = – 1! + (n + 1)!

$\Rightarrow$ S = (n + 1)! – 1

Example : 41

Sum the series to n terms : 4 + 4 4 + 4 4 4 + 4 4 4 4 + ... to n terms.

Solution

Let S_n = 4 + 44 + 444 + 4444 + ... to n terms

$\Rightarrow$ S_n = 4(1 + 11 + 111 + 1111 + ... n terms)

$\Rightarrow$ S_n = 4/9 {(10 – 1) + (100 – 1) + (1000 – 1) + ... n terms}

$\Rightarrow$ S_n = 4/9 {(10 + 10^2 + 10^3 + ... n terms) – (1 + 1 + 1 + ... n terms)}

$$S_n = \frac{4}{9}\left(\frac{10(10^n - 1)}{10 - 1} - n\right)$$

$$\Rightarrow \quad S_n = \frac{4}{81}\,[10\,(10^n - 1) - 9n]$$

Example : 1

What can you say about the roots of the following equations ?

(i) $x^2 + 2(3a + 5)x + 2(9a^2 + 25) = 0$

(ii) $(y - a)(y - b) + (y - b)(y - c) + (y - c)(y - a) = 0$

Solution :

(i) Calculate Discriminant D

$D = 4(3a + 5^2) - 8(9a^2 + 25)$

$D = -4(3a - 5)^2$

$\Rightarrow$ $D \leq 0$, so the roots are :

complex if $a \neq 5/3$ and real and equal if $a = 5/3$.

(ii) Simplifying the given equation ;

$3y^2 - 2(a + b + c)y + (ab + bc + ca) = 0$

$\Rightarrow$ $D = 4(a + b + c)^2 - 12(ab + bc + ca)$

$\Rightarrow$ $D = 4(a^2 + b^2 + c^2 - ab - bc - ca)$

Now using the identity

$$(a^2 + b^2 + c^2 - ab - bc - ca) = \frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2]$$

we get :

$D = 2[(a - b)^2 + (b - c)^2 + (c - a)^2]$

$\Rightarrow$ $D \geq 0$, so the roots are real

Note : if $D = 0$, then $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

$\Rightarrow$ $a = b = c$

$\Rightarrow$ if $a = b = c$, then the root are equal

Example : 2

Find the value of k, so that the equations $2x^2 + kx - 5 = 0$ and $x^2 - 3x - 4 = 0$ may have one root in common.

Solution :

Let α be common root of two equations.

Hence $2\alpha^2 + k\alpha - 5 = 0$ and $\alpha^2 - 3\alpha - 4 = 0$

Solving the two equations;

$$\frac{\alpha^2}{-4k-15} = \frac{-\alpha}{-8+5} = \frac{1}{-6-k}$$

$\Rightarrow$ $(-3)^2 = (4k + 15)(6 + k)$

$\Rightarrow$ $4k^2 + 39k + 81 = 0$

$\Rightarrow$ $k = -3$ or $k = -27/4$

Example : 3

If $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a root in common, find the relation between a, b and c.

Solution

Solve the two equations as done in last example,

$ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$

$$\frac{x^2}{ba-c^2} = \frac{-x}{a^2-bc} = \frac{1}{ac-b^2}$$

$\Rightarrow$ $(a^2 - bc)^2 = (ba - c^2)(ac - b^2)$

simplifying to get : $a(a^3 + b^3 + c^3 - 3abc) = 0$

$\Rightarrow$ $a = 0$ or $a^3 + b^3 + c^3 = 3abc$

This is the relation between a, b and c.

Example : 4

If α, β are the roots of $x^2 + px + q = 0$ and γ, δ are the roots of $x^2 + rx + s = 0$, evaluate the value of $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$ in terms of p, q, r, s. Hence deduce the condition that the equation have a common root.

Solution

Let α, β be the roots of $x^2 + px + q = 0$

$\Rightarrow \quad \alpha + \beta = -p \quad$ and $\quad \alpha\beta = q \quad$ ....(i)

γ, δ be the roots of $x^2 + rx + s = 0$

$\Rightarrow \quad \gamma + \delta = -r \quad$ and $\quad \gamma\delta = s \quad$ ....(ii)

Expanding $\quad (\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$

$= [\alpha^2 - (\gamma + \delta)\alpha + \gamma\delta]\ [\beta^2 - (\gamma + \delta)\beta + \gamma\delta]$

...............[using (i) and (ii)]

$= (\alpha^2 - r\alpha + s)(\beta^2 + r\beta + s)$

As α is a root of $x^2 + px + q = 0$

we have $\alpha^2 + p\alpha + q = 0$

and similarly $\beta^2 + p\beta + q = 0$

Substituting the values of α^2 and β^2 , and we get;

$(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$

$= (-p\alpha - q + r\alpha + s)(-p\beta - q + r\beta + s)$

$= [(r - p)\alpha + s - q]\ [(r - p)\beta + s - q]$

$= (r - p)^2 \alpha\beta + (s - q)^2 + (s - q)(r - p)(\alpha + \beta)$

$= (r - p)^2 q + (s - q)^2 - p(s - q)(r - p)$

$= (r - p)(rq - pq - ps + pq) + (s - q)^2$

$= (r - p)(qr - ps) + (s - q)^2$

If the equation have a common root then either

$\alpha = \gamma$ or $\alpha = \delta$ or $\beta = \gamma$ or $\beta = \delta$

i.e. $\quad (\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta) = 0$

$\Rightarrow \quad (s - q)^2 + (r - p)(qr - ps) = 0$

$\Rightarrow \quad (s - q)^2 = (r - p)(ps - qr)$

Example : 5

If the ratio of roots of the equation $x^2 + px + q = 0$ be equal to the ratio of roots of the equation $x^2 + bx + c = 0$, then prove that $p^2c = b^2q$.

Solution

Let α and β be the roots of $x^2 + px + q = 0$ and γ, δ be the roots of equation $x^2 + bx + c = 0$

$$\Rightarrow \quad \frac{\alpha}{\beta} = \frac{\gamma}{\delta} \qquad \Rightarrow \quad \frac{\alpha}{\gamma} = \frac{\beta}{\delta}$$

$$\Rightarrow \quad \frac{\alpha}{\gamma} = \frac{\beta}{\delta} = \frac{\alpha + \beta}{\gamma + \delta} = \frac{\sqrt{\alpha\beta}}{\sqrt{\gamma\delta}} \qquad \Rightarrow \quad \frac{\alpha + \beta}{\gamma + \delta} = \frac{\sqrt{\alpha\beta}}{\sqrt{\gamma\delta}}$$

$$\Rightarrow \quad \frac{-p}{-b} = \frac{\sqrt{q}}{\sqrt{c}} \qquad \Rightarrow \quad p^2c = b^2q$$

Another Method :

$$\frac{\alpha}{\beta} = \frac{\gamma}{\delta} \qquad \Rightarrow \quad \frac{(\alpha + \beta)^2}{(\alpha - \beta)^2} = \frac{(\gamma + \delta)^2}{(\gamma - \delta)^2}$$

$$\Rightarrow \quad \frac{(\alpha + \beta)^2}{(\alpha + \beta)^2 - (\alpha - \beta)^2} = \frac{(\gamma + \delta)^2}{(\gamma + \delta)^2 - (\gamma - \delta)^2} \qquad \Rightarrow \quad \frac{(\alpha + \beta)^2}{4\alpha\beta} = \frac{(\gamma + \delta)^2}{4\gamma\delta}$$

$$\Rightarrow \quad \frac{p^2}{4q} = \frac{b^2}{4c} \qquad \Rightarrow \quad p^2c = b^2q$$

Example : 6

If α is a root of $4x^2 + 2x - 1 = 0$, prove that $4\alpha^3 - 3\alpha$ is the other root.

Solution

If α is one root, then the sum of root = – 2/4 = – 1/2

$\Rightarrow$ other root = $\beta = -1/2 - \alpha$

Now we will try to prove that :

$-1/2 - \alpha$ is equal to $4\alpha^3 - 3\alpha$.

We have $4\alpha^2 + 2\alpha - 1 = 0$, because α is a root of $4x^2 + 2x - 1 = 0$

Now $4\alpha^3 - 3\alpha = \alpha (4\alpha^2 + 2\alpha - 1) - 2\alpha^2 - 2\alpha$

$= \alpha (0) - 1/2 (4\alpha^2 + 2\alpha - 1) - 1/2 - \alpha$

$= \alpha (0) - 1/2 (0) - 1/2 - \alpha = -1/2 - \alpha$

hence $4\alpha^3 - 3\alpha$ is the other root.

Example : 7

Find all the roots of the equation : $4x^4 - 24x^3 + 57x^2 + 18x - 45 = 0$ if one root is $3 + i\sqrt{6}$.

Solution

As the coefficients are real, complex roots will occur in conjugate pairs. Hence another root is $3 - i\sqrt{6}$

Let α, β be the remaining roots.

$\Rightarrow$ the four roots are $3 \pm i\sqrt{6}$, α, β

$\Rightarrow$ the factors

$= (x - 3 - i\sqrt{6})(x - 3 + i\sqrt{6})(x - \alpha)(x - \beta)$

$= [(x - 3)^2 + 6](x - \alpha)(x - \beta)$

$= (x^2 - 6x + 15)(x - \alpha)(x - \beta)$

Dividing $4x^4 - 24x^3 + 57x^2 + 18x - 45$ by $x^2 - 6x + 15$ or by inspection we can find the other factor of quadratic equation is $4x^2 - 3$

$\Rightarrow$ $4x^4 - 24x^3 + 57x^2 + 18x - 45 = (x^2 - 6x + 15)(4x^2 - 3)$

$\Rightarrow$ α, β are roots of $4x^2 - 3 = 0$

$\Rightarrow$ $\alpha, \beta = \pm\sqrt{3/2}$

Hence roots are $3 \pm i\sqrt{6}$, $\pm\sqrt{3/2}$

Example : 8

Show that f(x) can never lie between 5 and 9 if $x \in R$, where : $f(x) = \dfrac{x^2 + 34x - 71}{x^2 + 2x - 7}$

Solution

Let $\dfrac{x^2 + 34x - 71}{x^2 + 2x - 7} = k$

$\Rightarrow$ $x^2(1 - k) + (34 - 2k)x + 7k - 71 = 0$

As $x \in R$, discriminant ≥ 0

$\Rightarrow$ $(34 - 2k)^2 - 4(1 - k)(7k - 71) \geq 0$

$\Rightarrow$ $(17 - k)^2 - (1 - k)(7k - 71) \geq 0$

$\Rightarrow$ $8k^2 - 112k + 360 \geq 0$

$\Rightarrow$ $k^2 - 14k + 45 \geq 0$

$\Rightarrow$ $(k - 5)(k - 9) \geq 0$

$\Rightarrow$ $k \in (-\infty, 5] \cup [9, \infty)$

Hence k can never lie between 5 and 9

Example : 9

Find the values of m for which the expression : $\frac{2x^2 - 5x + 3}{4x - m}$ can take all real values for $x \in R$.

Solution

Let $\frac{2x^2 - 5x + 3}{4x - m}$ = k

$\Rightarrow$ $2x^2 - (4k + 5)x + 3 + mk = 0$

$\Rightarrow$ as $x \in R$, discriminant ≥ 0

$\Rightarrow$ $(4k + 5)^2 - 8(3 + mk) \geq 0$

$\Rightarrow$ $16k^2 + (40 - 8m)k + 1 \geq 0$

k can take values which satisfy this inequality. Hence k will take all real values if this inequality is true for all values of k.

A quadratic in k is positive for all values of k if coefficient of k^2 is positive and discriminant ≤ 0

$\Rightarrow$ $(40 - 8m)^2 - 4(16)(1) \leq 0$

$\Rightarrow$ $(5 - m)^2 - 1 \leq 0$

$\Rightarrow$ $(m - 5 - 1)(m - 5 + 1) \leq 0$

$\Rightarrow$ $(m - 6)(m - 4) \leq 0$

$\Rightarrow$ $m \in [4, 6]$

So for the given expression to take all real values, m should take values : $m \in [4, 6]$

Example : 10

Solve for x : $\frac{8x^2 + 16x - 51}{(2x - 3)(x + 4)} > 3$

Solution

$$\frac{8x^2 + 16x - 51}{(2x - 3)(x + 4)} - 3 > 0$$

$$\Rightarrow \quad \frac{8x^2 + 16x - 51 - 3(2x - 3)(x + 4)}{(2x - 3)(x + 4)} > 0$$

$$\Rightarrow \quad \frac{2x^2 + x - 15}{(2x - 3)(x + 4)} > 0$$

$$\Rightarrow \quad \frac{(2x - 5)(x + 3)}{(2x - 3)(x + 4)} > 0$$

Critical points are : x = – 4, –3, 3/2, 5/2

The solution from the number line is :

$$x \in (-\infty, -4) \cup \left(-3, \frac{3}{2}\right) \cup \left(\frac{5}{2}, \infty\right)$$

Example : 11

Find the values of m so that the inequality : $\left|\frac{x^2 + mx + 1}{x^2 + x + 1}\right| < 3$ holds for all $x \in R$.

Solution

We know that $|a| < b \quad \Rightarrow \quad -b < a < b$

Hence $\left|\frac{x^2 + mx + 1}{x^2 + x + 1}\right| < 3$

$$\Rightarrow \quad -3 < \frac{x^2 + mx + 1}{x^2 + x + 1} < 3$$

First consider $\dfrac{x^2 + mx + 1}{x^2 + x + 1} < 3$

$$\Rightarrow \quad \frac{(x^2 + mx + 1) - 3(x^2 + x + 1)}{x^2 + x + 1} < 0$$

$$\Rightarrow \quad \frac{-2x^2 + (m-3)x - 2}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} < 0$$

multiplying both sides by denominator, we get :

$\Rightarrow \quad -2x^2 + (m-3)x - 2 < 0$

(because denominator is always positive)

$\Rightarrow \quad 2x^2 - (m-3)x + 2 > 0$

A quadratic expression in x is always positive if :

coefficient of $x^2 > 0$ and $D < 0$

$\Rightarrow \quad (m-3)^2 - 4(2)(2) < 0$

$\Rightarrow \quad m^2 - 6m - 7 < 0$

$\Rightarrow \quad (m-7)(m+1) < 0$

$\Rightarrow \quad m \in (-1, 7)$ ....(i)

Now consider $-3 < \dfrac{x^2 + mx + 1}{x^2 + x + 1}$

$$\Rightarrow \quad \frac{(x^2 + mx + 1) + 3(x^2 + x + 1)}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} > 0$$

$\Rightarrow \quad 4x^2 + (m+3)x + 4 > 0$ $\qquad \dfrac{1}{x+1}$

For this to be true, for all $x \in R$, $D < 0$

$\Rightarrow \quad (m+3)^2 - 4(4)(4) < 0$

$\Rightarrow \quad m^2 + 6m - 55 < 0$

$\Rightarrow \quad (m-5)(m+11) < 0$

$\Rightarrow \quad m \in (-11, 5)$ ....(ii)

We will combine (i) and (ii), because both must be satisfied

$\Rightarrow$ The common solution is $m \in (-1, 5)$.

Example : 12

Let $y = \sqrt{\dfrac{2}{x^2 - x + 1} - \dfrac{1}{x+1} - \dfrac{(2x+1)}{x^3 + 1}}$; find all the real values of x for which y takes real values.

Solution

For y to take real values

$$\frac{2}{x^2 - x + 1} - \qquad - \frac{(2x+1)}{x^3 + 1} \geq 0$$

$$\Rightarrow \quad \frac{2(x+1) - (x^2 + 1 - x) - (2x+1)}{x^3 + 1} \geq 0$$

$$\Rightarrow \quad \frac{-x^2 + x}{(x+1)(x^2 - x + 1)} \geq 0$$

$$\Rightarrow \quad \frac{x(x-1)}{(x+1)(x^2 - x + 1)} \leq 0$$

As $x^2 - x + 1 > 0$ for all $x \in R$ (because $D < 0$, $a > 0$)

Multiply both sides by $x^2 - x + 1$

$\Rightarrow \quad \frac{x(x-1)}{(x+1)} \le 0$

Critical points are x = 0, x = 1, x = – 1

Expression is negative for

$\Rightarrow \quad x \in (-\infty, -1) \cup [0, 1]$

So real values of x for which y is real are

$x \in (-\infty, -1) \cup [0, 1]$

Example : 13

Find the values of a for which the inequality (x – 3a) (x – a – 3) < 0 is satisfied for all x such that $1 \le x \le 3$.

Solution

(x – 3a) (x – a – 3) < 0

Case – I :

Let 3a < a + 3 $\Rightarrow$ a < 3/2 ....(i)

Solution set of given inequality is $x \in (3a, a + 3)$

Now for given inequality to be true for all $x \in [1, 3]$, set [1, 3] should be the subset of (3a, a + 3)

i.e. 1 and 3 lie inside 3a and a + 3 on number line

So we can take, 3a < 1 and a + 3 > 3 ....(ii)

Combining (i) and (ii), we get :

$\Rightarrow \quad a \in (0, 1/3)$

Case – II :

Let 3a > a + 3 $\Rightarrow$ a > 3/2 ....(iii)

Solution set of given inequality is $x \in (a + 3, 3a)$

As in case–I, [1, 3] should be the subset of (a + 3, 3a)

i.e. a + 3 < 1 and 3a > 3 ....(iv)

Combining (iii) and (iv), we get :

$a \in \{\ \}$ i.e. No solution ....(vi)

Combining both cases, we get : $a \in (0, 1/3)$

Alternate Solution :

Let f(x) = (x – 3a) (x – a – 3)

for given equality to be true for all values of $x \in [1, 3]$, 1 and 3 should lie between the roots of f(x) = 0.

$\Rightarrow$ f(1) < 0 and f(3) < 0 ....[using section 4.1(f)]

Consider f(1) < 0 :

$\Rightarrow$ (1 – 3a) (1 – a – 3) < 0

$\Rightarrow$ (3a – 1) (a + 2) < 0

$\Rightarrow \quad a \in (-2, 1/3)$ ....(ii)

Consider f(3) < 0 :

$\Rightarrow$ (3 – 3a) (3 – a – 3) < 0

$\Rightarrow$ (a – 1) (a) < 0

$\Rightarrow \quad a \in (0, 1)$ ....(iii)

Combining (ii) and (iii) we get : $a \in (0, 1/3)$

Example : 14

Find all the values of m, for which both the roots of the equation $2x^2 + mx + m^2 - 5 = 0$ are less than 1.

Solution

Let $f(x) = 2x^2 + mx + m^2 - 5$

As both roots of f(x) = 0 are less than 1, we can take a f (1) > 0, –b/2a < 1 and $D \ge 0$

.........[using section 4.1(b)]

Consider a f(1) > 0 :

$\Rightarrow \quad 2[2 + m + m^2 - 5] > 0$

$\Rightarrow \quad m^2 + m - 3 > 0$

$\Rightarrow \quad m \in \left(-\infty, \frac{-1-\sqrt{13}}{2}\right) \cup \left(\frac{-1+\sqrt{13}}{2}, \infty\right)$ ....(i)

Consider – b/2a < 1 :

$\frac{-m}{4} < 1$

$\Rightarrow$ $m > -4$ ....(ii)

Consider D ≥ 0 :

$m^2 - 8(m^2 - 5) \geq 0$

$\Rightarrow$ $-7m^2 + 40 \geq 0$

$\Rightarrow$ $7m^2 - 40 \leq 0$

$\Rightarrow$ $m \in \left[-\sqrt{\frac{40}{7}}, \sqrt{\frac{40}{7}}\right]$ ....(iii)

Combining (i), (ii) and (iii) on the number line, we get :

$$m \in \left[-\sqrt{\frac{40}{7}}, \frac{-1-\sqrt{13}}{2}\right) \cup \left(\frac{\sqrt{13}-1}{2}, \sqrt{\frac{40}{7}}\right]$$

Example : 15

Suppose x_1 and x_2 are the roots of the equation $x^2 + 2(k-3)x + 9 = 0$. Find all values of k such that both 6 and 1 lie between x_1 and x_2.

Solution

Let $f(x) = x^2 + 2(k-3)x + 9$

As 1 and 6 lie between x_1 and x_2 , we have

a f (6) < 0, and a f(1) < 0

.... [using section 4.1 (f)]

a f(6) < 0

$\Rightarrow$ $36 + 2(k-3)(6) + 9 < 0$

$\Rightarrow$ $12k + 9 < 0$

$\Rightarrow$ $k < -3/4$ ....(i)

a f(1) < 0

$\Rightarrow$ $1 + 2(k-3) + 9 < 0$

$\Rightarrow$ $2k + 4 < 0$

$\Rightarrow$ $k < -2$

Combining (i), (ii) and (ii) on the number line, we get : $k \in (-\infty, -2)$

Example : 16

If 2, 3 are roots $2x^3 + mx^2 - 13x + n = 0$, find m, n and the third root of the equation.

Solution

Let α be the third root of the equation

Using section 4.2 (d) we can make the following equations,

$\Rightarrow$ $\alpha + 2 + 3 = -m/2$ (sum of roots)

$2\alpha + 3\alpha + 2(3) = -13/2$ (sum of roots taken two at a time)

$2.3 \,.\, \alpha = -n/2$ (product of roots)

Hence : $\alpha + 5 = -m/2$ ....(i)

$5\alpha + 6 = -13/2$ ....(ii)

$6\alpha = -n/2$ ....(iii)

Solving (i), (ii), (iii) for α, m and n we get; $\alpha = -5/2$, $m = -5$, $n = 30$

Example : 17

Find all the values of p for which the roots of the equation $(p-3)x^2 - 2px + 5p = 0$ are real and positive

Solution

Roots are real and positive if :

D ≥ 0, sum of the roots > 0 and product of the roots > 0

D ≥ 0

$\Rightarrow$ $4p^2 - 20p(p-3) \geq 0$

$\Rightarrow$ $-4p^2 + 15p \geq 0$

$\Rightarrow \quad 4p^2 - 15p \le 0$

$\Rightarrow \quad p \in [0, 15/4]$ ....(i)

Sum of the roots > 0

$\frac{2p}{p-3} > 0 \quad \Rightarrow \quad \frac{p}{p-3} > 0$

$\Rightarrow \quad p\,(p-3) > 0$

$\Rightarrow \quad p \in (-\infty, 0) \cup (3, \infty)$ ....(ii)

Product of the roots > 0

$\frac{5p}{p-3} > 0$

$\Rightarrow \quad \frac{p}{p-3} > 0$

$\Rightarrow \quad p\,(p-3) > 0$

$\Rightarrow \quad p \in (-\infty, 0) \cup (3, \infty)$ ....(iii)

Combining (i), (ii) and (iii) on the number line, we get $p \in (3, 15/4]$

Example : 18

If 1, a_1, a_2 ....a_{n-1} are nth roots of unity, then show that $(1 - a_1)(1 - a_2)(1 - a_3)$ .... $(1 - a_{n-1}) = n$.

Solution

The roots of equation $x^n = 1$ are called as the nth roots of unity

Hence 1, a_1, a_2, a_3, ...a_{n-1} are the roots of $x^n - 1 = 0$

$x^n - 1 = (x - 1)(x - a_1)(x - a_2)(x - a_3)$ .... $(x - a_{n-1})$

is an identity in x (i.e., true for all values of x)

$\Rightarrow \quad \frac{x^n - 1}{x - 1} = (x - a_1)(x - a_2)(x - a_3) \,.........\, (x - a_{n-1})$

$\Rightarrow \quad x^{n-1} + x^{n-2} + \,...........\, + x^0 = (x - a_1)(x - a_2)(x - a_3) \,........\, (x - a_{n-1})$

[using $x^n - y^n = (x - y)(x^{n-1} y^0 + x^{n-2} y^1 + \,.........\, + x^0 y^{n-1})$

substituting x = 1 in the above identity, we get;

$n = (1 - a_1)(1 - a_2) \,..........\, (1 - a_{n-1}) + 0$

$\Rightarrow \quad (1 - a_1)(1 - a_2) \,.........\, (1 - a_{n-1}) = n$.

Example : 19

Solve for x : $|x^2 + 2x - 8| + x - 2 = 0$

Solution

$|x^2 + 2x - 8| + x - 2 = 0$

Case – I

Let $(x - 2)(x + 4) \le 0$

$\Rightarrow \quad x \in [-4, 2]$ ....(i)

the given equation reduces to : $-(x - 2)(x + 4) + x - 2 = 0$

$\Rightarrow \quad x^2 + x - 6 = 0$

$\Rightarrow \quad x = -3, 2$

We accept both the values because they satisfy (i)

Case – II

Let $(x - 2)(x + 4) > 0$

$\Rightarrow \quad x \in (-\infty, -4) \cup (2, \infty)$ ....(ii)

the given equation reduces to : $(x - 2)(x + 4) + x - 2 = 0$

$\Rightarrow \quad (x - 2)(x + 5) = 0$

$\Rightarrow \quad x = -5, 2$

We reject x = 2, because it does not satisfy (ii)

Hence the solution is $x = -5$

Now combining both cases, the values of x satisfying the given equation are $x = -5, -3, 2$.

Example : 20

Solve for x : $x^2 + 2a|x - a| - 3a^2 = 0$ if $a < 0$

Solution

Case – I

Let $x \geq a$ or $x \in [a, \infty)$ and $a < 0$ ....(i)

$\Rightarrow$ the equation is $x^2 + 2a(x - a) - 3a^2 = 0$

$\Rightarrow$ $x^2 + 2ax - 5a^2 = 0$

$\Rightarrow$ $x = -(\sqrt{6} + 1)a, (\sqrt{6} - 1)a$

We reject $(\sqrt{6} - 1)a$ because it does not satisfy (i)

Hence one solution is $-(\sqrt{6} + 1)a$.

Case – II

Let $x < a$ or $x \in (-\infty, a)$ and $a < 0$ ....(ii)

$\Rightarrow$ the equation is $x^2 - 2a(x - a) - 3a^2 = 0$

$\Rightarrow$ $x^2 - 2ax - a^2 = 0$

$\Rightarrow$ $x = (1 + \sqrt{2})a, (1 - \sqrt{2})a$

We reject $x = (1 - \sqrt{2})a$ because it does not satisfy (ii). Hence one solution is $(1 + \sqrt{2})a$

Now combining both cases, we have the final solution as $x = -(\sqrt{6}+1)a, (1+\sqrt{2})a$

Example : 21

Solve the following equation for x : $\log_{2x+3}(6x^2 + 23x + 21) + \log_{3x+7}(4x^2 + 12x + 9) = 4$

Solution

$\log_{2x+3}(6x^2 + 23x + 21) + \log_{3x+7}(4x^2 + 12x + 9) = 4$

$\Rightarrow$ $\log_{2x+3}(2x + 3)(3x + 7) + \log_{3x+7}(2x + 3)^2 = 4$

$\Rightarrow$ $1 + \log_{2x+3}(3x + 7) + 2\log_{3x+7}(2x + 3) = 4$ .... [using : log(ab) = loga + logb]

$\Rightarrow$ $\log_{2x+3}(3x + 7) + \dfrac{2}{\log_{2x+3}(3x+7)} = 3$ .... [using $\log_a b = \dfrac{1}{\log_b a}$]

Let $\log_{2x+3}(3x + 7) = t$ ....(i)

$\Rightarrow$ $t + \dfrac{2}{t} = 3$

$\Rightarrow$ $t^2 - 3t + 2 = 0$

$\Rightarrow$ $(t - 1)(t - 2) = 0$

$\Rightarrow$ $t = 1, 2$

Substituting the values of t in (i), we get :

$\log_{2x+3}(3x + 7) = 1$ and $\log_{2x+3}(3x + 7) = 2$

$3x + 7 = 2x + 3$ and $(3x + 7) = (2x + 3)^2$

$\Rightarrow$ $x = -4$ and $4x^2 + 9x + 2 = 0$

$\Rightarrow$ $x = -4$ and $(x + 2)(4x + 1) = 0$

$\Rightarrow$ $x = -4$ and $x = -2, x = -1/4$

As $\log_a x$ is defined for $x > 0$ and $a > 0$ ($a \neq 1$), the possible values of x should satisfy all of the following inequalities :

$\Rightarrow$ $2x + 3 > 0$ and $3x + 7 > 0$

Also, $(2x + 3) \neq 1$ and $3x + 7 \neq 1$

Out of $x = -4$, $x = -2$ and $x = -1/4$, only $x = -1/4$, only $x = 1/4$ satisfies the above inequalities

So only solution is $x = -1/4$

Example : 22

Solve the following equality for x : $\log_{\left(x+\frac{5}{2}\right)}\left(\frac{x-5}{2x-3}\right)^2 < 0$

Solution

$$\log_{\left(x+\frac{5}{2}\right)}\left(\frac{x-5}{2x-3}\right)^2 < 0$$

If $\log_a b < 0$, then $0 < b < 1$ and $a > 1$ OR $b > 1$ and $0 < a < 1$

Case – I

Let $\left(x+\frac{5}{2}\right) > 1$ and $0 < \left(\frac{x-5}{2x-3}\right)^2 < 1$

Consider $\left(x+\frac{5}{2}\right) > 1$

$\Rightarrow \quad x > -3/2$ ....(i)

Consider $\left(\frac{x-5}{2x-3}\right)^2 < 1$

$\Rightarrow \quad (x-5)^2 < (2x-3)^2$
$\Rightarrow \quad x^2 + 25 - 10x < 4x^2 + 9 - 12x$
$\Rightarrow \quad 3x^2 - 2x - 16 > 0$
$\Rightarrow \quad (3x-8)(x+2) > 0$
$\Rightarrow \quad x \in (-\infty, -2) \cup (8/3, \infty)$ ....(ii)

Consider $\left(\frac{x-5}{2x-3}\right)^2 > 0$

$\Rightarrow \quad x \in R - \{3/2, 5\}$ ....(iii)

Combining (i), (ii) and (iii), we get :
$x \in (8/3, \infty) - \{5\}$

Case – II

Let : $0 < \left(x+\frac{5}{2}\right) < 1$ and $\left(\frac{x-5}{2x-3}\right)^2 < 1$

Consider : $0 < \left(x+\frac{5}{2}\right) < 1$

$\Rightarrow \quad -\frac{5}{2} < x < -\frac{3}{2}$ ....(iv)

Consider $\left(\frac{x-5}{2x-3}\right)^2 > 1$

$\Rightarrow \quad x \in (-2, 8/3) - \{3/2\}$ ....(v)

Combine (iv) and (v) to get : $x \in \left(-2, -\frac{3}{2}\right)$

Now combining both cases we have the final solution as :

$$x \in \left(-2, -\frac{3}{2}\right) \cup \left(\frac{8}{3}, \infty\right) - \{5\}$$

Example : 23

For what values of the parameter a the equation $x^4 + 2ax^3 + x^2 + 2ax + 1 = 0$ has at least two distinct negative roots.

Solution

The given equation is : $x^4 + 2ax^3 + x^2 + 2ax + 1 = 0$

Divide by x^2 to get : (because, x = 0 does not satisfy the equation)

$$x^2 + 2ax + 1 + \frac{2a}{x} + \frac{1}{x^2} = 0$$

$$\Rightarrow \quad x^2 + \frac{1}{x^2} + 2a\left(x + \frac{1}{x}\right) + 1 = 0$$

Let $\left(x + \frac{1}{x}\right) = t$

$\Rightarrow \quad (t^2 - 2) + 2at + 1 = 0$

$\Rightarrow \quad t^2 + 2at - 1 = 0$

$$\Rightarrow \quad t = \frac{-2a \pm \sqrt{4a^2 + 4}}{2}$$

$$\Rightarrow \quad t = -a \pm \sqrt{a^2 + 1}$$

So we get,

$$x + \frac{1}{x} = -a + \sqrt{a^2 + 1} \text{ and} \qquad \text{............(i)}$$

$$x + \frac{1}{x} = -a - \sqrt{a^2 + 1} \qquad \text{............(ii)}$$

Consider (i)

$$x + \frac{1}{x} = -a + \sqrt{a^2 + 1}$$

$$\Rightarrow \quad x^2 + \left(a - \sqrt{a^2 + 1}\right)x + 1 = 0$$

Sum of the roots = $\sqrt{a^2 + 1} - a$

It can be easily observed that for all $a \in R$ sum of the roots is positive

Product of the roots = 1 > 0

Product of roots is also positive for all $a \in R$

$\Rightarrow$ As sum of the roots is positive and product of roots is positive, none of the roots is negative

So for given equation to have atleast 2 roots negative both roots of equation (ii) should be negative

Consider (ii)

$$x + \frac{1}{x} = -a - \sqrt{a^2 + 1}$$

$$\Rightarrow \quad x^2 + \left(a + \sqrt{a^2 + 1}\right)x + 1 = 0$$

Sum of roots = $-\left(a + \sqrt{a^2 + 1}\right) < 0$ for all $a \in R$

Product of the roots = 1 > 0 for all $a \in R$

So for above equation to have both roots negative, D should be positive

i.e. .... [using section 4.1 (g)]

D > 0

$$\Rightarrow \quad \left(a + \sqrt{a^2 + 1}\right)^2 - 4 > 0$$

$\Rightarrow \quad \left(a+\sqrt{a^2+1}-2\right)\left(a+\sqrt{a^2+1}+2\right) > 0$

$\Rightarrow \quad \left(a+\sqrt{a^2+1}-2\right) > 0$ .... (As $\left(a+\sqrt{a^2+1}+2\right)$ is positive for all $a \in R$)

$\Rightarrow \quad \sqrt{a^2+1} > 2 - a$ ....(iii)

consider $a < 2 \quad \Rightarrow \quad a^2 + 1 > 4 + a^2 - 4a$

$\Rightarrow \quad 4a > 3 \quad \Rightarrow \quad a > 3/4$ ....(iv)

consider a – 2

$\Rightarrow$ for $a > 2$, RHS < 0 and LHS > 0

$\Rightarrow$ (iii) is true for all $a \geq 2$ ....(v)

Combining (iv) and (v) we get $a > 3/4$

Example : 24

Solve for real x : x(x⁻ – 1) (x + 2) + 1 = 0

Solution

$x(x^2 - 1)(x + 2) + 1 = 0$

$\Rightarrow \quad x(x - 1)(x + 1)(x + 2) + 1 = 0$

$\Rightarrow \quad (x^2 + x)(x^2 + x - 2) + 1 = 0$

Let $x^2 + x = y$

$\Rightarrow \quad y(y - 2) + 1 = 0$

$\Rightarrow \quad y^2 - 2y + 1 = 0$

$\Rightarrow \quad (y - 1)^2 = 0$

$\Rightarrow \quad y = 1$

So $\quad x^2 + x - 1 = 0$

$\Rightarrow \quad x^2 + x - 1 = 0$

$\Rightarrow \quad x = \dfrac{-1 \pm \sqrt{5}}{2}$

Example : 25

If each pair of the three equations $x^2 + p_1x + q_1 = 0$, $x^2 + p_2x + q_2 = 0$ and $x^2 + p_3x + q_3 = 0$ have a common roots, then prove that $p_1^2 + p_2^2 + p_3^2 + 4(q_1 + q_2 + q_3) = 2(p_1p_2 + p_2p_3 + p_3p_1)$.

Solution

Since each pair has a common root, the roots of the three equations can be taken as α, β ; β, γ and γ, α respectively.

First equation is : $x^2 + p_1x + q_1 = 0$

$\Rightarrow \quad \alpha + \beta = -p_1$ ....(i)

$\Rightarrow \quad \alpha\beta = q_1$ ....(ii)

Second equation is : $x^2 + p_2x + q_2 = 0$

$\Rightarrow \quad \beta + \gamma = -p_2$ ....(iii)

$\Rightarrow \quad \beta\gamma = q_2$ ....(iv)

Third equation is : $x^2 + p_3x + q_3 = 0$

$\Rightarrow \quad \alpha + \gamma - p_3$ ....(v)

$\Rightarrow \quad \alpha\gamma = q_3$ ....(vi)

On adding (i), (iii) and (v), we get :

$2(\alpha + \beta + \gamma) = -(p_1 + p_2 + p_3)$ ....(vii)

To prove that :

$p_1^2 + p_2^2 + p_3^2 + 4(p_1 + p_2 + p_3) = 2(p_1p_2 + p_2p_3 + p_3p_1)$

To prove that :

$p_1^2 + p_2^2 + p_3^2 = 2(p_1p_2 + p_2p_3 + p_3p_1) - 4(p_1 + p_2 + p_3)$

Add $2(p_1p_2 + p_2p_3 + p_3p_1)$ to both sides, we get :

$(p_1 + p_2 + p_3)^2 = 4(p_1p_2 + p_2p_3 + p_3p_1 - q_1 - q_2 - q_3)$

Consider RHS

$RHS = 4(p_1p_2 + p_2p_3 + p_3p_1 - q_1 - q_2 - q_3)$

Using (ii), (iv) and (vi), we get :

$= 4[(\alpha + \beta)(\beta + \gamma) + (\beta + \gamma)(\alpha + \gamma) + (\alpha + \gamma)(\alpha + \beta) - \alpha\beta - \beta\gamma - \gamma\alpha]$

$= 4(\alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma)$
$= 4(\alpha + \beta + \gamma)^2$ .... [using (7)]
$= (p_1 + p_2 + p_3)^2 = \text{LHS}$

Example : 26

Solve for real x : $x^2 + \dfrac{x^2}{(x+1)} = 3$

Solution

Use : $a^2 + b^2 = (a - b)^2 + 2ab$ to get

$$\left(x - \frac{x}{x+1}\right)^2 + \frac{2x^2}{(x+1)} - 3 = 0.$$

$$\Rightarrow \quad \left(\frac{x^2 + x - x}{x+1}\right)^2 + \frac{2x^2}{(x+1)} - 3 = 0$$

$$\Rightarrow \quad \left(\frac{x^2}{x+1}\right)^2 + \frac{2x^2}{(x+1)} - 3 = 0$$

Let $\dfrac{x^2}{(x+1)} = y$

$\Rightarrow \quad y^2 + 2y - 3 = 0$
$\Rightarrow \quad y = 1, -3$

$\Rightarrow \quad \dfrac{x^2}{(x+1)} = 1$ and $\dfrac{x^2}{(x+1)} = -3$

$\Rightarrow \quad x^2 + x - 1 = 0$ and $x^2 + 3x + 3 = 0$

$\Rightarrow \quad x = \dfrac{1 \pm \sqrt{5}}{2}$ and No real roots (D < 0)

So possible values of x are $\dfrac{1 \pm \sqrt{5}}{2}$

Example : 27

Solve for x : $2^{|x+1|} - 2^x = |2^x - 1| + 1$

Solution

Find critical points
$x + 1$ and $2^x - 1 = 0$
$\Rightarrow \quad x = -1$ and $x = 0$
so critical points are $x = 0$ and $x = -1$
Consider following cases :
$x \le -1$ ....(i)
$2^{-(x+1)} - 2^x = -(2^x - 1) + 1$
$2^{-x-1} - 2^x = -2^x + 2$
$\Rightarrow \quad 2^{-x-1} = 2$
$\Rightarrow \quad -x - 1 = 1$
$\Rightarrow \quad x = -2$
As $x = -2$ satisfies (i), one solution is $x = -2$
$-1 < x \le 0$ ....(ii)
$2^{x+1} - 2^x = -(2^x - 1) + 1$
$\Rightarrow \quad 2^{x+1} = 2$
$\Rightarrow \quad x + 1 = 1$
$\Rightarrow \quad x = 0$

As $x = 0$ satisfies (ii), second solution is $x = 0$

$x > 0$ ....(iii)

$2^{x+1} - 2^x = (2^x - 1) + 1$

$\Rightarrow$ $2^{x+1} = 2^{x+1}$

$\Rightarrow$ identity in x, i.e. true for all $x \in R$

On combining $x \in R$ with (iii), we get :

$x > 0$

Now combining all cases, we have the final solution as :

$x \geq 0$ and $x = -2$

Example : 1

A straight line drawn through point A (2, 1) making an angle π/4 with the +X-axis intersects another line x + 2y + 1 = 0 in point B. Find the length AB.

Solution

Let AB = r

from parametric form, the point B can be taken as :

$B = (x_A + r\cos\theta, y_A + r\sin\theta)$

$B = (2 + r\cos\pi/4, 1 + r\sin\pi/4)$

$B = (2 + r/\sqrt{2}, 1 + r/\sqrt{2})$

As B lies on x + 2y + 1 = 0, we have $\left(2+\frac{r}{\sqrt{2}}\right) + 2\left(1+\frac{r}{\sqrt{2}}\right) = -1$

$\Rightarrow \quad r = -\frac{5\sqrt{2}}{3}$

r is negative because the point B lies below the point A.

$\Rightarrow \quad AB = \frac{5\sqrt{2}}{3}$

Alternative Method :

Find the equation of AB from point-slope form and then solve with x + 2y + 1 = 0 simultaneously to get coordinates of AB. Then use distance formula to find AB.

Example : 2

If two opposite vertices of a square are (1, 2) and (5, 8), find the coordinates of its other vertices.

Solution

Let ABCD be the square and $A \equiv (1, 2)$ and $C \equiv (5, 8)$

Let P be the intersection of diagonals

$\Rightarrow \quad P \equiv [(1 + 5)/2, (2 + 8)/2]$

$\Rightarrow \quad P \equiv (3, 5)$

To find B and D, we will apply parametric form for the line BD with P as the given point

$PB = PD = \frac{1}{2} AC = \frac{1}{2}\sqrt{(8-2)^2 + (5-1)^2}$

$\Rightarrow \quad PB = PD = \sqrt{13}$

Slope (AC) $= \frac{8-2}{5-1} = \frac{3}{2}$

$\Rightarrow \quad$ slope (BD) $= -\frac{2}{3} = \tan\theta \quad \Rightarrow \quad \tan\theta$ is obtuse

Where θ is the angle between BD and +ve X-axis

$\Rightarrow \quad \cos\theta = -\frac{3}{\sqrt{13}}$ and $\sin\theta = \frac{2}{\sqrt{13}}$

using parametric form on BD with $P \equiv (x_1, y_1) \equiv (3, 5)$

Coordinates of D :

$r = +\sqrt{13}$ because D is above P.

$\Rightarrow \quad D \equiv (x_1 + r\cos\theta, y_1 + r\sin\theta)$

$\Rightarrow \quad D \equiv \left[3+\sqrt{13}\left(-\frac{3}{\sqrt{13}}\right), 5+\sqrt{13}\frac{2}{\sqrt{13}}\right]$

$\Rightarrow \quad D \equiv (0, 7)$

Coordinates of B :

$r = -\sqrt{13}$ because B is below P.

$\Rightarrow$ $B \equiv (x_1 + r\cos\theta,\ y_1 + r\sin\theta)$

$$\Rightarrow \quad B \equiv \left[3 - \sqrt{13}\left(-\frac{3}{\sqrt{13}}\right) 5 - \sqrt{13}\,\frac{2}{\sqrt{13}}\right]$$

$\Rightarrow$ $B \equiv (6, 3)$

Example : 3

Two opposite vertices of a square are (1, 2) and (5, 8). Find the equations of its side.

Solution

Let ABCD be the square

m = slope of AC = (8 – 2) / (5 – 1) = 3/2

lines AB and AD make an angle α = 45º with AC

$$m_1 = \text{slope (AD)} = \frac{m + \tan\alpha}{1 - m\,.\tan\alpha} = \frac{3/2 + \tan 45^o}{1 - 3/2\,.\tan 45^o} = -5$$

$$m_2 = \text{slope (AB)} = \frac{m - \tan\alpha}{1 + m\,.\tan\alpha} = \frac{3/2 - \tan 45^o}{1 + 3/2 \tan 45^o} = \frac{1}{5}$$

We also have AB || DC and AB || DC.

$\Rightarrow$ slope (DC) = 1/5 and slope (BC) = – 5

Now use $y - y_1$ = slope $(x - x_1)$ on each side

Equation of AB :

$y - 2 = 1/5\,(x - 1) \quad \Rightarrow \quad x - 5y + 9 = 0$

Equation of AD :

$y - 2 = -5\,(x - 1) \quad \Rightarrow \quad 5x + y - 7 = 0$

Equation of BC :

$y - 8 = -5\,(x - 5) \quad \Rightarrow \quad 5x + y - 33 = 0$

Equation of CD :

$y - 8 = 1/5\,(x - 5) \quad \Rightarrow \quad x - 5y + 35 = 0$

Alternative Method :

Find the coordinates of B and D on the pattern of illustration and then use two-point form of equation of line for each side.

Example : 4

The equation of the base of an equilateral triangle is x + y = 2 and its vertex is (2, – 1). Find the length and equations of its sides.

Solution

Let A ≡ (2, –1) and B, C be the other vertices of the equilateral triangle. Length of the perpendicular from A to BC (x + y – 2 = 0)

$$\Rightarrow \quad p = \frac{|\,2 + (-1) - 2\,|}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}$$

$$\text{Side} = \frac{p}{\sin 60^o} = \frac{1}{\sqrt{2}} \times \frac{2}{\sqrt{3}} = \sqrt{\frac{2}{3}}$$

Now AB and AC make equal angles α = 60º with line BC whose slope is m = – 1

$$m_1 = \text{slope (AC)} = \frac{m + \tan\alpha}{1 - m, \tan\alpha} = \frac{(-1) + \tan 60^o}{1 - (-1)\,.\tan 60^o} = 2 - \sqrt{3}$$

$$m_2 = \text{slope of (AB)} = \frac{m - \tan\alpha}{1 - m\,.\tan\alpha} = \frac{-1 - \tan 60^o}{1 + (-1)\,.\tan 60^o} = 2 + \sqrt{3}$$

Equation of AC :

$y = (-1) = (2 - \sqrt{3})\,(x - 2)$

$\Rightarrow \quad (2 - \sqrt{3})\, x - y - 5 + 2\sqrt{3} = 0$

Equation of AB :

$$y - (-1) = (2 + \sqrt{3})\,(x - 2)$$

$\Rightarrow \quad (2 + \sqrt{3})\, x - y - 5 - 2\sqrt{3} = 0$

Example : 5

Find the equations of straight lines passing through (–2, –7) and having an intercept of length 3 between the straight lines : 4x + 3y = 12, 4x + 3y = 3.

Solution

Let the required line cut the given parallel lines in points A and B.

$\Rightarrow \quad AB = 3$

Let AC be the perpendicular distance between the given lines

$$\Rightarrow \quad AC = \frac{|12 - 3|}{\sqrt{4^2 + 3^2}} = \frac{9}{5}$$

$$\Rightarrow \quad \sin\theta = \frac{AC}{AB} = \frac{9/5}{3} = \frac{3}{5}$$

hence the required line(s) cut the given parallel lines at an angle θ where :

$\sin\theta = 3/5 \quad \Rightarrow \quad \tan\theta = 3/4$

Let m_1 and m_2 be the slopes of required lines.

Slopes of the given parallel lines = m = –4/3

$$m_1 = \frac{m + \tan\theta}{1 - m\tan\theta} = \frac{-4/3 - 3/4}{1 + 4/3 \,.\, 3/4} = -\frac{7}{24}$$

$$m_2 = \frac{m - \tan\theta}{1 + m\,.\,\tan\theta} = \frac{-4/3 - 3/4}{1 - 4/3\,.\,3/4} = \text{undefined.}$$

Hence one line is parallel to Y-axis and passes through (–2, –7)

$\Rightarrow$ its equation is : y + 7 = –7/24 (x + 2)

$\Rightarrow$ 7x + 24y + 182 = 0

Example : 6

Two straight lines 3x + 4y = 5 and 4x – 3y = 15 intersect at point A. Points B and C are chosen on these two lines, such that AB = AC. Determine the possible equations of the line BC passing through the point (1, 2).

Solution

Through the point (1, 2) two lines L_1 and L_2 can be drawn and hence two equations are possible for line BC.

Let m be the slope of BC

AB = BC $\quad \Rightarrow \quad$ ΔABC is isosceles and hence acute angle between BC and AB is equal to the acute angle between BC and AC.

Acute angle between AB(3x 4y = 5) and BC is α :

$$\tan\alpha = \left|\frac{m - (-3/4)}{1 + m(-3/4)}\right|$$

Acute angle between AC (4x – 3y = 15) and BC is α :

$$\tan\alpha = \left|\frac{m - (4/3)}{1 + m(-3/4)}\right|$$

$$\Rightarrow \quad \left|\frac{m - (-3/4)}{1 + m(-3/4)}\right| = \left|\frac{m - (4/3)}{1 + m(4/3)}\right|$$

$$\Rightarrow \quad \frac{4m + 3}{4 - 3m} = \pm\frac{3m - 4}{3 + 4m} = \pm\frac{m - (4/3)}{1 + m(4/3)}$$

Taking + sign

$(4m + 3)(3 + 4m) = (3m - 4)(4 - 3m)$

$16m^2 + 24m + 9 = -9m^2 + 24m - 16$

$25m^2 = -25$ which is impossible

Taking + sign

$(4m + 3)(3 + 4m) = -(3m - 4)(4 - 3m)$

$16m^2 + 24m + 9 = 9m^2 - 24m + 16$

$\Rightarrow \quad 7m^2 + 48m - 7 = 0$

$\Rightarrow \quad (m + 7)(7m + 1) = 0$

$\Rightarrow \quad m = -7$ or $m = 1/7$

Equation of BC are :

$y - 2 = -7(x - 1)$ and $y - 2 = 1/7(x - 1)$

$\Rightarrow \quad 7x + y - 9 = 0$ and $x - 7y + 13 = 0$

Method 2 ;

As line BC makes equal angles with AB and AC, it must be parallel to one of the angle bisectors of AB and AC. By finding the equations of bisectors, we get the slope of BC.

Angle bisectors of AB and AC are :

$$\frac{3x - 4y - 5}{\sqrt{9+16}} = \pm \frac{4x - 3y - 15}{\sqrt{16+9}}$$

$\Rightarrow \quad x - 7y - 10 = 0$ and $7x + y - 20 = 0$

$\Rightarrow$ slopes are 1/7 and – 7

$\Rightarrow$ slopes of BC are $m = 1/7$ and $m = -7$

$\Rightarrow$ Equations are BC are

$y - 2 = 1/7(x - 1)$ and $y - 2 = -7(x - 1)$

$\Rightarrow \quad 7x + y - 9 = 0$ and $x - 7y + 13 = 0$

Example : 7

Lines $L_1 \equiv ax + by + c = 0$ and $L_2 \equiv \ell x + my + n = 0$ intersect at point P and make an angle θ with each other. Find the equation of the line L different from L_2 which passes through P makes the same angle with L_1 .

Solution

As L passes through the intersection of L_1 and L_2 , let its equation be :

$(ax + by + c) + k(\ell x + my + n) = 0$ ....(i)

where k is a parameter

As L_1 is the angle bisector of L and L_2 , any arbitrary point $A(x_1, y_1)$ on L_1 is equidistant from L and L_2 .

$$\Rightarrow \quad \frac{\ell x_1 + my_1 + n|}{\sqrt{\ell^2 + m^2}} = \frac{|ax_1 + by_1 + c + k(\ell x_1 + my_1 + n)}{\sqrt{(a + k\ell)^2 + (b + km)^2}}$$

But A lies on L_1 . hence it must satisfy the equation of L_1

$\Rightarrow \quad ax_1 + by_1 + c = 0$

$$\Rightarrow \quad \frac{|\ell x_1 + my_1 + n|}{\sqrt{\ell^2 + m^2}} = \frac{|0 + (\ell x_1 + my_1 + n)}{\sqrt{(a + k\ell)^2 + (b + km)^2}}$$

$\Rightarrow \quad k^2(\ell^2 + m^2) = (a + k\ell)^2 + (b + km)^2$

$$\Rightarrow \quad k = -\frac{a^2 + b^2}{2a\ell + 2bm}$$

$$\Rightarrow \quad (ax + by + c) - \left(\frac{a^2 + b^2}{2a\ell + 2bm}\right)(\ell x + my + n) = 0 \text{ is the equation of L.}$$

$\Rightarrow \quad (2a\ell + 2bm)(ax + by + c) - (a^2 + b^2)(\ell x + my + n) = 0$

Alternative Method :

Let S be the slope of line L.

$\Rightarrow \quad \tan\theta = \left|\dfrac{S-(-a/b)}{1+S(-a/b)}\right| = \left|\dfrac{(-\ell/m)-(-a/b)}{1+\dfrac{\ell a}{mb}}\right|$

($\because$ by taking +ve sign, we will get S = –ℓ/m which is not the slope of L)

We also have $S = -\left(\dfrac{a+k\ell}{b+km}\right)$ [equation (i)]

Substituting for S, we can value of k.

Example : 8

Find all points on x + y = 4 that lie at a unit distance from the line 4x + 3y – 10 = 0

Solution

Let P (t, 4 – t) be an arbitrary point on the line x + y = 4

distance of P from 4x + 3y – 10 = 0 is unity

$\Rightarrow \quad \dfrac{|4t+3(4-t)-10|}{\sqrt{16+9}} = 1$

$\Rightarrow \quad |t + 2| = 5$

$\Rightarrow \quad t = -2 \pm 5 = -7, 3$

⇒ points are (–7, 11) and (3, 1)

Draw the diagram yourself

Example : 9

One side of a rectangle lies on the line 4x + 7y + 5 = 0. Two of its vertices are (–3, 1) and (1, 1). Find the equations of other three sides.

Solution

One side is 4x + 7y + 5 = 0

⇒ slope of the four sides of rectangle are : $-\dfrac{4}{7}, \dfrac{7}{4}, -\dfrac{4}{7}, \dfrac{7}{4}$

Slope of Line joining (–3, 1) and (1, 1) = $\dfrac{1-1}{1+3} = 0$

Hence A(–3, 1) and C(1, 1) are opposite vertices. Let ABCD be the rectangle with AB lying along 4x + 7y + 5 = 0 (check that A lies on this line)

Equation of AD :

$y - 1 = 7/4\ (x + 3)$

$\Rightarrow \quad 7x - 4y + 25 = 0$

Equation of CB :

$y - 1 = 7/4\ (x - 1)$

$\Rightarrow \quad 7x - 4y - 3 = 0$

Equation of CD :

$y - 1 = -4/7\ (x - 1)$

$\Rightarrow \quad 4x + 7y - 11 = 0$

Example : 10

Find the coordinates of incentre of the triangle formed by 3x – 4y = 17; y = 4 and 12x + 5y = 12.

Solution

Let A, B and C be the vertices of the triangle Let us first find the equation of interior angle bisectors of the triangle ABC. The coordinates of vertices can be calculate as :

A ≡ (19/9, –8/3), B ≡ (11, 4) and C ≡ (–2/3, 4)

Interior Bisector of angle A :

bisectors of AB and AC are :

$$\frac{3x-4y-17}{5} = \pm\frac{12x+5y-12}{13}$$

21x + 77y + 161 = 0 and 99x – 27y – 281 = 0

$\Rightarrow$ $3x + 11y + 23 = 0$ and $99x - 27y - 281 = 0$

B and C must lie on opposite sides of the interior bisector

Consider $3x + 11y + 23 = 0$

for $B \equiv (11, 4)$: LHS = 3(11) + 11(4) + 23 = 100

for $C \equiv (-2/3, 4)$: LHS = – 2 + 44 + 23 = 65

Both have same sign and hence B, C are one same side.

$\Rightarrow$ this is exterior bisector.

Hence the interior bisector of angle A is :

$99x - 27y - 281 = 0$ ....(i)

Interior bisector of angle B :

following the same procedure, we get the equation of interior bisector of B as :

$3x + 9y + 3 = 0$ ....(ii)

Solving (i) and (ii) simultaneously, we get the coordinates of incentre :

$$I = \left(\frac{29}{9}, \frac{38}{27}\right)$$

Example : 11

The ends AB of a straight line segment of constant length C slide upon the fixed rectangular axes OX and OY respectively. If The rectangle OAPB be completed, then show that the locus of the foot of perpendicular drawn from P to AB is $x^{2/3} + y^{2/3} = C^{2/3}$.

Solution

Let $A \equiv (a, 0)$ and $B \equiv (0, b)$

$\Rightarrow$ $P \equiv (a, b)$

$PQ \perp AB$

We have to find the locus of the point Q.

Let $Q \equiv (x_1, y_1)$

AB = C $\Rightarrow$ $a^2 + b^2 = c^2$ ....(i)

$PQ \perp AB$ $\Rightarrow$ slope (PQ) × slope (AB) = – 1

$$\Rightarrow \quad \left(\frac{b - y_1}{a - x_1}\right) \times \left(\frac{0 - b}{a - 0}\right) = -1$$

$\Rightarrow$ $ax_1 - by_1 = a^2 - b^2$ ....(ii)

Q lies on AB whose equation is $\frac{x}{a} + \frac{y}{b} = 1$

$$\Rightarrow \quad \frac{x_1}{a} + \frac{y_1}{b} = 1$$

$\Rightarrow$ $bx_1 + ay_1 = ab$ ....(iii)

In the problem, C is a fixed quantity while a, b are changing, we will eliminate a, b from (i), (ii) and (iii) to get the locus. By solving (ii) and (iii), we get :

$$x_1 = \frac{a^3}{a^2 + b^2} \quad \text{and} \quad y_1 = \frac{b^3}{a^2 + b^2}$$

consider $$x_1^{2/3} + y_1^{2/3} = \frac{b^2 + a^2}{(a^2 + b^2)^{2/3}} = (a^2 + b^2)^{1/3} = (C)^{2/3}$$

$\Rightarrow$ $x_1^{2/3} + y_1^{2/3} = C^{2/3}$

$\Rightarrow$ $x^{2/3} + y^{2/3} = C^{2/3}$ is the equation of required locus.

Alternative method :

Let angle OAB = θ

$\Rightarrow$ OA = C cos θ and OB = C sin θ = AP

From ΔAPQ :

AQ = (C sin θ) sin θ = $C\sin^2\theta$

Draw $QM \perp OA$

From ΔAQM :

$AM = AQ \cos\theta = C \sin^2\theta \cos\theta$
$QM = AQ \sin\theta = C \sin^3\theta$
$QM = y_1 = C \sin^3\theta$
and $OM = x_1 = OA - AM = C\cos\theta - C\sin^2\theta\cos\theta$
$\Rightarrow$ $x_1 = C\cos\theta\,(1 - \sin^2\theta) = C\cos^3\theta$
$\Rightarrow$ $x_1 = C\cos^3\theta$ and $y_1 = C\sin^3\theta$. We will eliminate θ
Substituting for $\cos\theta$, $\sin\theta$ in $\sin^2\theta + \cos^2\theta = 1$, we get :

$$\left(\frac{x_1}{C}\right)^{2/3} + \left(\frac{y_1}{C}\right)^{2/3} = 1$$

$\Rightarrow$ $x_1^{2/3} + y_1^{2/3} = C^{2/3}$
$\Rightarrow$ $x^{2/3} + y^{2/3} = C^{2/3}$ is the locus of Q.

Example : 12

A variable line is draw through O to cut two fixed straight lines L_1 and L_2 in R and S. A point P is chosen on the variable line such that : $\frac{m+n}{OP} = \frac{m}{OR} + \frac{n}{OS}$. Show that the locus of P is a straight line passing through intersection of L_1 and L_2

Solution

Let the fixed point O be at origin.
Let $L_1 \equiv ax + by + c = 0$, $L_2 \equiv Lx + My + N = 0$ and $P \equiv (x_1, y_1)$
As lines L_1 and L_2 are fixed, (a, b, c, L, M, N) are fixed quantities.
Parametric form is likely to be used because distance of P, R and S from a fixed point are involved.
Let θ be the angle made by the variable line ORS with + ve X-axis. Note that θ is a changing quantity and we will have to eliminate it later
Let $OR = r_1$; $OS = r_2$ and $OP = r$
Note that r_1, r_2 r are also changing quantities.
Using parametric form, we have :
$R \equiv (r_1\cos\theta, r_1\sin\theta)$, $S \equiv (r_2\cos\theta, r_2\sin\theta)$
$P \equiv (r\cos\theta, r\sin\theta) \equiv (x_1, y_1)$
As R lies on L_1 , $ar_1\cos\theta + br_1\sin\theta + c = 0$

$$\Rightarrow \quad r_1 = \frac{-c}{a\cos\theta + b\sin\theta}$$

As S lies on L_2 , $Lr_2\cos\theta + Mr_2\sin\theta + N = 0$

$$\Rightarrow \quad r_2 = \frac{-N}{L\cos\theta + M\sin\theta}$$

Substituting in $\frac{m+n}{OP} = \frac{m}{OR} + \frac{n}{OS}$

$$\Rightarrow \quad \frac{m+n}{r} = \frac{m}{r_1} + \frac{n}{r_2}$$

$$\Rightarrow \quad \frac{(m+n)}{r} = -\frac{m(a\cos\theta + b\sin\theta)}{c} - \frac{n(L\cos\theta + M\sin\theta)}{N}$$

Put $\cos\theta = \frac{x_1}{r}$ and $\sin\theta = \frac{y_1}{r}$ to eliminate θ

$$\Rightarrow \quad \frac{m+n}{r} = -\frac{m}{c}\left[\frac{ax_1}{r} + \frac{by_1}{r}\right] - \frac{n}{N}\left[\frac{Lx_1}{r} + \frac{My_1}{r}\right]$$

$$\Rightarrow \quad (m+n) = -\frac{m}{c}(ax_1 + by_1) - \frac{n}{N}(Lx_1 + My_1)$$

$$\Rightarrow \quad (ax_1 + by_1 + c) + \frac{nc}{mN}(Lx_1 + My_1 + N) = 0$$

The above equation is the locus of P which represents a straight line passing through the intersection of L_1 and L_2

Example : 13

Let (h, k) be a fixed point, where h > 0, k > 0. A straight line passing through this point cuts the positive direction of the coordinates axes at the points P and Q. Find the minimum area of the triangle OPQ, O being the origin.

Solution

Equation of any line passing through the fixed point (h, k) and having slope m can be taken as :

$y - k = m(x - h)$ ....(i)

Put y = 0 in (i) to get OP i.e. $X_{intercept} = OP = h - \frac{k}{m}$

Put x = 0 in (i) to get OQ i.e. $Y_{intercept} = OQ = k - mh$

Area of triangle OPQ = $A(m)\ \frac{1}{2}\left(h - \frac{k}{m}\right)(k - mh) = \frac{1}{2}\left(2hk - mh^2 - \frac{k^2}{m}\right)$

$\Rightarrow \quad A(m) = \frac{1}{2}\left(2hk - mh^2 - \frac{k^2}{m}\right)$ ....(ii)

To minimise A(m), Put $A'(m) = 0$

$\Rightarrow \quad A'(m) = \frac{1}{2}\left(-h^2 + \frac{k^2}{m^2}\right) = 0 \quad \Rightarrow \quad m = \pm\frac{k}{h}$

$A''(m) = -\frac{k^2}{m^2} \quad \Rightarrow \quad A''\left(\frac{-k}{h}\right) = \frac{h^3}{k} > 0$

$\Rightarrow$ for m = –k/h, A(m) is minimum.

Put m = –k/h is (ii) to get minimum area.

$\Rightarrow$ Minimum Area of $\Delta OPQ = \frac{1}{2}$ [2hk + kh + hk[= 2hk]

Example : 14

A rectangle PQRS has its side PQ parallel to the line y = mx and vertices P, Q and S lie on the lines y = a, x = b and x = – b, respectively. Find the locus of the vertex R.

Solution

Let coordinates of P be (t, a) and R be (x_1, y_1)

Slope of PQ = m (given)

Slope of PS = – 1/(slope of PQ) = –1/m

Equation of Pθ ≡ y – a = m (x – t) ....(i)

As Q lies on x = b line, put x = b in (i) to get Q.

$\Rightarrow$ Q ≡ [b, a + m (b – t)]

Equation of PS ≡ y – a = – 1/m (x – t) ....(ii)

As S lies on x = – b line, put x = –b in (ii) to get S.

$\Rightarrow$ S ≡ [–b, a + 1/m (b + t)]

Slope of RS = $\dfrac{y_1 - a\frac{-1}{m}(b+t)}{x_1 + b} = m$ ....(iii)

$\Rightarrow$ $b + t = m(y_1 - a) - m^2(x + b)$

Slope of RQ = $\dfrac{y_1 - a - m(b-t)}{x_1 - b} = -\dfrac{1}{m}$

$\Rightarrow \quad \dfrac{m(y_1 - a) + (x_1 - b)}{m^2} = b - t$

Add (iii) and (iv) to eliminate t

$\Rightarrow \quad 2b = m\,(y_1 - a) - m^2\,(x + b) + \dfrac{m(y_1 - a) + (x_1 - b)}{m^2}$

$\Rightarrow \quad$ Locus is : $my + (1 - m^2)\,x - am - b\,(1 + m^2) = 0$

Example : 15

Let ABC be a triangle with AB = AC. If D is the midpoint of BC, E the foot of the perpendicular drawn from D to AC and F the midpoint of DE, prove the AF is perpendicular to BE.

Solution

Let vertex A of the triangle be at origin and AC as x-axis. Let the coordinates of C and B be (4a, 0) and (4b, 4c) respectively.

Then the coordinates of points D, E and F will be (2a + 2b, 2c), (2a + 2b, 0) and (2a + 2b, c) respectively.

Since AB = AC, we will have $(4c)^2 + (4b)^2 = (4a^2)$

$\Rightarrow \quad b^2 + c^2 = a^2$ ....(i)

Now, Slope of BE = $\dfrac{0 - 4c}{(2a + 2b) - 4b} = \dfrac{2c}{b - a}$

Slope of AF = $\dfrac{c - 0}{(2a + 2b) - 0} = \dfrac{c}{2(b + a)}$

Slope of BE × AF = $\dfrac{c^2}{b^2 - a^2} = -1$ [using (i)]

Hence AF ⊥ BE

Example : 16

A line through A (–5, –4) meets the lines x + 3y + 2 = 0, 2x + y + 4 = 0 and x – y – 5 = 0 at the points B, C and D respectively. If $(15/AB)^2 + (10/AC)^2 = (6/AD)^2$, find the equation of the line.

Solution

The parametric form of the line passing through A(–5, –4) is

$x = -5 + r\cos\theta$

$y = -4 + r\sin\theta$ ....(i)

where r is the distance of any other point P(x, y) on this line from A.

Equation (i) meets the line x + 3y + 2 = 0 at B.

Let AB = r_1

$\Rightarrow$ The coordinates of B are $(-5 + r_1\cos\theta,\ -4 + r_1\sin\theta)$

Since B lies on x + 3y + 2 = 0, we get

$(-5 + r_1\cos\theta) + 3\,(-4 + r_1\sin\theta) + 2 = 0$

$\Rightarrow \quad r_1 = \dfrac{15}{\cos\theta + 3\sin\theta}$ ....(ii)

Equation (i) meets the line 2x + y + 4 = 0 at C.

Let Ac = r_2

$\Rightarrow$ The coordinates of C are $(-5 + r_2\cos\theta,\ -4 + r_2\sin\theta)$

Since C lies on 2x + y + 4 = 0, we get

$2(-5 + r_2\cos\theta) + (-4 + r_2\sin\theta) + 4 = 0$

$\Rightarrow \quad r_2 = \dfrac{10}{2\cos\theta + \sin\theta}$ ....(iii)

Similarly, $r_3 = \dfrac{6}{\cos\theta - \sin\theta}$ where r_3 = AD ....(iv)

It is given that : $\left(\dfrac{15}{AB}\right)^2 + \left(\dfrac{10}{AC}\right)^2 = \left(\dfrac{6}{AD}\right)^2$

$\Rightarrow \quad \left(\dfrac{15}{r_1}\right)^2 + \left(\dfrac{10}{r_2}\right)^2 = \left(\dfrac{6}{r_3}\right)^2$

Substituting r_1, r_2 and r_3 from equation (ii), (iii) and (iv), we get

$(\cos\theta + 3\sin\theta)^2 + (2\cos\theta + \sin\theta)^2 = (\cos\theta - \sin\theta)^2$

$\Rightarrow$ $(\cos^2\theta + 9\sin^2\theta + 6\cos\theta\sin\theta) + (4\cos^2\theta + \sin^2\theta + 4\cos\theta\sin\theta)$

$= \cos^2\theta + \sin^2\theta - 2\cos\theta\sin\theta$

$\Rightarrow$ $4\cos^2\theta + 9\sin^2\theta + 12\cos\theta\sin\theta = 0$

$\Rightarrow$ $(2\cos\theta + 3\sin\theta)^2 = 0$

$\Rightarrow$ $2\cos\theta + 3\sin\theta = 0$ $\Rightarrow$ $\tan\theta = -2/3$

$\Rightarrow$ slope of the line $= -2/3$

Hence equation of required line is : $y + 5 = -2/3\,(x + 4)$

$\Rightarrow$ $3y + 2x + 23 = 0$

Example : 17

Using the methods of co-ordinates geometry, show that $\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = -1$, where P, Q, R the points of intersection of a line L with the sides BC, CA, AB of a triangle ABC respectively.

Solution

Let A (x_1, y_1), B (x_2, y_2) and C(x_3, y_3) be the vertices of the ΔABC.

Let the equation of the line L be $ax + by + c = 0$

Let L divide BC at P in the ratio m : 1 i.e. $\frac{BP}{PC} = \frac{m}{1}$

Using section formula, the coordinates of P are $\left(\frac{x_1 + mx_3}{1+m}, \frac{y_2 + my_3}{1+m}\right)$

As P lies on the line L

$$a\left(\frac{x_2 + mx_3}{1+m}\right) + b\left(\frac{y_2 + my_3}{1+m}\right) + c = 0$$

$\Rightarrow$ $m(ax_3 + by_3 + c) + (ax_2 + by_2 + c) = 0$

$\Rightarrow$ $\frac{m}{1} = -\left(\frac{ax_2 + by_2 + c}{ax_3 + by_3 + c}\right)$

$\Rightarrow$ $\frac{BP}{PC} = -\left(\frac{ax_2 + by_2 + c}{ax_3 + by_3 + c}\right)$ ....(i)

Similarly $\frac{CQ}{QA} = \frac{ax_3 + by_3 + c}{ax_1 + by_1 + c}$ ....(ii)

and $\frac{AR}{RB} = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$ ....(iii)

Multiple (i), (ii) and (iii) to get : $\frac{BP}{PC}, \frac{CQ}{QA}, \frac{AR}{RB} = -1$

Example : 18

The vertices of a triangle are A$(x_1, x_1\tan\theta_1)$, B$(x_2, x_2\tan\theta_2)$, and C$(x_3, x_3\tan\theta_3)$. If the circumcentre of ΔABC coincides with the origin and H(x', y') is the orthocentre, show that : $\frac{y'}{x'} = \frac{\sin\theta_1 + \sin\theta_2 + \sin\theta_3}{\cos\theta_1 + \cos\theta_2 + \cos\theta_3}$

Solution

Let circumcentre of the triangle ABC = r

Since origin is the circumcentre of ΔABC, OA = OB = OC = r

Using Distance Formula,

$x_1^2 + x_1^2\tan^2\theta_1 = x_2^2 + x_2^2\tan^2\theta_2 = x_3^2 + x_3^2\tan^2\theta_3$

$\Rightarrow$ $x_1\sec\theta_1 = x_2\cos\theta_2 = x_3 = \sec\theta_3 = r$

$\Rightarrow \quad x_1 = r\cos\theta_1$, $x_2 = \sec\theta_2$, $x_3 = r\cos\theta_3$

Therefore, the coordinates of the vertices of the triangle are :

$A \equiv (r\cos\theta_1, r\sin\theta_1)$

$B \equiv (r\cos\theta_2, r\sin\theta_2)$ and

$C \equiv (r\cos\theta_3, r\sin\theta_3)$

In triangle, we know that the circumcentre (O), centroid (G) and orthocentre (H) are collinear.

Using this result,

Slope of OH = Slope of GO

$$\Rightarrow \quad \frac{y'-0}{x'-0} = \frac{(\text{y cordinate of G})-0}{(\text{x cordinate of G})-0}$$

$$\Rightarrow \quad \frac{y'}{x'} = \frac{\sin\theta_1+\sin\theta_2+\sin\theta_3}{\cos\theta_1+\cos\theta_2+\cos\theta_3} \quad \text{Hence proved}$$

Example : 19

Find the coordinates of the points at unit distance from the lines : $3x - 4y + 1 = 0$, $8x + 6y + 1 = 0$

Solution

Let $L_1 \equiv 3x - 4y + 1 = 0$ and $L_2 \equiv 8x + 6y + 1 = 0$

In diagram, A, B, C and D are four points which lie at a unit distance from the two lines. You can also observe that A, B, C and D lie on angle bisectors of L_1 and L_2 .

Let (h, k) be the coordinates of a point of unit distance from each of the given lines.

$$\Rightarrow \quad \frac{|3h-4k+1|}{\sqrt{3^2+4^2}} \quad \text{and} \quad \frac{|8h+6k+1|}{\sqrt{8^2+6^2}}$$

$\Rightarrow \quad 3h - 4k + 1 = \pm 5$ and $8h + 6k + 1 = \pm 10$

$\Rightarrow \quad 3h - 4k - 4 = 0$ ....(i)

$3h - 4k + 6 = 0$ ....(ii)

$8h + 6k - 9 = 0$ ....(iii)

and $8h + 6k + 11 = 0$ ....(iv)

Solve (i) and (iii) to get : $(h, k) \equiv \left(\frac{6}{5}, \frac{-1}{10}\right)$

Solve (i) and (iv) to get : $(h, k) \equiv \left(\frac{-2}{5}, \frac{-13}{10}\right)$

Solve (ii) and (iii) to get : $(h, k) \equiv \left(0, \frac{3}{2}\right)$

Solve (ii) and (iv) to get : $(h, k) \equiv \left(\frac{-8}{5}, \frac{3}{10}\right)$

Hence the required four points are $\left(\frac{6}{5}, \frac{-1}{10}\right)$, $\left(\frac{-2}{5}, \frac{-13}{10}\right)$, $\left(0, \frac{3}{2}\right)$ and $\left(\frac{-8}{5}, \frac{3}{10}\right)$

Example : 20

Show that the area of the parallelogram formed by the line $3y - 2x = a$; $2y - 3x + a = 0$; $2x - 3y + 3a = 0$ and $3x - 2y = 2a$ is $\left(\frac{2a^2}{5}\right)$

Solution

The equations of four sides of the line are :

$2x - 3y + a = 0$ ....(i)

$-3x + 2y + a = 0$ ....(ii)

$2x - 3y + 3a = 0$ ....(iii)

$-3x + 2y + 2a = 0$ ....(iv)

Area of the parallelogram formed by above sides = $\frac{p_1p_2}{\sin\theta}$ ....(v)

where p_1 = perpendicular distance between parallel sides (i) and (iii),
p_2 = perpendicular distance between parallel sides (ii) and (iv),
θ = angle between adjacent sides (i) and (ii)

Find p_1

p_1 = perpendicular distance between (i) and (iii) = $\frac{|a-3a|}{\sqrt{2^2+(-3)^2}} = \frac{|2a|}{\sqrt{13}}$

Find p_2

p_2 = perpendicular distance between (ii) and (iv) = $\frac{|a-2a|}{\sqrt{2^2+(-3)^2}} = \frac{|a|}{\sqrt{13}}$

Find $\sin\theta$

If θ is the angle between (i) and (ii), then

$$\tan\theta = \frac{m_1-m_2}{1+m_1m_2} = \left|\frac{2/3-3/2}{1+(2/3).(3/2)}\right|$$

$\Rightarrow$ $\tan\theta = 5/12$

$\Rightarrow$ $\sin\theta = 5/13$

On substituting the values of p_1, p_2 and $\sin\theta$ in (v), we get

Area of the parallelogram formed by above sides = $\frac{\frac{|2a|}{\sqrt{13}} - \frac{|a|}{\sqrt{13}}}{5/13}$

$\Rightarrow$ Area of parallelogram = $2a^2$ /5q. units

Example : 21

The line joining the points A(2, 0); B(3, 1) is rotated about A in the anticlockwise direction through an angle of 15º. Find the equation of the line in the new position. If B goes to C in the new position, what will be the co-ordinate of C?

Solution

Slope of AB = $\frac{1-0}{3-2}$ = 1 = tan 45º

$\Rightarrow$ $\angle BAX = 45^\circ$

Now line AB is rotated through an angle of 15º

$\Rightarrow$ $\angle CAX = 60^\circ$ and

AC = AB $\Rightarrow$ AC = $\sqrt{2}$

Equation of line AC in parametric form is :

$x = 2 + r\cos 60^\circ$
$y = 0 + r\sin 60^\circ$ ....(i)

Since AC = $\sqrt{2}$, pur r = $\sqrt{2}$ in (i) to get the coordinates of point C, i.e.

coordinates of C are $\left(\frac{4+\sqrt{2}}{2}, \frac{\sqrt{6}}{2}\right)$

Example : 22

Prove that two of the straight lines represented by the equation $ax^3 + bx^2y + cxy^2 + dy^3 = 0$ will be at right angles, if $a^2 + ac + bd + d^2 = 0$.

Solution

$ax^3 + bx^2y + cxy^2 + dy^3 = 0$ ....(i)

Equation (i) is a homogeneous equation of third degree in x and y

$\Rightarrow$ It represents combined equations of three straight lines passing through origin

Divide (i) by x^3 $\Rightarrow$ $a + b\,(y/x) + c\,(y/x)^2 + d\,(y/x)^3 = 0$

Put $(y/x) = m$

$\Rightarrow$ $a + bm + cm^2 + dm^3 = 0$

$\Rightarrow$ $dm^3 + cm^2 + bm + a = 0$

This is a cubic equation in 'm' with three roots m_1, m_2, m_3 [i.e. slopes of the three lines]

product of roots = $m_1\, m_2\, m_3 = -\,a/d$ ....(ii)

product of roots taken two at a time = $m_1\, m_2 + m_2\, m_3 + m_1\, m_3 = b/d$ ....(iii)

sum of roots = $m_1 + m_2 + m_3 = -c/d$ ....(iv)

If any two lines are perpendicular to each other, then :

$m_1\, m_2 = -\,1$ ....(v)

Solving (ii) and (v), we get

$m_3 = a/d$

On substituting the value of m_3 in (iv), we get

$m_1 + m_2 = -\,(a + c)/d$ ....(vi)

Solve (v) and (iii) and substitute the value of m_3 to get :

$m_3\,(m_1 + m_2) = (b + d)/d$

On substituting the value of $m_1 + m_2$ from (vi) in above equation, we get

$(a/d)\,[-(a + c)/d] = (b + d)/d$

$\Rightarrow$ $-a^2 - ac = bd + d^2$

$\Rightarrow$ $a^2 + ac + bd + d^2 = 0$

Hence proved

Example : 23

The sides of a triangle are, $L_r \equiv x \cos \theta_r + y \sin \theta_r - a_r = 0$, r = 1, 2, 3. Show that the orthocentre of the triangle is given by : $L1 \cos(\theta_2 - \theta_3) = L_2 \cos(\theta_3 - \theta_1) = L_3 \cos(\theta_1 - \theta_2)$.

Solution

Equation of any line through the point of intersection of $L_1 = 0$ and $L_2 = 0$ is

$L_1 + kL_2 = 0$, where k is a parameter.

$\Rightarrow$ $(\cos \theta_1 + k \cos \theta_2)\, x + (\sin \theta_1 + k \sin \theta_2)\, y - (a_1 + k\, a_2) = 0$ ....(i)

Line (i) will be perpendicular to $L_3 \equiv x \cos q_3 + y \sin \theta_3 - a_3 = 0$ if

[slope of (i)] × [slope of L_3] = – 1

$-\,[(\cos \theta_1 + k \cos\theta_2) / (\sin \theta_1 + k \sin \theta_2)]\,.\,[-(\cos \theta_3) / (\sin \theta_3)] = -\,1$

$\Rightarrow$ $k = -[\cos(\theta_3 - \theta_1)] / [\cos(\theta_2 - \theta_3)]$

On substituting the value of k in (i), we get the equation of one altitude as :

$L_1 \cos(\theta_2 - \theta_3) = L_2 \cos(\theta_3 - \theta_1)$ ....(ii)

Similarly, we can obtain the equations of second altitudes as :

$L_2 \cos(\theta_3 - \theta_1) = L_3 \cos(\theta_1 - \theta_2)$ ....(iii)

Solving the equations of altitudes (ii) and (iii), the orthocentre of the triangle is given by,

$L_1 \cos(\theta_2 - \theta_3) = L_2 \cos(\theta_3 - \theta_1) = L_3 \cos(\theta_1 - \theta_2)$.

Trigonometry

Example : 1

Find the maximum and minimum value of :

(i) $\sin\theta + \cos\theta$

(ii) $\sqrt{3}\sin\theta - \cos\theta$

(iii) $5\sin\theta + 12\cos\theta + 7$

Solution

Given expressions are in the form of a $\sin\theta$ + b $\cos\theta$

Express this in terms of one t-ratio by dividing and multiplying by $\sqrt{a^2+b^2}$

(i) $\sin\theta\cos\theta = 1 . \sin\theta + 1 . \cos\theta$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta\right)$$

$$= \sqrt{2}\left(\sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4}\right)$$

$$= \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$$

Now sine of angle must be between – 1 and 1

$$\Rightarrow \quad -1 \le \sin\left(\theta + \frac{\pi}{4}\right) \le 1$$

$$\Rightarrow \quad -\sqrt{2} \le \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) \le \sqrt{2}$$

So maximum value of $\sin\theta + \cos\theta$ is $\sqrt{2}$ and minimum value of $\sin\theta + \cos\theta$ is $-\sqrt{2}$

(ii) $$\sqrt{3}\sin\theta - \cos\theta = 2\left(\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta\right)$$

$$= 2\left(\sin\theta\cos\frac{\pi}{6} - \cos\theta\sin\frac{\pi}{6}\right)$$

$$= 2\sin\left(\theta - \frac{\pi}{6}\right)$$

$$\text{as } -1 \therefore \sin\left(\theta - \frac{\pi}{6}\right) \le 1$$

$$\Rightarrow \quad -2 \le 2\sin\left(\theta - \frac{\pi}{6}\right) \le 2$$

so maximum value is 2 and minimum value is – 2

(iii) Consider $5\sin\theta + 12\cos\theta = 13\,[5/13 \sin\theta + 12/13 \cos\theta]$

construct a triangle with sides, 5, 12, 13. If α is an angle of triangle,

then $\cos\alpha = 5/13$, $\sin\alpha$ 12/13,

$\Rightarrow$ $5\sin\theta + 12\cos\theta = 13\,[\sin\theta\cos\alpha + \cos\theta - \sin\alpha]$

$5\sin\theta + 12\cos\theta + 7 = 13\,[\sin(\theta + \alpha)] + 7$

as $-1 \le \sin(\theta + \alpha) \le 1$

$\Rightarrow$ $-13 \le 13\sin(\theta + \alpha) \le 13$

$-13 + 7 \le 13\sin(\theta + \alpha) + 7 \le 13 + 7$

So maximum value s 20 and minimum value is – 6.

Example : 2

Show that sin π/13 is a root of $8x^3 - 4x^2 + - 4x + 1 = 0$

Solution

Let $\theta = \pi/14$

$\Rightarrow \quad 4\theta = \pi/2 - 3\theta$

$\Rightarrow \quad \sin 4\theta = \sin [\pi/2 - 2\theta] = \cos 3\theta$

$\Rightarrow \quad 2[2 \sin \theta \cos \theta] \cos 2\theta = \cos \theta [4 \cos^2\theta - 3]$

$\Rightarrow \quad 4 \sin \theta [1 - 2 \sin^2\theta] = 4 - 4 \sin^2\theta - 3$

$\Rightarrow \quad 8 \sin^3\theta - 4 \sin^2\theta - 4 \sin \theta + 1 = 0$

$\Rightarrow \quad \sin \theta$ is root of $8x^3 - 4x^2 - 4x + 1 = 0$

Example : 3

If α and β are roots of $a \tan \theta + b \sec \theta = c$, find the value of :

(i) $\tan [\alpha + \beta]$ (ii) $\cos [\alpha + \beta]$

Solution

(i) To find tan (α + β), we need tan α + tan β and tan α tan β, so express the given equation in terms of a quadratic in tan θ where sum of roots is tan α + tan β and product of roots in tan α tan β

Consider $a \tan \theta + b \sec \theta = c$

$\Rightarrow \quad (c - a \tan \theta)^2 = b^2 \sec^2\theta$

$\Rightarrow \quad c^2 + a^2 \tan^2\theta - 2ac \tan \theta = b^2 + b^2 \tan^2\theta$

$\Rightarrow \quad (a^2 - b^2) \tan^2\theta - 2ac \tan \theta + c^2 - b^2 = 0$

$$\tan \alpha + \tan \beta = \text{sum of roots} = \frac{2ac}{a^2 - b^2}$$

$$\tan \alpha \tan \beta = \text{product of roots} = \frac{c^2 - b^2}{a^2 - b^2}$$

$$\Rightarrow \quad \tan (\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta} = \frac{\frac{2ac}{a^2 - b^2}}{1 - \frac{c^2 - b^2}{a^2 - b^2}} = \frac{2ac}{a^2 - c^2}$$

(ii) To find cos (α + β), express given equation as :

1. quadratic in cos θ to find cos α cos β
2. quadratic in sin θ to find sin α sin β

Given equation can be written as :

$a \sin \theta + b = c \cos \theta$

$\Rightarrow \quad a^2 \sin^2\theta + b^2 + 2ab \sin \theta = c^2 \cos^2\theta = c^2 (1 - \sin^2\theta)$

$\Rightarrow \quad (a^2 + c^2) \sin^2\theta + 2ab \sin \theta \sin \theta + b^2 - c^2 = 0$

$$\text{Hence product of roots of } \sin \alpha \sin \beta = \frac{b^2 - c^2}{a^2 + c^2}$$

Given equation can be written as

$a \sin \theta + b = \cos \theta$

$\Rightarrow \quad a^2 \sin^2\theta + b^2 + 2ab \sin \theta = c^2 \cos^2\theta = c^2 (1 - \sin^2\theta)$

$\Rightarrow \quad (a^2 + c^2) \sin^2\theta + 2ab \sin \theta + b^2 - c^2 = 0$

$$\text{Hence product of roots of } \sin \alpha \sin \beta = \frac{b^2 - c^2}{a^2 + c^2}$$

Given equation can be written as

$a \sin \theta + b = c \cos \theta$

$\Rightarrow \quad a^2 \sin^2\theta = (c \cos \theta - b)^2$

$\Rightarrow \quad a^2 (1 - \cos^2\theta) = c^2 \cos^2\theta + b^2 - 2bc \cos \theta$

$\Rightarrow \quad (a^2 + c^2) \cos^2 \theta - 2bc \cos \theta + b^2 - a^2 = 0$

$$\text{so product of roots} = \cos \alpha \cos \beta = \frac{b^2 - a^2}{a^2 + c^2}$$

Now cos (α + β) = cos α cos β – sin α sin β

$$= \frac{b^2 - a^2}{a^2 + c^2} - \frac{b^2 - c^2}{a^2 + c^2} = \frac{c^2 - a^2}{a^2 + c^2}$$

Example : 4

If m tan (θ – 30º) = n tan (θ + 120º), then show that : cos 2θ = $\frac{m+n}{2(m-n)}$

Solution

$$\frac{\tan(\theta + 120^o)}{\tan(\theta - 30^o)} = \frac{m}{n}$$

$$\Rightarrow \quad \frac{\sin(\theta+120^o)\cos(\theta-30^o)}{\cos(\theta+120^o)\sin(\theta-30^o)} = \frac{m}{n}$$

$$\Rightarrow \quad \frac{\sin(\theta+120^o)\cos(\theta-30^o) - \cos(\theta+120^o)\sin(\theta-30^o)}{\sin(\theta+120^o)\cos(\theta-30^o) + \cos(\theta+120^o)\sin(\theta-30^o)} = \frac{m-n}{m+n}$$

$$\Rightarrow \quad \frac{\sin 150^o}{\sin(2\theta+90^o)} = \frac{m-n}{m+n}$$

$$\Rightarrow \quad \frac{\sin[(\theta+120^o)-(\theta-30^o)]}{\sin[(\theta+120^o)+(\theta-30^o)]} = \frac{m-n}{m+n}$$

$$\Rightarrow \quad \frac{1}{2}(m+n) = (m-n)\cos 2\theta$$

$$\Rightarrow \quad \cos 2\theta = \frac{m+n}{2(m-n)}$$

Example : 5

Show that : $\sin^2 B = \sin^2 A + \sin^2 (A - B) - 2 \sin A \cos B \sin (A - B)$

Solution

Starting from RHS :

RHS $= \sin^2 A + \sin^2(A - B) - 2 \sin A \cos B \sin (A - B)$

$= \sin^2 A + \sin^2(A - B) - [\sin (A + B) + \sin (A - B)] \sin (A - B)$

$= \sin^2 A + \sin^2(A - B) - \sin (A + B) \sin (A - B) - \sin^2 (A - B)$

$= \sin^2 A - [\sin^2 A - \sin^2 B]$

$= \sin^2 B$ = LHS

Example : 6

If $\tan \frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{\phi}{2}$, show that : $\cos \phi = \frac{\cos\theta - e}{1 - e\cos\theta}$

Solution

We have to find cos φ in terms of e and cos θ, so try to convert tan θ/2 to cos φ

$$\tan^2 \frac{\theta}{2} = \frac{1-e}{1+e} \tan^2 \frac{\phi}{2}$$

$$\Rightarrow \quad \tan^2 \frac{\phi}{2} = \frac{1+e}{1-e} \tan^2 \frac{\theta}{2} = \frac{1+e}{1-e}\left(\frac{1-\cos\theta}{1+\cos\theta}\right)$$

$$\Rightarrow \quad \frac{\tan^2 \frac{\phi}{2}}{1} = \frac{1+e-\cos\theta - e\cos\theta}{1-e+\cos\theta - e\cos\theta}$$

$$\Rightarrow \qquad \frac{1-\tan^2\phi/2}{1+\tan^2\phi/2} = \frac{(1+e+\cos\theta-e\cos\theta)-(1+e\cos\theta-e\cos\theta)}{(1-e+\cos\theta-e\cos\theta)+(1+e-\cos\theta-e\cos\theta)}$$

$$\Rightarrow \qquad \cos\phi = \frac{-2e+2\cos\theta}{2-2e\cos\theta} = \frac{\cos\theta-e}{1-e\cos\theta}$$

Example : 7

If $\tan\beta = \dfrac{\tan\alpha+\tan\gamma}{1+\tan\alpha\tan\gamma}$, prove that : $\sin 2\beta = \dfrac{\sin 2\alpha+\sin 2\gamma}{1+\sin 2\alpha\sin 2\gamma}$

Solution

We are given $\tan\beta$ in terms of α and γ , so we have to express $\sin 2\beta$ in terms of α, γ . Hence we will start with $\sin 2\beta = (2\tan\beta) / (1+\tan^2\beta)$ and substitute for $\tan\beta$ in RHS. Also, as the final expression does not contain $\tan\alpha$ and $\tan\gamma$, so express $\tan\beta$ in terms of sine and consine.

$$\tan\beta = \frac{\sin\alpha\cos\gamma+\cos\gamma\sin\alpha}{\cos\alpha\cos\gamma+\sin\alpha\sin\gamma} = \frac{\sin(\alpha+\gamma)}{\cos(\alpha-\gamma)}$$

$$\text{Now } \sin\beta = \frac{2\tan\theta}{1+\tan^2\beta}$$

$$\Rightarrow \qquad \sin 2\beta = \frac{2\dfrac{\sin(\alpha+\gamma)}{\cos(\alpha-\gamma)}}{1+\dfrac{\sin^2(\alpha+\gamma)}{\cos^2(\alpha-\gamma)}} \quad \frac{2\sin(\alpha+\gamma)\cos(\alpha-\gamma)}{\cos^2(\alpha-\gamma)+\sin^2(\alpha+\gamma)}$$

$$= \frac{\sin\left[\overline{\alpha+\gamma}+\overline{\alpha-\gamma}\right]+\sin\left[\overline{\alpha+\gamma}-\overline{\alpha-\gamma}\right]}{1+\sin^2(\alpha+\gamma)-\sin^2(\alpha-\gamma)}$$

$$= \frac{\sin 2\alpha+\sin 2\gamma}{1+\sin\left[\overline{\alpha+\gamma}+\overline{\alpha-\gamma}\right]\sin\left[\overline{\alpha+\gamma}+\overline{\alpha-\gamma}\right]}$$

$$\Rightarrow \qquad \sin 2\beta = \frac{\sin 2\alpha+\sin 2\gamma}{1+\sin 2\alpha\sin 2\gamma}$$

Example : 8

If $2\tan\alpha = 3\tan\beta$, then show that : $\tan(\alpha-\beta) = \dfrac{\sin 2\beta}{5-\cos 2\beta}$

Solution

We have to express $\tan(\alpha-\beta)$ in terms of β only. Staring with standard result of $\tan(\alpha-\beta)$ and substituting for $\tan\alpha = 3/2\tan\beta$ in RHS, we have :

$$\Rightarrow \qquad \tan(\alpha-\beta) = \frac{\tan\alpha+\tan\beta}{1+\tan\alpha\tan\beta} = \frac{3/2\tan\beta-\tan\beta}{1+3/2\tan^2\beta}$$

$$\Rightarrow \qquad \tan(\alpha-\beta) = \frac{\tan\beta}{2+3\tan^2\beta} = \frac{\sin\beta\cos\beta}{2\cos^2\beta+3\sin^2\beta} = \frac{2\sin\beta\cos\beta}{4\cos^2\beta+6\sin^2\beta} = \frac{\sin 2\beta}{2(1+\cos 2\beta)+3(1-\cos 2\beta)}$$

$$\Rightarrow \qquad \tan(\alpha-\beta) = \frac{\sin 2\beta}{5-\cos 2\beta}$$

Example : 9

Prove that $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = \cot\alpha$.

Solution

For this problem, use the result $\cos\alpha - \tan\alpha = 2\cot 2\alpha$

Now we can express the above relation as :

$$\tan\alpha = \cot\alpha - 2\cot 2\alpha$$

Replacing α by 2α :

$$\tan 2\alpha = \cot 2\alpha - 2\cot 4\alpha$$

Replacing α by 4α :

$$\tan 4\alpha = \cot 4\alpha - 2\cot 8\alpha$$

Multiplying these equations by 1, 2, 4 respectively and adding them together, we get

$$\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha = \cot\alpha - 8\cot 8\alpha$$

$$\Rightarrow \quad \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = \cot\alpha$$

Example : 10

Show that $\cos\frac{\pi}{33}\cos\frac{2\pi}{33}\cos\frac{4\pi}{33}\cos\frac{8\pi}{33}\cos\frac{16\pi}{33} = \frac{1}{32}$

Solution

If $\theta = \pi/33$, observe that pattern $\cos\theta\cos 2\theta\cos 4\theta\cos 16\theta$

In this pattern $\sin 2\alpha = 2\sin\alpha\cos\alpha$ will be used repeatedly in LHS, so multiply and divide by $2\sin\pi/33$.

$$\text{LHS} = \frac{\left(2\sin\frac{\pi}{33}\cos\frac{\pi}{33}\right)\left(\cos\frac{2\pi}{33}\cos\frac{4\pi}{33}\cos\frac{8\pi}{33}\cos\frac{16\pi}{33}\right)}{2\sin\frac{\pi}{33}}$$

$$= \frac{2\left(\sin\frac{2\pi}{33}\cos\frac{2\pi}{33}\right)\left(\cos\frac{4\pi}{33}\cos\frac{8\pi}{33}\cos\frac{16\pi}{33}\right)}{2.2\sin\frac{\pi}{33}}$$

.... following the same pattern .... we have

$$\text{LHS} = \frac{\sin\frac{32\pi}{33}}{2^5 \, . \sin\frac{\pi}{33}} = \frac{\sin\left(\pi = \frac{\pi}{33}\right)}{32\sin\frac{\pi}{33}} = \frac{1}{32}$$

Example : 11

Show that $\tan 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ = 1/16$

Solution

Note that $(66 + 6)/2 = 36^\circ$ $(66 - 6)/2 = 30^\circ$. Hence $\sin 6^\circ$ and $\sin 66^\circ$ should be combined.

$$\text{LHS} = 1/4\,[2\sin 6^\circ \sin 66^\circ]\,[2\sin 42^\circ \sin 78^\circ]$$

$$= 1/4\,[\cos(6^\circ + 66^\circ) - \cos(6^\circ + 66^\circ)]\,[\cos(42^\circ - 78^\circ) - \cos(42^\circ + 78^\circ)]$$

$$= 1/4\,[\cos 60^\circ - \cos 72^\circ]\,[\cos 36^\circ - \cos 120^\circ]$$

Substituting the values, we get

$$\text{LHS} = \frac{1}{4}\left(\frac{1}{2} - \frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4} + \frac{1}{2}\right) = \frac{1}{4}\left(\frac{2-\sqrt{5}+1}{4}\right)\left(\frac{\sqrt{5}+1+2}{4}\right)$$

$$= \frac{1}{64}\,(3 - \sqrt{5})\,(3 + \sqrt{5}) = \frac{1}{16} \text{ RHS}$$

Example : 12

If $\alpha = 2\pi/7$, show that $\tan\alpha \tan 2\alpha \tan 4\alpha \tan\alpha = -7$

Solution

$$\text{LHS} = \frac{\sin\alpha\sin2\alpha\cos4\alpha + \sin2\alpha\sin4\alpha\cos\alpha + \sin4\alpha\sin\alpha\cos2\alpha}{\cos\alpha\cos2\alpha\cos4\alpha}$$

We will use the formula for cos (A + B + C)

cos (A + B + C) = cos A cos B cos C – sin A sin B cos C – sin B sin C cos A – sin A cos B

$$\Rightarrow \quad \text{LHS} = \frac{\cos\alpha\cos2\alpha\cos4\alpha(\alpha+2\alpha+4\alpha)}{\cos\alpha\cos2\alpha\cos4\alpha}$$

$$= 1 - \frac{\cos7\alpha}{\cos\alpha\cos2\alpha\cos4\alpha} = 1 - \frac{\cos2\pi(2\sin\alpha)}{2\sin\alpha\cos\alpha\cos2\alpha\cos4\alpha}$$

$$= 1 - \frac{4\sin\alpha}{2\sin\alpha\cos2\alpha\cos4\alpha}$$

$$= 1 - \frac{8\sin\alpha}{2\sin4\alpha\cos4\alpha}$$

$$= 1 - \frac{8\sin\alpha}{\sin8\alpha} = 1 - \frac{8\sin\alpha}{\sin(2\pi+\alpha)} = 1 - \frac{8\sin\alpha}{\sin\alpha} = -7$$

Example : 13

Show that $\cos 2\pi/7 + \cos4\pi/7 + \cos 6\pi/7 = -1/2$.

Solution

$$\text{LHS} = \frac{2\sin\frac{\pi}{7}\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}\right)}{2\sin\frac{\pi}{7}}$$

$$= \frac{1}{2\sin\frac{\pi}{7}}\left[\left(\sin\frac{3\pi}{7}-\sin\frac{\pi}{7}\right)+\left(\sin\frac{5\pi}{7}-\sin\frac{3\pi}{7}\right)+\left(\sin\frac{7\pi}{7}-\sin\frac{5\pi}{7}\right)\right]$$

$$= \frac{\sin\pi - \sin\frac{\pi}{7}}{2\sin\frac{\pi}{7}} = -\frac{1}{2}$$

Alternative Method :

We can also use the relation :

$$\cos a + \cos(a+d) + \ldots\ldots\ldots + \left(a+\overline{n-1}\,d\right) = \frac{\sin nd/2}{\sin d/2}\cos\left(\frac{2a+\overline{n-1}\,d}{2}\right)$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin3\left(\frac{2\pi/7}{2}\right)}{\sin\frac{2\pi/7}{2}}\cos\left(\frac{\frac{4\pi}{7}+2\left(\frac{2\pi}{7}\right)}{2}\right) = \frac{\sin3\pi/7}{\sin\pi/7}\cos\left(\frac{4\pi}{7}\right) = \frac{\sin\pi+\sin(-\pi/7)}{2\sin\pi/7} = -\frac{1}{2}$$

Example : 14

Let cos α cos β cos ϕ = cos γ cos θ and sin α = 2 sin $\phi/2$ sin $\theta/2$, then prove that $\tan^3\alpha/2 = \tan^2\beta/2 \tan^2\gamma/2$.

Solution

From the given three equations, we have to eliminate two variables, θ and ϕ

cos α = cos β cos ϕ = cos γ cos θ

$\Rightarrow \quad \cos\phi = \frac{\cos\alpha}{\cos\beta}$; $\cos\theta = \frac{\cos\alpha}{\cos\gamma}$

$\Rightarrow \quad 2\sin^2\frac{\phi}{2} = 1 - \frac{\cos\alpha}{\cos\beta}$; $\quad 2\sin^2\frac{\theta}{2} = 1 - \frac{\cos\alpha}{\cos\gamma}$

substitute these is $\sin\alpha = 2\sin\frac{\phi}{2}\sin\frac{\theta}{2}$

$\Rightarrow \quad \sin\alpha = \sqrt{\left(1-\frac{\cos\alpha}{\cos\beta}\right)\left(1-\frac{\cos\alpha}{\cos\gamma}\right)}$

$\Rightarrow \quad \sin^2\alpha \left(1-\frac{\cos\alpha}{\cos\beta}-\frac{\cos\alpha}{\cos\gamma}+\frac{\cos^2\alpha}{\cos\beta\cos\gamma}\right)$

$\Rightarrow \quad \cos\alpha\left(1+\frac{1}{\cos\beta\cos\gamma}\right) = \frac{\cos\beta+\cos\gamma}{\cos\beta\cos\gamma}$

$\Rightarrow \quad \cos\alpha(\cos\beta\cos\alpha + 1) = \cos\beta + \cos\gamma$

$\Rightarrow \quad \cos\alpha = \frac{\cos\beta\cos\gamma}{1+\cos\beta\cos\gamma}$

Using component and dividendo, we get :

$$\frac{1-\cos\alpha}{1+\cos\alpha} = \frac{1+\cos\beta\cos\gamma-\cos\beta-\cos\gamma}{1+\cos\beta\cos\gamma+\cos\beta+\cos\gamma}$$

$\Rightarrow \quad \tan^2\frac{\alpha}{2} = \frac{(1-\cos\beta)(1-\cos\gamma)}{(1+\cos\beta)(1+\cos\gamma)}$

$\Rightarrow \quad \tan^2\frac{\alpha}{2} = \tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2}$

Example : 15

If $\frac{\sin^4\alpha}{a} + \frac{\cos^4\alpha}{b} = \frac{1}{a+b}$, then show that : $\frac{\sin^8\alpha}{a^3} + \frac{\cos^8\alpha}{b^3} = \frac{1}{(a+b)^3}$

Solution

Express the given equation in quadratic in terms of $\sin^2\alpha$

$$\frac{\sin^4\alpha}{a} + \frac{\cos^4\alpha}{b} = \frac{1}{a+b}$$

$\Rightarrow \quad \frac{\sin^4\alpha}{a} + \frac{(\sin^2\alpha)^2}{b} = \frac{1}{a+b}$

$\Rightarrow \quad (a+b)^2\sin^4\alpha - 2a(a+b)\sin^2\alpha + a^2 = 0$

$\Rightarrow \quad [(a+b)\sin^2\alpha - a]^2 = 0$

$\Rightarrow \quad \sin^2\alpha = \frac{a}{a+b}$

$\Rightarrow \quad \cos^2\alpha = \frac{b}{a+b}$

Now LHS of the equation to be proved is :

$$= \frac{\sin^8 \alpha}{a^3} + \frac{\cos^8 \alpha}{b^3}$$

$$= \frac{a^4}{a^3(a+b)^4} + \frac{b^4}{b^3(a+b)^4}$$

$$= \frac{a+b}{(a+b)^4} = \frac{1}{(a+b)^3} = \text{RHS}$$

Example : 16

If $\frac{\cos^4 x}{\cos^2 y} = \frac{\sin^4 x}{\sin^2 y} = 1$, then prove that : $\frac{\cos^4 y}{\cos^2 x} = \frac{\sin^4 y}{\sin^2 x} = 1$

Solution

Consider $\frac{\cos^4 x}{\cos^2 y} = \frac{\sin^4 x}{\sin^2 y} = 1$

$$\Rightarrow \quad \left(\frac{\cos^4 x}{\cos^2 y} - \cos^2 x\right) + \left(\frac{\sin^4 x}{\sin^2 y} - \sin^2 x\right) = 0$$

$$\Rightarrow \quad \frac{\cos^2 x}{\cos^2 y}(\cos^2 x - \cos^2 y) + \frac{\sin^2 x}{\sin^2 y}(\sin^2 x - \sin^2 y) = 0$$

$$\Rightarrow \quad \frac{\cos^2 x}{\cos^2 y}(\cos^2 x - \cos^2 y) + \frac{\sin^2 x}{\sin^2 y}(\cos^2 x - \cos^2 y) = 0$$

$$\Rightarrow \quad (\cos^2 x - \cos^2 y)\left[\frac{\cos^2 x}{\cos^2 y} - \frac{\sin^2 x}{\sin^2 y}\right] = 0$$

$\Rightarrow \quad \cos^2 x = \cos^2 y$ or $\tan^2 x = \tan^2 y$ ....(i)

$$= \quad \frac{\cos^4 y}{\cos^2 x} + \frac{\sin^4 y}{\sin^2 x}$$

$$= \quad \frac{\cos^4 x}{\cos^2 x} + \frac{\sin^4 x}{\sin^2 x} \qquad \text{(using i)}$$

$= \quad \cos^2 x + \sin^2 x = 1 = \text{RHS}$

Example : 17

Show that $1 + \sin^2\alpha + \sin^2\beta > \sin \alpha + \sin \beta + \sin \alpha \sin \beta$

Solution

Consider the expression :

$a^2 + b^2 + c^2 - ab - bc - ca$

$1/2 [(a - b)^2 + (b - c)^2 + (c - a)^2]$

which is positive

$\Rightarrow \quad (a^2 + b^2 + c^2 - ab - bc - ca) > 0$ If a, b, c are unequal

Taking $a = 1$, $b = \sin \alpha$ $c = \sin \beta$, we get

$1 + \sin^2\alpha + \sin^2\beta - \sin \alpha - \sin \alpha \sin \beta - \sin \beta > 0$

$\Rightarrow \quad 1 + \sin^2\alpha + \sin^2\beta > \sin \alpha + \sin \beta + \sin \alpha \sin \beta$

Example : 18

If $\dfrac{ax}{\cos\theta} + \dfrac{by}{\sin\theta} = a^2 - b^2$ and $\dfrac{ax\sin\theta}{\cos^2\theta} - \dfrac{by\cos\theta}{\sin^2\theta} = 0$, then show that $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$.

Solution

From $\dfrac{ax\sin\theta}{\cos^2\theta} - \dfrac{by\cos\theta}{\sin^2\theta} = 0 \quad \Rightarrow \quad \tan\theta = \left(\dfrac{by}{ax}\right)^{1/3}$

Substituting this value of tan θ in the other given condition.

We have : $ax \sec\theta + by \operatorname{cosec}\theta = a^2 - b^2$

$\Rightarrow \quad ax\sqrt{1+\tan^2\theta} + by\sqrt{1+\cot^2\theta} = a^2 - b^2$

$\Rightarrow \quad ax\sqrt{1+\left(\dfrac{by}{ax}\right)^{2/3}} + by\sqrt{1+\left(\dfrac{ax}{by}\right)^{2/3}} = a^2 - b^2$

$\Rightarrow \quad \dfrac{ax}{(ax)^{1/3}}\sqrt{(ax)^{2/3}+(by)^{2/3}} + \dfrac{by}{(by)^{1/3}}\sqrt{(ax)^{2/3}+(by)^{2/3}} = a^2 - b^2$

$\Rightarrow \quad [(ax)^{2/3} + (by)^{2/3}]\sqrt{(ax)^{2/3}+(by)^{2/3}} = a^2 - b^2$

$\Rightarrow \quad ((ax)^{2/3} + (by)^{2/3})^{3/2} = a^2 - b^2$

$\Rightarrow \quad (ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$

Example : 19

If a, b, c are unequal. Eliminate θ from : $a\cos\theta + b\sin\theta = c$ and $a\cos^2\theta + 2a\sin\theta\cos\theta + b\sin^2\theta = c$.

Solution

Consider $a\cos\theta + b\sin\theta = c$

$\Rightarrow \quad a^2\cos\theta + b^2\sin^2\theta + 2ab\sin\theta\cos\theta = c^2$

$\Rightarrow \quad (a^2 - c^2)\cos^2\theta + 2ab\sin\theta\cos\theta + (b^2 - c^2)\sin^2\theta = 0$ ....(i)

Now consider $a\cos^2\theta + 2ab\sin\theta\cos\theta + (b^2 - c^2)\sin^2\theta = c$

$\Rightarrow \quad (a - c)\cos^2\theta + 2a\sin\theta\cos\theta + (b - c)\sin^2\theta = 0$ ....(ii)

Use cross-multiplication method on (i) and (ii).

$(a^2 - c^2)\cos^2\theta + 2ab\sin\theta\cos\theta + (b^2 - c^2)\sin^2\theta = 0$

$(a - c)\cos^2\theta + 2a\sin\theta\cos\theta + (b - c)\sin^2\theta = 0$

$$\Rightarrow \quad \frac{\cos^2\theta}{2ab(b-c)-2a(b^2-c^2)} = \frac{-\sin\theta\cos\theta}{(b-c)(a^2-c^2)-(a-c)(b^2-c^2)} = \frac{\sin^2\theta}{2a(a^2-c^2)-2ab(a-c)}$$

$$\Rightarrow \quad \frac{\cos^2\theta}{2ab(b-c)(-c)} = \frac{-\sin\theta\cos\theta}{(b-c)(a-c)(a-b)} = \frac{\sin^2\theta}{2a(a-c)(a+c-b)}$$

$\Rightarrow \quad (a - b)^2 (b - c)^2 (a - c)^2 = 4a^2c (b - c) (c - a) (a + c - b)$

$\Rightarrow \quad (a - b)^2 (b - c) (c - a) = 4a^2c (a + c - b)$

Example : 20

If $m^2 + m^2 + 2m = m'\cos\theta = 1$, $n^2 + n'^2 + 2nn'\cos\theta = 1$ and $mn + m'n' + (mm' + m'n)\cos\theta = 0$, then prove that : $m^2 + n^2 = \operatorname{cosec}^2\theta$

Solution

Consider the first given condition :

$m^2 + m'^2 + 2mm'\cos\theta = 1$

$\Rightarrow \quad m^2(\sin^2\theta + \cos^2\theta) + m'^2 + 2mm'\cos\theta = 1$

$\Rightarrow \quad m^2\cos^2\theta + m'^2 + 2mm'\cos\theta = 1 - m^2\sin^2\theta$

$\Rightarrow \quad (m\cos\theta + m')^2 = 1 - m^2\sin^2\theta$ ....(i)

Similarly using the second given condition, we can get

$(n\cos\theta + n')^2 = 1 - n^2\sin^2\theta$ ....(ii)

By multiplying (i) and (ii) we can prove the required relation.

$(m \cos\theta + m')^2 (n \cos\theta + n')^2 = (1 - m^2 \sin^2\theta)(1 - n^2 \sin^2\theta)$

$\Rightarrow \quad [mn \cos^2\theta + m'n' + (m'n + mn')\cos\theta]^2 = 1 - m^2 \sin^2\theta - n^2 \sin^2\theta + m^2 n^2 \sin^4\theta$

Using the third given condition in LHS, we get :

$[mn \cos^2\theta - mn]^2 = 1 - m^2 \sin^2\theta - n^2 \sin^2\theta + m^2 n^2 \sin^4\theta$

$m^2n^2 \sin^4\theta = 1 - \sin^2\theta (m^2 + n^2) + m^2n^2 \sin^4\theta$

$\Rightarrow \quad m^2 + n^2 = \text{cosec}^2\theta$

Example : 21

If $\tan\theta = m/n$ and $\theta = 3\phi$ $(0 < \theta < \pi/2)$ show that : $\dfrac{m}{\sin\phi} - \dfrac{n}{\cos\phi} = 2\sqrt{m^2 + n^2}$

Solution

$\tan\theta = m/n$

$\Rightarrow \quad \sin\theta = \dfrac{m}{\sqrt{m^2 + n^2}}$ and $\cos\theta = \dfrac{n}{\sqrt{m^2 + n^2}}$

LHS of given equation $= \dfrac{m}{\sin\phi} - \dfrac{n}{\cos\phi} = \sqrt{m^2 + n^2}\left(\dfrac{\sin\theta}{\sin\phi} - \dfrac{\cos\theta}{\cos\phi}\right)$

$= \sqrt{m^2 + n^2}\left(\dfrac{\sin(\theta - \phi)}{\sin\phi\cos\phi}\right) = 2\sqrt{m^2 + n^2}\ \dfrac{\sin(3\theta - \phi)}{2\sin\phi\cos\phi}$

$= 2\sqrt{m^2 + n^2}\ \dfrac{\sin 2\phi}{2\sin\phi\cos\phi} = 2\sqrt{m^2 + n^2} = \text{RHS}$

Example : 22

If $A + B + C = \pi$, then show that

(i) $\sin^2A + \sin^2B - \sin^2C = 2 \sin A \sin B \cos C$

(ii) $\cos^2A/2 + \cos^2 B/2 + \cos^2C/2 = 2 + 2 \sin A/2 \sin B/2 \sin C/2$

(iii) $\sin^2A + \sin^2B + \sin^2C = 2 + 2 \cos A \cos B \cos C$

Solution

(i) Starting from LHS

$= \sin^2A + \sin^2B - \sin^2C$

$= \sin^2A + \sin(B + C)\sin(B - C)$

$= \sin^2A + \sin(\pi - A)\sin(B - C)$

$= \sin A[\sin A + \sin(B - C)$

$= \sin A[\sin(\pi - (B + C) + \sin(B - C)]$

$= \sin A[\sin(B + C) + \sin(B - C)]$

$= \sin A[2 \sin B \cos C] = 2 \sin A \sin B \cos C = \text{RHS}$

(ii) LHS $= \cos^2A/2 + (1 - \sin^2B/2) + \cos^2C/2$

$= 1 + (\cos^2A/2 - \sin^2B/2) + \cos^2C/2$

$= 1 + \cos(A + B)/2 \cos(A - B)/2 + \cos^2C/2$

$= 1 + \sin C/2 \cos(A - B)/2 + 1 - \sin^2C/2$

$= 2 + \sin C/2 [\cos(A - B)/2 - \sin C/2]$

$= 2 + \sin C/2 [\cos(A - B)/2 = \cos(A + B)/2$

$= 2 + 2 \sin C/2 \sin A/2 \sin B/2 = \text{RHS}$

(iii) LHS $= \sin^2A + \sin^2B + \sin^2C$

$= 1 - (\cos^2A - \sin^2B) + \sin^2C$

$= 1 - \cos(A + B)\cos(A - B) + \sin^2C$

$= 1 + \cos C \cos(A - B) + 1 - \cos^2C$

$= 2 + \cos C[\cos(A - B) - \cos C]$

$= 2 + \cos C[\cos(A - B) + \cos(A + B)]$

$= 2 + 2 \cos C \cos A \cos B = \text{RHS}$

Example : 23

If A + B + C = π , Show that cos A/2 + cos B/2 + cos C/2 = 4 cos (π – A)/4 cos (π – B)/4 cos (π – C)/4.

Solution

LHS $= \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = 2\cos\frac{A+B}{4}\cos\frac{A-B}{4} + \cos\frac{C}{2}$

$= 2\cos\frac{\pi-C}{4}\cos\frac{A-B}{4} + \sin\frac{\pi-C}{2}$

$= 2\cos\frac{\pi-C}{4}\cos\frac{A-B}{4} + 2\sin\frac{\pi-C}{4}\cos\frac{\pi-C}{4}$

$= 2\cos\frac{\pi-C}{4}\left[\cos\frac{A-B}{4} + \sin\frac{\pi-C}{4}\right]$

$= 2\cos\frac{\pi-C}{4}\left[\cos\frac{A-B}{4} + \cos\left(\frac{\pi}{2} - \frac{\pi-C}{4}\right)\right]$

$= 2\cos\frac{\pi-C}{4}\left[2\cos\frac{\pi+A+C-B}{8}\cos\frac{A-B-\pi-C}{8}\right]$

$= 4\cos\frac{\pi-C}{4}\left[\cos\frac{A-C}{4}\cos\frac{B-C}{4}\right]$

$= 4\cos\frac{\pi-C}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-A}{4}$ = RHS

Example : 24

If x + y + z = xyz, then show that $\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{8xyz}{(1-x^2)(1-y^2)(1-z^2)}$

Solution

Let x = tan A, y tan B, z = tan C

$\Rightarrow$ $\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A} = \frac{x+y+z-xyz}{1-xy-yz-zx}$

$\Rightarrow$ A + B + C = nπ = (n ∈ I) (tan θ = 0 ⇒ θ = nπ)

$\Rightarrow$ 2A + 2B + 2C = 2nπ

$\Rightarrow$ tan (2A + 2B + 2C) = tan 2nπ = 0

$\Rightarrow$ $\frac{\tan 2A + \tan 2B + \tan 2C - \tan 2A \tan 2B \tan 2C}{1 - \tan 2A \tan 2B - \tan 2B \tan 2C - \tan 2C \tan 2A} = 0$

$\Rightarrow$ tan 2A + tan 2B + tan 2C = tan 2A tan 2B tan 2C

$\Rightarrow$ $\frac{2\tan A}{1-\tan^2 A} + \frac{2\tan B}{1-\tan^2 B} + \frac{2\tan C}{1-\tan^2 C} = \frac{2\tan A}{1-\tan^2 A} \cdot \frac{2\tan B}{1-\tan^2 B} \cdot \frac{2\tan C}{1-\tan^2 C}$

$\Rightarrow$ $\frac{2x}{1-x^2} + \quad + \frac{2z}{1-z^2} =$

Example : 25

If xy + yz + zx = 1, the prove that $\frac{x}{1+x^2}$ +   + $\frac{z}{1+z^2}$ =

Solution

Let = tan A/2, y = B/2, z = tan C/2

$\Rightarrow$ tan A/2 tan B/2 + tan B/2 tan C/2 + tan C/2 tan A/2 = 1

$\Rightarrow$ $\tan\left(\frac{A}{2}+\frac{B}{2}+\frac{C}{2}\right)$ is undefined

$\Rightarrow$ $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$ $\Rightarrow$ $A + B + C = \pi$

Using the relation : $\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha+\beta+\gamma) = 4\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta+\gamma}{2}\right)\sin\left(\frac{\gamma+\alpha}{2}\right)$

Substitute α + A , β = B γ = C

$\Rightarrow$ $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$$\Rightarrow \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}} + \frac{2\tan\frac{B}{2}}{1+\tan^2\frac{B}{2}} + \frac{2\tan\frac{C}{2}}{1+\tan^2\frac{C}{2}} = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$\Rightarrow \frac{2x}{1+x^2} + \quad + \frac{2z}{1+z^2} = 4 \quad \cdot \frac{1}{\sqrt{1+y^2}} \cdot \frac{1}{\sqrt{1+z^2}}$$

$$\Rightarrow \frac{x}{1+x^2} + \quad + \frac{z}{1+z^2} = \frac{2}{\sqrt{(1+x^2)(1+y^2)(1+z^2)}}$$

Example : 26

If A + B + C = π , the show that :

sin 3A $\cos^3$ (B – C) + sin 3B $\cos^3$ (C – A) + sin 3C $\cos^3$ (A – B) sin 3A sin 3B sin 3C

Solution

LHS = sin 3A cos3(B – C) + sin 3B $\cos^3$ (C – A) + sin 3C $\cos^3$ (A – B)

$$= \sum \sin 3A \cos^3(B-C)$$

$$= \frac{1}{4}\sum \sin 3A[3\cos(B-C) + \cos(3B-3C)]$$

$$= \frac{3}{4}\sum \sin 3A\cos(B-C) + \frac{1}{4}\sum \sin 3A\cos(3B-3C)$$

Substitute 3A = 3π – (3B + 3C)

$$= \frac{3}{4}\sum \sin(3B+3C)\cos(B-C) + \frac{1}{4}\sum \sin(3B+3C)\cos(3B-3C)$$

$$= \frac{3}{8}\sum [\sin(4B+2C) + \sin(2B+4C)] + \frac{1}{8}\sum [\sin 6B + \sin 6C]$$

$$= \frac{3}{8}(0) + \frac{2}{8}(4\sin 3C\sin 3B\sin 3A) = \sin 3A\sin 3B\sin 3C$$

Example : 27

Find the values of x lying between 0 and 2 and satisfying the equation sin x + sin 3x = 0.

Solution

The given equation is sin x + sin 3x = 0

$2\sin\frac{x+3x}{2}\cos\frac{x-3x}{2} = 0$

$\Rightarrow$ 2 sin 2x cos x = 0

Hence sin 2x = 0 or cos x = 0

$\Rightarrow$ $2x = n\pi$, $n \in I$ or $x = (2n+1)\pi/2$, $n \in I$

$\Rightarrow$ $x = n\pi/2$, $n \in I$ or $x = (2n+1)\pi/2$, $n \in I$

(i) n = 0 $\Rightarrow$ $x = 0, \pi/2$

(ii) n = 1 $\Rightarrow$ $x = \pi/2, 3\pi/2$

(iii) n = 2 $\Rightarrow$ $x = \pi, 5\pi/2$

(iv) n = 3 $\Rightarrow$ $x = 3\pi/2, 7\pi/2$

Hence for $0 < x < 2\pi$, the solution is :

$x = \pi/2, \pi, 3\pi/2$

Example : 28

Find the values of θ satisfying sin θ = sin α

Solution

$\sin\theta = \sin\alpha$

$\Rightarrow$ $\sin\theta - \sin\alpha = 0$

$\Rightarrow$ $2\cos\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0$

$\Rightarrow$ $\cos\frac{\theta+\alpha}{2} = 0$ or $\sin\frac{\theta-\alpha}{2} = 0$

$\Rightarrow$ $\frac{\theta+\alpha}{2} = (2\ell+1)\frac{\pi}{2}$ or $\frac{\theta-\alpha}{2} = n\pi$ (where ℓ, n are integers)

$\theta = (2\ell+1)\pi - \alpha$ or $\theta = 2n\pi + \alpha$

θ = (odd no.) π – α or θ = (even no.) π + α

$\theta = (\text{integer})\pi + (-1)^{\text{integer}}\alpha$

$\theta = n\pi + (-1)^n\alpha$; $n \in I$

Example : 29

Find the values of θ satisfying cos θ = cos α in the interval $0 \le \theta \le \pi$.

Solution

$\cos\theta = \cos\alpha$

$\Rightarrow$ $\cos\theta - \cos\alpha = 0$

$\Rightarrow$ $-2\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0$

$\Rightarrow$ $\sin\frac{\theta+\alpha}{2} = n\pi$ or $\frac{\theta-\alpha}{2} = n\pi$

$\Rightarrow$ $\theta = 2n\pi - \alpha$ or $\theta = 2n\pi + \alpha$

combining the two values :

$\theta = 2n\pi \pm \alpha$; $n \in I$

Example : 30

Find the values of θ satisfying $\tan\theta = \tan\alpha$

Solution

$\tan\theta = \tan\alpha$

$\Rightarrow \quad \dfrac{\sin\theta}{\cos\theta} = \dfrac{\sin\alpha}{\cos\alpha}$

$\Rightarrow \quad \sin\theta\cos\alpha - \cos\theta\sin\alpha = 0$

$\Rightarrow \quad \sin(\theta - \alpha) = 0$

$\Rightarrow \quad \theta - \alpha = n\pi$, $n \in I$

$\Rightarrow \quad \theta = n\pi + \alpha$, $n \in I$

Note : The following results should be committed to memory before proceeding further.

(i) $\sin\theta = \sin\alpha \quad \Rightarrow \quad \theta = n\pi + (-1)^n\alpha \;\; n \in I$

(ii) $\cos\theta = \cos\alpha \quad \Rightarrow \quad \theta = 2n\pi \pm \alpha$, $n \in I$

(iii) $\tan\theta = \tan\alpha \quad \Rightarrow \quad \theta = n\pi + \alpha$, $n \in I$

Every trigonometric equation should be manipulated so that it reduces to any of the above results.

Example : 31

Solve the equation $\cos x + \cos 2x + \cos 4x = 0$, where $0 \le x \le \pi$

Solution

$\cos x + (\cos 2x + \cos 4x) = 0$

$\Rightarrow \quad \cos x + 2\cos 3x \cos x = 0$

$\Rightarrow \quad \cos x (1 + 2\cos 3x) = 0$

$\Rightarrow \quad \cos x = 0 \quad$ or $\quad 1 + 2\cos 3x = 0$

$\Rightarrow \quad \cos x = 0 \quad$ or $\quad \cos 3x = -1/2 = \cos 2\pi/3$

$\Rightarrow \quad x = (2n + 1)\pi/2 \quad$ or $\quad 3x = 2n\pi \pm 2\pi/3$

$\Rightarrow \quad x = (2n + 1)\pi/2$, $n \in I \quad$ or $\quad x = 2n\pi/3 \pm 2\pi/9$, $n \in I$

This is the general solution of the equation. To get particular solution satisfying $0 \le x \le p$, we will substitute integral values of n.

(i) $n = 0 \quad \Rightarrow \quad x = \pi/2, \pm 2\pi/9$

(ii) $n = 1 \quad \Rightarrow \quad x = 3\pi/2, 8\pi/9, 4\pi/9$

(iii) $n = 2 \quad \Rightarrow \quad x = 5\pi/2, 14\pi/9, 10\pi/9$ (greater than π)

(iv) $n = -1 \quad \Rightarrow \quad x = -\pi/2, -2\pi/3 \pm 2\pi/9$ (less than 0)

Hence the values for $0 \le x \le p$ are

$x = \pi/2, 2\pi/9 , 4\pi/9, 8\pi/9$

Example : 32

Solve the equation $\sin x = \cos 4x$ for $0 \le x \le p$

Solution

$\sin x = \cos 4x$

$\Rightarrow \quad \cos 4x = \cos(\pi/2 - x)$

$\Rightarrow \quad 4x = 2n\pi \pm (\pi/2 - x)$

$\Rightarrow \quad 4x = 2n\pi + \pi/2 - x \quad$ or $\quad 4x = 2n\pi - \pi/2 + x$

$\Rightarrow \quad x = 2n\pi/5 = \pi/10 \quad$ or $\quad x = 2n\pi/3 - \pi/6$

(i) $n = 0 \quad \Rightarrow \quad x = \pi/10, -\pi/6$

(ii) $n = 1 \quad \Rightarrow \quad x = \pi/2$

(iii) $n = 2 \quad \Rightarrow \quad x = 9\pi/10, 7\pi/6$

(iv) $n = 3 \quad \Rightarrow \quad x = 13\pi/10, 11\pi/6$ (greater than π)

(v) $n = -1 \quad \Rightarrow \quad x = -3\pi/10, -5\pi/6$ (less than 0)

Hence the required solution for $0 \le x \le \pi$ is : $x = \pi/10, \pi/2, 9\pi/10$

Example : 33

Solve the equation $\sqrt{3}\sin x + \cos x = 1$ in the interval $0 \le x \le 2p$.

Solution

For the equation of the type $a\sin\theta + b\cos\theta = c$,

write $a\sin\theta + b\cos\theta$ as $\sqrt{a^2 + b^2}\,\cos(\theta - \alpha)$

$\sqrt{3}\sin x + \cos x = 1$

⇒ $2(\sqrt{3}/2 \sin x + 1/2 \cos x) = 1$
⇒ $2(\cos \pi/3 \cos x + \sin \pi/3 \sin x) = 1$
⇒ $2 \cos (x - \pi/3) = 1$
⇒ $\cos (x - \pi/3) = \cos \pi/3$
⇒ $x - \pi/3 = 2n\pi \pm \pi/3$
⇒ $x = 2n\pi \pm \pi/3 + \pi/3$
(i) $n = 0$ ⇒ $x = 0, 2\pi/3$
(ii) $n = 1$ ⇒ $x = 2\pi + 2\pi/3, 2\pi$
(iii) $n = 2$ ⇒ $x = 4\pi + 2\pi/3, 3\pi$ (greater than 2π)
(iv) $n = -1$ ⇒ $x = -4\pi/3, -2\pi$ (less than 0)
Hence the required values of x are $0, 2\pi/3, 2\pi$

Example : 34

Solve $\tan \theta + \tan 2\theta + \tan 3\theta = 0$ for general values of θ .

Solution

Using tan (A + B), $\tan \theta + \tan 2\theta = \tan 3\theta (1 - \tan \theta \tan 2\theta)$
Hence the equation can be written as :
$\tan 3\theta (1 - \tan \theta \tan 2\theta) + \tan 3\theta = 0$
$\tan 3\theta (2 - \tan \theta \tan 2\theta) = 0$
⇒ $\tan 3\theta = 0$ or $\tan \theta \tan 2\theta = 2$
⇒ $3\theta = n\pi$ or $2 \tan^2\theta = 2 (1 - \tan^2\theta)$
⇒ $\theta = n\pi/3, n \in I$ or $\tan \theta = \pm 1/\sqrt{2}$
⇒ $\theta = n\pi/3, n \in I$ or $\theta = n\pi \pm \tan^{-1} 1/\sqrt{2}$

Example : 35

Solve the equation $\sin x + \cos x = \sin 2x - 1$

Solution

Let $t = \sin x + \cos x$
⇒ $t^2 = 1 + 2 \sin x \cos x$
⇒ $\sin 2x = t^2 - 1$
Hence the given equation is :
$t = (t^2 - 1) - 1$
⇒ $t^2 - t - 2 = 0$
Solving the equation, $(t - 2) (t + 1) = 0$
⇒ $t = 2$ or $t = -1$
⇒ $\sin x + \cos x = 2$ or $\sin x + \cos x = -1$
⇒ $\sqrt{2} \cos (x - \pi/4) = 2$ or $\sqrt{2} \cos (x - \pi/4) = -1$
⇒ $\cos (x - \pi/4) = \sqrt{2}$ or $\cos (x - \pi/4) = -1/\sqrt{2}$
As $-1 \le \cos \theta \le 1$, $\cos (x - \pi/4) = \sqrt{2}$ is impossible.
⇒ $\cos (x - \pi/4) = -1/\sqrt{2}$ is the only possibility.
⇒ $\cos (x - \pi/4) = \cos (\pi - \pi/4)$
⇒ $x - \pi/4 = 2n\pi \pm 3\pi/4$
⇒ $x = 2n\pi \pm 3\pi/4 + \pi/4$ is the general solution

Example : 36

Solve $\sin^4x + \cos^4x = 7/2 \sin x \cos x$

Solution

$\sin^4x + \cos^4x = 7/2 \sin x \cos x$
⇒ $(\sin^2x + \cos^2x)^2 - 2 \sin^2x \cos^2x = 7/2 \sin x \cos x$
let $t = 2 \sin x \cos x = \sin 2x$

⇒ $$1 - \frac{2t^2}{4} = \frac{7t}{4}$$

⇒ $2t^2 + 7t - 4 = 0$
⇒ $(2t - 1) (t + 4) = 0$
⇒ $t = 1/2$ or $t = -4$
⇒ $\sin 2x = 1/2$ or $\sin 2x = -4$ (impossible)

$\Rightarrow$ $\sin 2x = \sin \pi/6$

$\Rightarrow$ $2x = n\pi + (-1)^n \pi/6, n \in I$

$\Rightarrow$ $x = n\pi/2 + (-1)^n \pi/12$ is the general solution.

Example : 37

Put $\cos 2x = 2\cos^2 x - 1$

$\Rightarrow$ $3 - 2\cos x - 4\sin x - (2\cos^2 x - 1) + \sin 2x = 0$

$\Rightarrow$ $(4 - 4\sin x) - 2\cos^2 x - 2\cos x + \sin 2x = 0$

$\Rightarrow$ $4(1 - \sin x) - 2(1 - \sin^2 x) - 2\cos x (1 - \sin x) = 0$

$\Rightarrow$ $(1 - \sin x)(2 - 2\sin x - 2\cos x) = 0$

$\Rightarrow$ $\sin x = 1$ or $\sin x + \cos x = 1$

$\Rightarrow$ $\sin x = \sin \pi/2$ or $\sqrt{2}\cos(x - \pi/4) = 1$

$\Rightarrow$ $x = n\pi + (-1)^n \pi/2$ or $x - \pi/4 = 2n\pi \pm \pi/4$

$\Rightarrow$ $x = n\pi + (-1)^n \pi/2$ or $x = 2n\pi \pm \pi/4 + \pi/4$

$\Rightarrow$ $x = n\pi + (-1)^n \pi/2$ or $x = 2n\pi, 2n\pi + \pi/2$

Combining the two, we get $x = 2n\pi, 2n\pi + \pi/2$

Example : 38

Solve the inequality $\sin x + \cos 2x > 1$ if $0 \le x \le \pi/2$

Solution

Let $\sin x = t$

$\Rightarrow$ $\cos 2x = 1 - 2t^2$

$\Rightarrow$ the inequality is : $t + 1 - 2t^2 > 1$

$\Rightarrow$ $t - 2t^2 > 0$

$\Rightarrow$ $2t^2 - t < 0$

$\Rightarrow$ $t(2t - 1) < 0$

$\Rightarrow$ $(t - 0)(t - 1/2) < 0$

$\Rightarrow$ $0 < t < 1/2$

$\Rightarrow$ $0 < \sin x < 1/2$

In $0 \le x \le \pi/2$, this means that $0 < x < \pi/6$ is the solution

Example : 39

Find the principal and general solution of the equation : $\sqrt{3}/2 \sin x - \cos x = \cos^2 x$

Solution

$\sqrt{3}/2 \sin x - \cos x = \cos^2 x$

squaring, $3(1 - \cos^2 x) - 4\cos^2 x (1 + \cos x)^2 = 0$

$\Rightarrow$ $(1 + \cos x)[3 - 3\cos x - 4\cos^2 x (1 + \cos x)] = 0$

$\Rightarrow$ $(1 + \cos x)[4\cos^3 x + 4\cos^2 x + 3\cos x - 3] = 0$

$(1 + \cos x)[2\cos x - 1][2\cos^2 x + 3\cos x + 3] = 0$

The quadratic factor has no real roots

$\Rightarrow$ $\cos x = -1$ or $\cos x = 1/2$

$\Rightarrow$ $x = (2n - 1)\pi$ or $x = 2n\pi \pm \pi/3$

As we have squared the original equation, we must check whether these values satisfy the given equation. On checking, it is found that both solutions are accepted.

$\Rightarrow$ $x = (2n - 1)\pi, 2n\pi \pm \pi/3$ where $n \in I$

Example : 40

Solve for x ; $\sec 4x - \sec 2x = 2$; $-\pi \le x \le \pi$

Solution

$\sec 4x - \sec 2x = 2$

$\Rightarrow$ $$\frac{1}{\cos 4x} - \frac{1}{\cos 2x} = 2$$

$\Rightarrow$ $\cos 2x - \cos 4x = 2\cos 2x \cos 4x$

$\Rightarrow$ $\cos 2x - \cos 4x = \cos 6x + \cos 2x$

$\Rightarrow$ $\cos 6x + \cos 4x = 0$

$\Rightarrow$ $2\cos 5x \cos x = 0$

$\Rightarrow$ $\cos 5x = 0$ or $\cos x = 0$

$\Rightarrow$ $5x = n\pi + \pi/2$ or $x = n\pi + \pi/2$

$\Rightarrow$ $x = n\pi/5 + \pi/10$ or $x = n\pi + \pi/2$

Consider : $x = n\pi/5 + \pi/10$:

$n = 0 \Rightarrow x = \pi/10$

$n = \pm 1 \Rightarrow x = 3\pi/10, -\pi/10$

$n = \pm 2 \Rightarrow x = \pi/2, -3\pi/10$

$n = \pm 3 \Rightarrow x = 7\pi/10, -\pi/2$

$n = \pm 4 \Rightarrow x = 9\pi/10, -7\pi/10$

$n = \pm 5 \Rightarrow x = -9\pi/10$

Consider : $x = n\pi + \pi/2$

$n = 0 \Rightarrow x = \pi/2$

$n = \pm 1 \Rightarrow x = -\pi/2$

These are the only values of x in $[-\pi, \pi]$

Example : 41

Solve the following equation for x if a is a constant. sin 3a = 4 sin a sin (x + a) sin (x – a)

Solution

$\sin 3\alpha = 4 \sin \alpha (\sin^2 x - \sin^2\alpha)$

$\Rightarrow$ $3 \sin \alpha - 4 \sin^2\alpha = 4 \sin \alpha \sin^2 x - 4 \sin^3\alpha$

$\Rightarrow$ $3 \sin \alpha - 4 \sin \alpha \sin^2 x = 0$

$\Rightarrow$ $\sin \alpha (3 - 5 \sin^2 x) = 0$

If $\sin \alpha = 0$, then the equation is true for all real values of x.

If $\sin \alpha \neq 0$, then $3 - 4 \sin^2 x = 0$

$\Rightarrow$ $\sin^2 x = \sin^2 \pi/3$

$\Rightarrow$ $x = n\pi \pm \pi/3$

Note : The following results are very useful

1. $\cos \theta = 0 \Rightarrow \theta = n\pi + \pi/2$

2. $\sin \theta = 1 \Rightarrow \theta = 2n\pi + \pi/2$

3. $\sin \theta = -1 \Rightarrow \theta = 2n\pi - \pi/2$

4. $\cos \theta = 1 \Rightarrow \theta = 2n\pi$

5. $\cos \theta = -1 \Rightarrow \theta = 2n\pi + \pi$

6. $\sin^2 \theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$

7. $\cos^2 \theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$

8. $\tan^2 \theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$

Example : 42

If tan (A – B) = 1, sec (A + B) = 2√3, calculate the smallest positive values and the most general values of A and B.

Solution

Smallest positive values

Let $A, B \in (0, 2\pi)$

$\Rightarrow$ $(A + B) > (A - B)$

Now $\tan (A - B) = 1$

$\Rightarrow$ $(A - B) = \pi/4, 5\pi/4$

$\sec (A + B) = 2/\sqrt{3}$

$\Rightarrow$ $(A + B) = \pi/6, 11\pi/6$

As (A + B) > (A – B), these are two possibilities :

1. $A - B = \pi/4$ & $A + B = 11\pi/6$

2. $A - B = 5\pi/4$ & $A + B = 11\pi/6$

From (i), we get : $A = \dfrac{25\pi}{24}$ and $B = \dfrac{19\pi}{24}$

From (ii), we get : $A = \dfrac{37\pi}{24}$ and $B = \dfrac{7\pi}{24}$

General Values

$\tan(A - B) = 1 \quad \Rightarrow \quad A - B = n\pi + \pi/4$

$\sec(A + B) = \frac{2}{\sqrt{3}} \quad \Rightarrow \quad A - B = n\pi + \pi/4$

Taking $A - B = n\pi + \frac{\pi}{4}$ and $A + B = 2k\pi + \frac{\pi}{6}$ we get :

$A = \frac{(2k+n)\pi}{2} + \frac{5\pi}{24}$ and $B = \frac{(2k-n)\pi}{2} - \frac{\pi}{24}$

Taking $A - B = n\pi + \frac{\pi}{4}$ and $A + B = 2k\pi - \frac{\pi}{6}$ we get :

$A = \frac{(2k+n)\pi}{2} + \frac{\pi}{24}$ and $B = \frac{(2k-n)\pi}{2} - \frac{5\pi}{24}$

Example : 43

Solve for θ : $\tan\left(\frac{\pi}{2}\sin\theta\right) = \cot\left(\frac{\pi}{2}\cos\theta\right)$

Solution

$\tan\left(\frac{\pi}{2}\sin\theta\right) = \tan\left(\frac{\pi}{2} - \frac{\pi}{2}\cos\theta\right)$

$\Rightarrow \quad \frac{\pi}{2}\sin\theta = n\pi + \frac{\pi}{2} - \frac{\pi}{2}\cos\theta$

$\Rightarrow \quad \sin\theta + \cos\theta = 2n + 1$

$\Rightarrow \quad \cos\left(\theta - \frac{\pi}{4}\right) = \frac{2n+1}{\sqrt{2}}$

As cosine lies between 1 and – 1, n = 0, – 1 are the only

Possible values of n for $-1 \le \frac{2n+1}{\sqrt{2}} \le 1$

$\Rightarrow \quad \cos\left(\theta - \frac{\pi}{4}\right) = \pm\frac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta - \frac{\pi}{4} = 2k\pi \pm \cos^{-1}\left(\pm\frac{1}{\sqrt{2}}\right)$

$\Rightarrow \quad \theta = 2k\pi + \frac{\pi}{4} \pm \cos^{-1}\left(\pm\frac{1}{\sqrt{2}}\right)$

$\Rightarrow \quad \theta = 2k\pi \pm \frac{\pi}{2}, 2k\pi, 2k\pi + \pi$

For $\theta = 2k\pi \pm \pi/2$, the equation becomes undefined.

Hence the solution is : $\theta = 2k\pi, 2k\pi + \pi$

$\Rightarrow \quad \theta = m\pi$, where $m \in I$

Example : 44

If sin A = sin B and cos A = cos B, find the value of A is terms of B.

Solution

$\sin A - \sin B \quad \Rightarrow \quad 2\cos\frac{A+B}{2}\sin\frac{A-B}{2} = 0$

$\cos A = \cos B \quad \Rightarrow \quad 2\sin\frac{A+B}{2}\sin\frac{A-B}{2} = 0$

Both the equation will be satisfied if $\sin\frac{A-B}{2} = 0$

$\Rightarrow \quad \frac{A-B}{2} = n\pi$

$\Rightarrow \quad A = 2n\pi + B \quad$ where $n \in I$

Example : 45

Evaluate : (i) $\sin^{-1}\sin 4\pi/3$ (ii) $\cos^{-1}\cos 5\pi/4$ (iii) $\tan^{-1}\tan 2\pi/3$

Solution

(i) $4\pi/3$ does not lie in the principal value branch of $\sin^{-1}x$. Hence $\sin^{-1}\sin 4\pi/3 \neq 4\pi/3$

$\sin^{-1}\sin 4\pi/3 = \sin^{-1}\sin(\pi + \pi/3)$
$= \sin^{-1}(-\sin\pi/3)$
$= \sin^{-1}\sin\pi/3 = -\pi/3$

(ii) $\cos^{-1}\cos 5\pi/4 = \cos^{-1}\cos(\pi + \pi/4)$
$= \cos^{-1}(-\cos\pi/4)$
$= \pi - \cos^{-1}\cos\pi/4$
$= \pi - \pi/4$
$= 3\pi/4$

(iii) $\tan^{-1}\tan 2\pi/3 = \tan^{-1}\tan(\pi - \pi/3)$
$= \tan^{-1}(-\tan\pi/3)$
$= \tan^{-1}\tan\pi/3$
$= -\pi/3$

Example : 46

Show that $\tan^{-1}1/3 + \tan^{-1}1/2 = \pi/4$

Solution

LHS $= \tan^{-1}1/3 + \tan^{-1}1/2$

$$= \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\frac{1}{2}}\right) \qquad \left(\because \frac{1}{3}\frac{1}{2} < 1\right)$$

$$= \tan^{-1}\left(\frac{5/6}{5/6}\right) = \tan^{-1}1 = \frac{\pi}{4} = \text{RHS}$$

Example : 47

Show that : $\cos^{-1}9 + \operatorname{cosec}^{-1}\frac{\sqrt{41}}{4} = \frac{\pi}{4}$

Solution

$$\text{LHS} = \cos^{-1}9 + \operatorname{cosec}^{-1}\frac{\sqrt{41}}{4}$$

$$= \tan^{-1}\frac{1}{9} + \sin^{-1}\frac{4}{\sqrt{41}}$$

$$= \tan^{-1}\frac{1}{9} + \tan^{-1}\frac{4}{5} \qquad \left(\text{using } \sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{9}+\frac{4}{5}}{1-\frac{1}{9}\frac{4}{5}}\right) \qquad \left(\because\ xy = \frac{1}{9}\times\frac{4}{5} < 1\right)$$

$$= \tan^{-1}\left(\frac{41}{41}\right) \qquad = \tan^{-1}1 = \frac{\pi}{4}$$

Example : 48

Show that :

(i) $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$, $-1 < x < 1$

(ii) $2\tan^{-1}x = \sin^{-1}$, $-1 < x < 1$

(iii) $2\tan^{-1}x = \cos^{-1}$, $x > 0$

Solution

(i) Let $x = \tan\theta$, $\frac{\pi}{4} < \theta < \frac{\pi}{4}$ (using $-1 < x < 1$)

$$\text{RHS} = \tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta} = \tan^{-1}\tan 2\theta$$

$= 2\theta = 2\tan^{-1}x =$ LHS [$\because$ $2\theta \in (-\pi/2, \pi/2)$ lies in the principal value branch of $\tan^{-1}x$]

(ii) Let $x = \tan\theta$ $\Rightarrow$ $\frac{\pi}{4} < \theta < \frac{\pi}{4}$ (using $-1 < x < 1$) $\frac{1\ 2x^2}{1+x^2}$

$$\text{RHS} = \tan^{-1}\frac{2\tan\theta}{1+\tan^2\theta} = \sin^{-1}\sin 2\theta$$

$= 2\theta$ [$\because$ $2\theta \in (-\pi/2, \pi/2)$ lies in the principal value branch of $\sin^{-1}x$]

$= 2\tan^{-1}x =$ LHS

(iii) Let $x = \tan\theta$, $0 < \theta < \pi/2$ (using $x > 0$)

$$\text{RHS} = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$= \cos^{-1}\cos 2\theta$

$= 2\theta$ [$\because$ $2\theta \in (0, \pi)$ lies in the principal value branch of $\cos^{-1}x$]

$= 2\tan^{-1}x =$ LHS

Example : 49

Prove that

(i) $\tan^{-1}\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right) = \frac{\pi}{4} - \theta$

(ii) $\tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right) = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x^2$

Solution

(i) LHS $= \tan^{-1}\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right) = \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$ (dividing by $\cos\theta$)

$= \tan^{-1} \tan\left(\frac{\pi}{4} - \theta\right) = \frac{\pi}{4} - \theta$ [for $\theta \in (-\pi/4, 3\pi/4)$]

(ii) Put $x^2 = \cos 2\theta$ in LHS

$$\text{LHS} = \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}\right) = \tan^{-1}\left(\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\right)$$

$$= \tan^{-1} \tan\left(\frac{\pi}{4} - \theta\right)$$

$= \frac{\pi}{4} - \theta$ [for $\theta \in (-\pi/4, 3\pi/4)$]

$= \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x^2$ [using $x^2 = \cos 2\theta$]

RHS

Example : 50

Show that : $2\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right] = \cos^{-1}\left[\frac{1-\left(\frac{a-b}{a+b}\right)\left(\frac{1-\cos\theta}{1+\cos\theta}\right)}{1+\left(\frac{a-b}{a+b}\right)\left(\frac{1-\cos\theta}{1+\cos\theta}\right)}\right]$

$$= \cos^{-1}\left[\frac{(a+b)(1+\cos\theta) - (a-b)(1-\cos\theta)}{(a+b)(1+\cos\theta) + (a-b)(1-\cos\theta)}\right] = \cos^{-1}\left[\frac{a\cos\theta + b}{a + b\cos\theta}\right] = \text{RHS}$$

Example : 51

Show that : $2\tan^{-1}\left[\tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)\tan\left(\frac{\pi}{4} - \frac{\beta}{2}\right)\right] = \tan^{-1}\left[\frac{\cos\alpha\cos\beta}{\sin\beta + \sin\alpha}\right]$

Solution

LHS $= 2\tan^{-1}\left[\sqrt{\frac{1-\sin\alpha}{1+\sin\alpha}}\sqrt{\frac{1-\sin\beta}{1+\sin\beta}}\right]$ Using $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$

$$= \tan^{-1}\left[\frac{2\sqrt{\left(\frac{1-\sin\alpha}{1+\sin\alpha}\right)\left(\frac{1-\sin\beta}{1+\sin\beta}\right)}}{1-\left(\frac{1-\sin\alpha}{1+\sin\alpha}\right)\left(\frac{1-\sin\beta}{1+\sin\beta}\right)}\right]$$

$$= \tan^{-1}\left[\frac{2\sqrt{(1-\sin^2\alpha)(1-\sin^{-2}\beta)}}{(1+\sin\alpha)(1+\sin\beta) - (1-\sin\alpha)(1-\sin\beta)}\right]$$

$$= \tan^{-1}\left[\frac{2\cos\alpha\cos\beta}{2\sin\alpha + 2\sin\beta}\right] = \text{RHS}$$

Example : 52

Show that : $\sin \cot^{-1} \cos \tan^{-1} x = \sqrt{\dfrac{x^2+1}{x^2+2}}$

Solution

Let $x = \tan\theta$ ⇒ $\cos\theta = \dfrac{1}{\sqrt{1+x^2}}$

LHS = $\sin \cot^{-1} \cos\theta = \sin \cot^{-1} \dfrac{1}{\sqrt{1+x^2}}$

Let $\dfrac{1}{\sqrt{1+x^2}} = \cot\alpha$ ⇒ $\sin\alpha = \dfrac{\sqrt{1+x^2}}{\sqrt{1+x^2+1}}$

⇒ LHS = $\sin \cot^{-1} \cos\alpha = \sin\alpha = \sqrt{\dfrac{1+x^2}{2+x^2}}$ = RHS

Example : 53

If $\sin^{-1} \dfrac{2n}{1+a^2} + \sin^{-1} \dfrac{2b}{1+b^2} = 2\tan^{-1}$, then show that $x = \dfrac{a+b}{1-ab}$

Solution

The given relation is : $\sin^{-1} \dfrac{2a}{1+a^2} + \sin^{-1} \dfrac{2b}{1+b^2} = 2\tan^{-1} x$

⇒ $2\tan^{-1} a + 2\tan^{-1} b = 2\tan^{-1} x$

⇒ $\tan^{-1}\left(\dfrac{a+b}{1-ab}\right) = \tan^{-1} x$

⇒ $x = \dfrac{a+b}{1-ab}$

Example : 54

Solve for x : $\tan^{-1} \dfrac{1}{2x+1} + \tan^{-1} \dfrac{1}{4x+1} = \tan^{-1} \dfrac{2}{x^2}$.

Solution

Equating the tan of both sides $\tan\left[\tan^{-1}\dfrac{1}{2x+1} + \tan^{-1}\dfrac{1}{4x+1}\right] = \tan^{-1} \dfrac{2}{x^2}$

⇒ $\dfrac{\dfrac{1}{2x+1} + \dfrac{1}{4x+1}}{1 - \dfrac{1}{(2x+1)(4x+1)}} = \dfrac{2}{x^2}$

⇒ $\dfrac{6x+2}{(2x+1)(4x+1)-1} = \dfrac{2}{x^2}$

⇒ $(3x + 1)\, x^2 = 8x^2 + 6x$

⇒ $3x^3 - 7x^2 - 6x = 0$

⇒ $x = 0, 3, -2/3$

$x = 0$ and $-2/3$ are rejected because they don't satisfy the equation

Note that for $x = 0$, RHS is undefined

⇒ the only solution is $x = 3$.

Example : 55

Show that $\sum_{n=1}^{\infty} \tan^{-1}\frac{1}{1+n+n^2} = \frac{\pi}{4}$

Solution

$$\text{LHS} = \sum_{n=1}^{\infty} \tan^{-1}\frac{1}{1+n+n^2}$$

$$= \sum_{n=1}^{\infty} \tan^{-1}\frac{\overline{n+1}-n}{1+(n+1)n}$$

$$= \sum_{n=1}^{\infty} \left[\tan^{-1}(n+1) - \tan^{-1}n\right]$$

$= (\tan^{-1}2 - \tan^{-1}1) + (\tan^{-1}3 - \tan^{-1}2) + \ldots\ldots\ldots.$

$= -\tan^{-1}1 + \tan^{-1}\infty$

$= -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4} = \text{RHS}$

Example : 1

Prove the following results :

(i) $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

(ii) $r_1 = s\tan\frac{A}{2}$, $r_2 = s\tan\frac{B}{2}$, $r_3 = s\tan\frac{C}{2}$

(iii) $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Solution

(i) $r = \frac{\Delta}{s} = (s-a)\ \frac{\Delta}{s(s-a)}$

$\Rightarrow \quad r = (s-a)\tan\frac{A}{2} \qquad \left(\text{using } \cot\frac{A}{2} = \frac{s(s-a)}{\Delta}\right)$

other results follows by symmetry.

(ii) $r_1 = \frac{\Delta}{s-a} = \frac{s\Delta}{s(s-a)} = s\tan\frac{A}{2}$

Other results follow by symmetry.

(iii) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$; $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$; $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ba}}$

multiply the three results to get :

$\Rightarrow \quad \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{(s-a)(s-b)(s-c)}{abc}$

$\Rightarrow \quad \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \left(\frac{\Delta^2}{s}\right)\left(\frac{1}{4R\Delta}\right)$

$\sum \quad 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\Rightarrow \quad \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \left(\frac{\Delta}{s}\right)\left(\frac{1}{4R}\right)$

$\Rightarrow \quad r = \frac{\Delta}{s} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Example : 2

Show that in a triangle ΔABC : a cot A b cot B + c cot C = 2 (R + r).

Solution

LHS = $\sum 2R\sin A\cot A = 2R \qquad \cos A$

$\Rightarrow \quad$ LHS = 2R (cos A + cos B + cos C)

$\Rightarrow \quad$ LHS = 2R

$\Rightarrow \quad$ LHS = $2R + 8R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\Rightarrow \quad$ LHS = 2R + 2r = RHS (using the result of last Ex.)

Example : 3

Show that : $\frac{r_1}{bc} + \frac{r_2}{ca} \frac{r_3}{ba} = \frac{1}{r} - \frac{1}{2R}$

Solution

$$LHS = \frac{\Delta}{abc}\left(\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\right)$$

$$LHS = \frac{\Delta}{abc}\left(\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\right) + \frac{1}{2R} - \frac{1}{2R}$$

$$LHS = \frac{\Delta}{abc}\left(\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\right) + \frac{2\Delta}{abc} - \frac{1}{2R}$$

$$LHS = \frac{\Delta}{abc}\left(\frac{a}{s-a}+1+\frac{b}{s-b}+1+\frac{c}{s-c}\right) - \frac{1}{2R}$$

$$LHS = \frac{\Delta}{abc}\left(\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\right) - \frac{1}{2R}$$

$$LHS = \frac{\Delta}{abc}\left(\frac{s(2s-a-b)}{(s-a)(s-b)}+\frac{c}{s-c}\right) - \frac{1}{2R}$$

$$LHS = \frac{\Delta}{ab}\left(\frac{s^2-sc+s^2-as-bs+ab}{(s-a)(s-b)(s-c)}\right) - \frac{1}{2R}$$

$$LHS = \frac{\Delta}{ab}\left(\frac{2s^2-s(2s)+ab}{(s-a)(s-b)(s-c)}\right) - \frac{1}{2R}$$

$$LHS = \frac{\Delta}{ab}\ \frac{\Delta}{(s-a)(s-b)(s-c)} - \frac{1}{2R}$$

$$LHS = \frac{\Delta s}{\Delta^2} - \frac{1}{2R} = \frac{1}{r} - \frac{1}{2R} = RHS$$

Example : 4

In a ΔABC, show that :

1. $c^2 = (a-b)^2 \cos^2 \frac{C}{2} + (a+b)^2 \sin^2 \frac{C}{2}$

2. $a \sin\left(\frac{A}{2}+B\right) = (b+c)\sin\frac{A}{2}$

3. $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C = a + b + c$

Solution

1. $RHS = (a-b)^2\left(\frac{1+\cos C}{2}\right) = (a+b)^2\left(\frac{1-\cos C}{2}\right)$

2. $RHS = \frac{1}{2}[(a-b)^2 + (a+b)^2] + \frac{1}{2}\cos C\,[(a-b)^2 - (a+b)^2]$

$RHS\ a^2 + b^2 + \frac{1}{2}\cos C\,(-4ab) = c^2$ (using cosine rule)

Note : Try to prove the same identity using sine rule on RHS

2. LHS = a sin $\left(\frac{A}{2}+B\right)$ = 2R sin A sin $\left(\frac{A}{2}+B\right)$ (using sine rule)

LHS = 2R $\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)$ sin $\left(\frac{A}{2}+B\right)$

LHS = 2R sin $\frac{A}{2}$ $\left[2\cos\frac{A}{2}\sin\left(\frac{A}{2}+B\right)\right]$

LHS = 2R sin $\frac{A}{2}$ [sin (A + B) – sin (–B)]

LHS = 2R sin $\frac{A}{2}$ [sin C + sin B]

LHS = sin $\frac{A}{2}$ [2R sin C + 2R sin B]

LHS = sin $\frac{A}{2}$ (c + b) = RHS

Note : Try to prove the same identity using RHS

3. LHS = (b + c) cos A + (c + a) cos B + (a + b) cos C
LHS = [c cos B + b cos C] + a [a cos C + c cos A] + [b cos A + a cos B]
LHS = a + b + c = RHS

Example :

In a ΔABC, prove that $(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$

Solution

Starting from LHS

$= \sum (b^2 - c^2)\ \cot A$

$= 4R^2 \sum (\sin^2 B - \sin^2 C)\ \cot A$ (using sine rule)

$= 4R^2 \sum \sin(B - C)\sin(B - C)\ \cot A$

$= 4R^2 \sum \sin A \sin(B - C)\frac{\cos A}{\sin A}$

$= -2R^2 \sum 2\cos(B + C)\sin(B - C)$ (using cos A = – cos (B + C)

$= -2R^2 \sum (\sin 2B - \sin 2C)$

$= -2R^2$ [(sin 2B – sin 2C) + (sin 2C – sin 2A) + (sin 2A – sin 2B)]

= 0 = RHS

Example : 6

In a ΔABC, show that : (a + b + c) $\left[\tan\frac{A}{2}+\tan\frac{B}{2}\right]$ = 2c cot $\frac{C}{2}$

Solution

Starting from LHS

= (a + b + c) $\left[\frac{(s-b)(s-c)}{\Delta}+\frac{(s-c)(s-a)}{\Delta}\right]$

= $\left(\frac{a+b+c}{\Delta}\right)$ (s – c) [s – b + s – a]

$= \left(\frac{s-c}{\Delta}\right)$ (a + b + c) (c)

$= \frac{(s-c)}{\Delta}$ = 2c $\left[\frac{s(s-c)}{\Delta}\right]$ = 2c cot $\frac{C}{2}$ = RHS

Example : 7

In a ΔABC, prove that :

(i) $r_1 + r_2 + r_3 - r = 4R$

(ii) $rr_1 + rr_2 + rr_3 = ab + bc + ca - s^2$

Solution

(i) Starting from LHS

$= \left(\frac{\Delta}{s-a}+\frac{\Delta}{s-b}\right) + \left(\frac{\Delta}{s-c}-\frac{\Delta}{s}\right)$

$= \Delta \frac{(2s-\overline{a+b})}{(s-a)(s-b)} + \frac{\Delta(s-\overline{s-c})}{s(s-c)}$

$= \frac{\Delta c}{(s-a)(s-b)} + \frac{\Delta c}{s(s-c)}$

$= \frac{\Delta c}{s(s-a)(s-b)(s-c)}$ [ss – c) +) s – a) (s – b)]

$= \frac{c}{\Delta}$ [2s² – 2s² + ab] = $\frac{abc}{\Delta}$ = 4 $\left(\frac{abc}{4\Delta}\right)$ = 4R

(ii) Starting from LHS

$= \frac{\Delta^2}{s}\left[\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right]$

$= \frac{\Delta^2}{s}\left[\frac{\sum(s-b)(s-c)}{(s-a)(s-b)(s-c)}\right]$

$= 3s^2 - 2s\,(a + b + c) + bc + ca + ab$

$= 3s^2 - 4s^2 + bc + ca + ab$

$= ab + bc + ca - s^2$ = RHS

Example : 8

In a ΔABC, show that : $\frac{(a+b+c)^2}{a^2+b^2+c^2} = \frac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{\cot A+\cot B+\cot C}$

Solution

Starting from RHS

$$= \frac{\frac{s(s-a)}{\Delta}+\frac{s(s-b)}{\Delta}+\frac{s(s-c)}{\Delta}}{\frac{b^2+c^2-a^2}{4\Delta}+\frac{c^2+a^2-b^2}{4\Delta}+\frac{a^2+b^2-c^2}{4\Delta}} = \frac{4s[s-a+s-b+s-c]}{b^2+c^2+a^2}$$

$$= \frac{4s(3s-2s)}{a^2+b^2+c^2} = \frac{4s^2}{a^2+b^2+c^2} = \frac{(a+b+c)^2}{a^2+b^2+c^2} \text{ LHS}$$

Example : 9

If a^2 , b^2 , c^2 in a ΔABC are in A.P. Prove that cot A, cot B and cot C are also in A.P.

Solution

cot A, cot B and cot C are in A.P. if :

cot A – cot B = cot B – cot C

$$\Rightarrow \quad \frac{\cos A}{\sin A} - \frac{\cos B}{\sin B} = \frac{\cos B}{\sin B} - \frac{\cos C}{\sin C}$$

$$\Rightarrow \quad \frac{\sin(B-A)}{\sin A \sin B} = \frac{\sin(C-B)}{\sin B \sin C}$$

$\Rightarrow \quad \sin (B - A) \sin C = \sin (C - B) \sin A$

$\Rightarrow \quad \sin (B - A) \sin (B + A) = \sin (C - B) \sin (C + B)$

$\Rightarrow \quad \sin^2 B - \sin^2 A = \sin^2 C - \sin^2 B$

$$\Rightarrow \quad \frac{b^2}{4R^2} - \frac{a^2}{4R^2} = \frac{c^2}{4R^2} - \frac{b^2}{4R^2} \qquad \text{(using sine rule)}$$

$\Rightarrow \quad b^2 - a^2 = c^2 - b^2 \qquad \Rightarrow \quad ab^2 = a^2 + c^2$

$\Rightarrow \quad a^2$, b^2 , c^2 are in A.P.

$\Rightarrow$ cot A, cot B and cot C are also in A.P.

Example : 10

If x, y, z are respectively the perpendiculars from circumcentre to the sides of the triangle ABC prove that

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}$$

Solution

We known that : $x = R \cos A$, $y = R \cos B$, $z = R \cos C$

Consider LHS :

$$= \frac{a}{R\cos A} + \frac{b}{R\cos B} + \frac{c}{R\cos C}$$

$$= \frac{2R\sin A}{R\cos A} + \frac{2R\sin B}{R\cos B} = \frac{2R\cos C}{R\cos C}$$

$= (\tan A + \tan B + \tan C)$

$= (\tan A \tan B \tan C) \qquad (\because A + B + C = \pi)$

$$= 2\left[\frac{\sin A}{\cos A}\frac{\sin B}{\cos B}\frac{\sin C}{\cos C}\right]$$

$$= \frac{2}{8R^3}\left[\frac{abc}{\cos A \cos B \cos C}\right] \qquad \text{(using sine rule)}$$

$$= \frac{1}{4}\left[\frac{abc}{(R\cos A)(R\cos B)(R\cos C)}\right] = \frac{1}{4}\,\frac{abc}{xyz} = \text{RHS}$$

Example : 11

I is the incentre of ΔABC and P_1 , P_2 , P_3 are respectively the radii of the circumcircle of DIBC, ΔICA and IAB, prove that : $P_1 P_2 P_3 = 2R^2 r$.

Solution

$$\angle BIC = \pi - \frac{1}{2}(B + C) = \pi - \frac{1}{2}(\pi - A) = \frac{\pi}{2} + \frac{A}{2}$$

Circumradius of ΔABC is :

$$P_1 = \frac{BC}{2\sin \angle BIC} = \frac{BC}{2\sin\left(\frac{\pi}{2} + \frac{A}{2}\right)} = \frac{a}{2\cos\frac{A}{2}}$$

Similarly we can show that : $P_2 = \dfrac{b}{2\cos\frac{B}{2}}$ and $P_3 = \dfrac{c}{2\cos\frac{C}{2}}$

$$\Rightarrow \quad P_1P_2P_3 = \frac{abc}{8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{8R^2\sin A\sin B\sin C}{8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$= \frac{8R^3\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = 8R^3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$= 2R^2 r$ = RHS

Example : 12

(Ptolemy Theorem) If ABCD is cyclic quadrilateral, show that AC . BD = AB . CD + BC . AD

Solution

Let AB = a, BC = b, CD = c, DA = d

using cosine rule in ΔABC and ΔADC, we get :

$AC^2 = a^2 + b^2 - 2ab\cos B$

$AC^2 = c^2 + d^2 - 2cd\cos D$

and $B + D = \pi$

$\Rightarrow$ $\cos B + \cos D = 0$

$\Rightarrow$ $AC^2(cd + ab) = (a^2 + b^2)cd + (c^2 + d^2)ab$

$$\Rightarrow \quad AC^2 = \frac{(a^2cd + c^2ab) + (b^2cd + d^2ab)}{cd + ab}$$

Similarly by taking another diagonal BD, we can show that :

$$BD^2 = \frac{(ba + cd)(bd + ca)}{da + bc}$$

Multiplying the two equations

$\Rightarrow$ $(AD\ .\ BD)^2 = (ac + bd)^2$

$\Rightarrow$ AC . BD = ac + bd

$\Rightarrow$ AC . BD = AB . CD + BC . AD

Example : 13

Show that : $\left[\cot\frac{A}{2} + \cot\frac{B}{2}\right]\left[a\sin^2\frac{B}{2} + b\sin^2\frac{A}{2}\right] = c\cot\frac{C}{2}$

Solution

Taking LHS :

$$= \left[\frac{s(s-a)}{\Delta} + \frac{s(s-b)}{\Delta}\right]\left[\frac{a(s-c)(s-a)}{ca} + \frac{b(s-b)(s-c)}{bc}\right]$$

$$= \frac{s}{\Delta}[2s - a - b]\left(\frac{s-c}{c}\right)(2s - a - b)$$

$$= \frac{s(s-c)}{\Delta c}c^2 = c\frac{s(s-c)}{\Delta} = c\cot\frac{C}{2} = \text{RHS}$$

Example : 14

In a ΔABC, show that $a^3 \cos(B - C) + b^3 \cos(C - A) + c^3 \cos(A - B) = 3abc$

Solution

Given expression $= \sum a^3 \cos(B - C)$

$= \sum a^2 (2R \sin A) \cos(B - C)$

$= R \sum a^2 (2R \sin \overline{B + C} \cos \overline{B - C})$

$= R \sum a^2 (\sin 2B + \sin 2C)$

$= 2R \sum a^2 (\sin B + \sin B + \sin C \cos C) = \sum a^2 (b \cos B + c \cos C)$

$= a^2 (\underline{b \cos B} + \overline{c \cos C}) + b^2 (c \cos C + \underline{a \cos A}) + c^2 (\overline{a \cos A} + b \cos B)$

$= ab (a \cos B + b \cos A) + ac (a \cos C + a \cos A) + bc (b \cos C + c \cos B)$

$= abc + acb + bca$ (using projection formula)

$= 3abc = \text{RHS}$

Example : 15

If the sides a, b, c of a ΔABC are in A.P., then prove that $\cot \frac{A}{2}$, $\cot \frac{B}{2}$ and $\cot \frac{C}{2}$ are also in A.P.

Solution

a, b, c are in A.P. $\Rightarrow$ $a - b = b - c$

$\Rightarrow \sin A - \sin B = \sin B - \sin C$

$\Rightarrow 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} = 2 \cos \frac{B+C}{2} \sin \frac{B-C}{2}$

$\Rightarrow \sin \frac{C}{2} \sin \frac{A-B}{2} = \sin \frac{A}{2} \sin \frac{B-C}{2}$

$$\Rightarrow \frac{\sin\left(\frac{A}{2} - \frac{B}{2}\right)}{\sin \frac{A}{2} \sin \frac{B}{2}} = \frac{\sin\left(\frac{B}{2} - \frac{C}{2}\right)}{\sin \frac{B}{2} \sin \frac{C}{2}}$$

$\Rightarrow \cot \frac{B}{2} - \cot \frac{A}{2} = \cot \frac{C}{2} - \cot \frac{B}{2}$

$\Rightarrow \cot \frac{A}{2}, \cot \frac{B}{2}, \cot \frac{C}{2}$ are in A.P.

Example : 16

In a ΔABC, prove that $A = B$ if : $a \tan A + b \tan B = (a + b) \tan \left(\frac{A+B}{2}\right)$

Solution

Rearranging the terms of the given expression as follows :

$\Rightarrow a \tan A - a \tan \frac{A+B}{2} - b \tan \frac{A+B}{2} - b \tan B$

$$\Rightarrow \frac{a \sin\left(A - \frac{A+B}{2}\right)}{\cos A \cos \frac{A+B}{2}} = \frac{b \sin\left(\frac{A+B}{2} - B\right)}{\cos \frac{A+B}{2} \cos B}$$

$$\Rightarrow \quad \frac{2R\sin A\sin\left(\frac{A-B}{2}\right)}{\cos A} = \frac{2R\sin B\sin\left(\frac{A-B}{2}\right)}{\cos B}$$

$$\Rightarrow \quad \sin\left(\frac{A-B}{2}\right)[\tan A - \tan B] = 0$$

$$\Rightarrow \quad \sin\left(\frac{A-B}{2}\right) = 0 \text{ or } \tan A - \tan B = 0$$

$\Rightarrow$ A = B

Example : 17

If the sides of a triangle are in A.P. and the greatest angle exceeds the smallest angle by α , show that the sides are in the ratio 1 – x : 1 : 1 + x; where $x = \sqrt{\frac{1-\cos\alpha}{7-\cos\alpha}}$

Solution

Let A > B > C

$\Rightarrow$ A – C = α and ab = a + c

We will first find the values of sin B/2 and cos B/2

2b = a + c

$\Rightarrow$ 2 sin B = sin A + sin C

$$\Rightarrow \quad 4\sin\frac{B}{2}\cos\frac{B}{2} = 2\sin\frac{A+C}{2}\cos\frac{A-C}{2}$$

$$\Rightarrow \quad 4\sin\frac{B}{2}\cos\frac{B}{2} = 2\cos\frac{B}{2}\cos\frac{\alpha}{2}$$

$$\Rightarrow \quad \sin\frac{B}{2} = \frac{1}{2}\cos\frac{\alpha}{2} \quad \Rightarrow \quad \sin\frac{B}{2} = \sqrt{\frac{1+\cos\alpha}{2\sqrt{2}}}$$

$$\Rightarrow \quad \cos\frac{B}{2} = \sqrt{1-\sin^2\frac{B}{2}} = \frac{\sqrt{7-\cos\alpha}}{2\sqrt{2}} \qquad \text{...........(i)}$$

Consider

$$\frac{a}{c} = \frac{\sin A}{\sin C} \qquad \text{(using sine rule)}$$

$$\Rightarrow \quad \frac{a+c}{a-c} = \frac{\sin A + \sin C}{\sin A - \sin C}$$

$$\Rightarrow \quad \frac{a+c}{a-c} = \frac{2\sin B}{2\cos\frac{A+C}{2}\sin\frac{A-c}{2}}$$

$$\Rightarrow \quad \frac{a+c}{a-c} = \frac{2\left(2\sin\frac{B}{2}\cos\frac{B}{2}\right)}{2\sin\frac{B}{2}\sin\frac{\alpha}{2}}$$

$$\Rightarrow \quad \frac{a+c}{a-c} = 2\,\frac{\cos B/2}{\sin\alpha/2}$$

$$\Rightarrow \quad \frac{a+c}{a-c} = \frac{2\left(\frac{\sqrt{7-\cos\alpha}}{2\sqrt{2}}\right)}{\sin\alpha/2} \qquad \text{(using (i))}$$

$$\Rightarrow \quad \frac{a+c}{a-c} = \frac{\sqrt{7-\cos\alpha}}{\sqrt{1-\cos\alpha}}$$

$$\Rightarrow \quad \frac{a+c}{a-c} = \frac{1}{x}$$

$$\Rightarrow \quad \frac{a}{c} = \frac{1+x}{1-x}$$

$$\Rightarrow \quad \frac{a}{1+x} = \frac{c}{1-x}$$

$$\Rightarrow \quad \frac{a}{1+x} = \frac{c}{1-x} = \frac{a+c}{2}$$

$$\Rightarrow \quad \frac{a}{1+x} = \frac{c}{1-x} = \frac{2b}{2}$$

$$\Rightarrow \quad \frac{a}{1+x} = \frac{b}{1} = \frac{c}{1-x}$$

Example : 18

Δ is the mid point of BC in a ΔABC. If AD is perpendicular to AC, show that : cos A cos C = $\frac{2(c^2-a^2)}{3ac}$

Solution

The value of cos C can be found by cosine rule in ΔABC or ΔADC

From ΔABC : cos C = $\frac{a^2+b^2-c^2}{2ab}$

From ΔADC : cos C = $\frac{a}{a/2}$

$$\Rightarrow \quad \frac{2b}{a} = \frac{a^2+b^2-c^2}{2ab}$$

$$\Rightarrow \quad b^2 = \frac{a^2-c^2}{3} \qquad \text{(i)}$$

= LHS = cos A cos C = $\left(\frac{b^2+c^2-a^2}{2bc}\right)\left(\frac{b}{a/2}\right)$

$$= \frac{b^2+c^2-a^2}{ac} = \frac{\frac{a^2-c^2}{3}+c^2-a^2}{ac} \qquad \text{(using (i))}$$

$$= \frac{2(c^2-a^2)}{3ac} = \text{RHS}$$

Example : 19

Let O be a point inside a ∠ABC such that ∠OAB = ∠OBC = ∠OCA = ω , Show that

(i) $\cos\omega = \cot A + \cot B + \cot C$

(ii) $\text{cosec}^2\omega = \text{cosec}^2A + \text{cosec}^2B + \text{cosec}^2C$

Solution

Apply the sine rule in ΔOBC

$$\Rightarrow \quad \frac{OB}{a} = \frac{\sin(c-\omega)}{\sin(\pi-\omega+C-\omega)}$$

$$\Rightarrow \quad \frac{OB}{a} = \frac{\sin(C-\omega)}{\sin C} \qquad \text{..........(i)}$$

Applying sine rule in ΔOAB and proceeding similarly :

$$\Rightarrow \quad \frac{OB}{c} = \frac{\sin\omega}{\sin B}$$

Divide (i) by (ii) to get :

$$\frac{c}{a} = \frac{\sin(C-\omega)\sin B}{\sin\omega\sin C} \quad \text{(sine rule in } \Delta ABC)$$

$$\Rightarrow \quad \frac{\sin C}{\sin A \sin B} = \frac{\sin(C-\omega)}{\sin\omega\sin C}$$

$$\Rightarrow \quad \frac{\sin(A+B)}{\sin A \sin B} = \frac{\sin(C-\omega)}{\sin\omega\sin C}$$

$\Rightarrow$ $\cot B + \cot A = \cot\omega - \cot C$

$\Rightarrow$ $\cot\omega = \cot A + \cot B + \cot C$

(ii) Squaring the above result :

$\cot^2\omega = (\cot A + \cot B + \cot C)^2$

$\Rightarrow$ $\text{cosec}^2\omega - 1 = \sum \cot^2 A + 2 \quad \cot A \cot B$ $\sum \frac{+ c^2 - a^2}{2bc}$

$\Rightarrow$ $\text{cosec}^2\omega - 1 = \quad (\text{cosec}^2A - 1) + 2 \quad (\because \text{ in a } \Delta \quad \cot A \cot B = 1)$

$\Rightarrow$ $\text{cosec}^2\omega = \quad \text{cosec}^2A - 3 + 2$

$\Rightarrow$ $\text{cosec}^2\omega = \text{cosec}^2A + \text{cosec}^2B + \text{cosec}^2C$

Example : 20

For a triangle ABC, it is given that : cos A + cos B + cos C = 3/2. Prove that the triangle is equilateral.

Solution

Consider cos A + cos B + cos C = 3/2

$$\Rightarrow \quad + \frac{c^2+a^2+b^2}{2ca} + \frac{a^2+b^2+c^2}{2ab} = \frac{3}{2}$$

$\Rightarrow$ $2(b^2 + c^2 - a^2) + b(c^2 + a^2 - b^2) + c(a^2 + b^2 + c^2) = 3abc$

$\Rightarrow$ $a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2) = a^3 + b^3 + c^3 + 3abc$

$\Rightarrow$ $a(b^2 + c^2 - 2bc) + b(c^2 + a^2 - 2ac) + c(a^2 + b^2 - 2ab) = a^3 + b^3 - 3abc$

$\Rightarrow$ $a(b - c)^2 + b(c - a)^2 + c(a - b)^2 - 1/2\,(a + b + c)\,[(b - c)^2 + (c - a)^2 + (a - b)^2] = 0$

$\Rightarrow$ $(b - c)^2 (b + c - a) + (c - a)^2 (c + a - b) + (a - b)^2 (a + b - c) = 0$

$\Rightarrow$ ∵ sum of two sides > third side

$\Rightarrow$ All terms in LHS are non-negative

$\Rightarrow$ each term = 0

$\Rightarrow$ $b - c = c - a = a - b = 0$

$\Rightarrow$ $a = b = c$

$\Rightarrow$ ΔABC is a equilateral.

Example : 21

If a = 100, c = 100 √2 and A = 30º, solve the triangle.

Solution

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 - 2b(100\sqrt{2})\cos 30^\circ + (100\sqrt{2})^2 - 100^2 = 0$

$b^2 - 100\sqrt{6}\, b + 10000 = 0$

$$b = \frac{100\sqrt{6} \pm 100\sqrt{2}}{2} = 50\sqrt{2}\left(\sqrt{3} \pm 1\right)$$

$b_1 = 50\sqrt{2}(\sqrt{3} - 1)$, $b_2 = 50\sqrt{2}(\sqrt{3} + 1)$

$$\sin C = \frac{c \sin A}{a} = \frac{100\sqrt{2}\sin 30^\circ}{100} = \frac{1}{\sqrt{2}}$$

$C_1 = 135^\circ$ and $C_2 = 45^\circ$

$B_1 = 180 - (135^\circ + 30^\circ) = 15^\circ$

$B_2 = 180 - (45^\circ + 30^\circ) = 105^\circ$

Example : 22

In the ambiguous case, if the remaining angles of the triangle formed with a, b and A be B1 , C_1 and B_2 , C_2,

then prove that : $\frac{\sin C_2}{\sin B_1} + \frac{\sin C_2}{\sin_2}$ = 2 cos A.

Solution

$$\sin B_1 - \sin B_2 = \frac{b \sin A}{a} \qquad \text{(using sine rule)}$$

$$\sin C_1 = \frac{c_1 \sin A}{a} \text{ and } \sin C_2 = \frac{c_2 \sin A}{a}$$

$$\Rightarrow \quad \text{LHS} = \frac{\frac{c_1 \sin A}{a}}{\frac{b \sin A}{a}} + \frac{\frac{c_2 \sin A}{a}}{\frac{b \sin A}{a}}$$

$$\Rightarrow \quad \text{LHS} = \frac{c_1 + c_2}{b} = \frac{2b \cos A}{b} = 2 \cos A$$

Example : 23

In a ΔABC; a, c, A are given and $b_1 = 2b_2$, where b_1 and b_2 are two values of the third side : then prove that:

$3a = c\sqrt{1 + 8\sin^2 A}$

Solution

$a^2 = b^2 + c^2 - 2bc \cos A$

consider this equation as a quadratic in b.

$\Rightarrow \quad b^2 - (2c \cos A)\, b + c^2 - a^2 = 0$

$\Rightarrow \quad b_1 + b_2 = 2c \cos A$

and $\quad b_1 - b_2 = c^2 - a^2$

and $\quad b_1 = 2b_2$

$\Rightarrow \quad 3b_1 = 2c \cos A$ and $2b_1^2 = c^2 - a^2$

$$\Rightarrow \quad 2\left(\frac{2c \cos A}{3}\right)^2 c^2 - a^2$$

$\Rightarrow \quad 8c^2 \cos^2 A = 9c^2 - 9a^2$

$\Rightarrow \quad 8c^2(1 - \sin^2 A) = 9c^2 - 9a^2$

$\Rightarrow \quad 9a^2 = c^2 + 8c^2 \sin^2 A$

$\Rightarrow \quad 3a = c\sqrt{1 + 8\sin^2 A}$

Example : 24

A man observes, that when he moves up a distance c meters on a slope, the angle of depression of a point on the horizontal plane from the base of the slope is 30º; and when he moves up further a distance c meters the angle of depression of that point is 45º. Obtain the angle of elevation of the slope with the horizontal.

Solution

Let the point A be observed from Q and R

$\Rightarrow$ PQ = QR = c

Apply m – n theorem in ΔAPR. Q divides PR in ratio c : c

$\Rightarrow$ $(c + c) \cot (\theta - 30^\circ) = c \cot 15^\circ - c \cot 30^\circ$

$\Rightarrow$ $2 \cot (\theta - 30^\circ) = 2 + \sqrt{3} - \sqrt{3}$

$\Rightarrow$ $2 \cot (\theta - 30^\circ) = 2$

$\Rightarrow$ $\cot (\theta - 30^\circ) = 1$

$\Rightarrow$ $\theta - 30^\circ = 45^\circ$ $\Rightarrow$ $\theta = 75^\circ$

Example : 25

A vertical pole (more than 100 ft high) consists of two portions, the lower being one third of the whole. If the upper portion subtends an angle $\tan^{-1}(1/2)$ at a point in the horizontal plane through the foot of the pole and at a distance of 40ft from it, find the height of the pole.

Solution

Let PQ be the tower and R be the point dividing PQ in 1 : 2

Angle subtended at A = $\alpha = \tan^{-1}1/2$

$$\Rightarrow \quad \alpha = \tan^{-1}\frac{PQ}{AP} - \tan^{-1}\frac{PR}{AP}$$

$$\Rightarrow \quad \tan^{-1}\frac{1}{2} = \tan^{-1}\frac{h}{40} - \tan^{-1}\frac{h/3}{40}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{\frac{h}{40} - \frac{h}{120}}{1 + \frac{h^2}{4800}}$$

$$\Rightarrow \quad 1 + \frac{h^2}{4800} = \frac{h}{20} - \frac{h}{60}$$

$\Rightarrow$ $h^2 - 160h + 4800 = 0$

$\Rightarrow$ h = 40, 120

$\Rightarrow$ h = 120 ft. (as h > 100ft)

Example : 26

A 2 metre long object is fired vertically upwards from the mid-point of two locations A and B, 8 metres apart. The speed of the object after t seconds is given by ds/dt = 2t + 1 m/s. Let α and β be the angles subtended by the object at A and B respectively after one and two seconds. Find the value of $\cos(\alpha - \beta)$.

Solution

At t = 1 s :

$$OP = s = \int_0^1 (2t + 1)\, dt = 2m$$

$$\Rightarrow \quad \alpha = \tan^{-1}\left(\frac{OP + PQ}{OA}\right) - \tan^{-1}\left(\frac{OP}{OA}\right)$$

$$\Rightarrow \quad \alpha = \tan^{-1}\left(\frac{2 + 2}{4}\right) - \tan^{-1}\left(\frac{2}{4}\right)$$

$$\Rightarrow \qquad \tan\alpha = \frac{1}{3}$$

At t = 2 s :

$$OP = \int_0^2 (2t+1)\, dt = 6m.$$

$$\Rightarrow \qquad \beta = \tan^{-1}\left(\frac{6+2}{4}\right) - \tan^{-1}\left(\frac{6}{4}\right)$$

$$\Rightarrow \qquad \tan\beta = \frac{2-3/2}{1+2.3/2}$$

$$\Rightarrow \qquad \tan\beta = \frac{1}{8}$$

$$\Rightarrow \qquad \tan\alpha = \frac{1}{3} \text{ and } \tan\beta = \frac{1}{8}$$

$$\Rightarrow \qquad \sin\alpha = \frac{1}{\sqrt{10}},\ \cos\alpha = \frac{3}{\sqrt{10}}$$

$$\Rightarrow \qquad \sin\beta = \frac{1}{\sqrt{65}},\ \cos\beta = \frac{8}{\sqrt{65}}$$

$$\Rightarrow \qquad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \frac{3}{\sqrt{10}}\ \frac{8}{\sqrt{65}} + \frac{1}{\sqrt{10}}\ \frac{1}{\sqrt{65}} = \frac{5}{\sqrt{26}}$$

Example : 27

A man observes two objects in a straight line in the west. On walking a distance c to the north, the object subtend as angle α in front of him and on walking a further distance of c to the north, they subtend an angle β . Prove that the distance between the objects is : $\frac{3c}{2\cot\beta - \cot\alpha}$

Solution

Let x = distance between objects A and B.
y = distance of B from initial position of man.
The man starts from O and observes angle α and β at P and Q respectively as shown.

$$\alpha = \tan^{-1}\frac{x+y}{c} - \tan^{-1}\frac{y}{c}$$

$$\beta = \tan^{-1}\frac{x+y}{2c}\ \tan^{-1}\frac{y}{2c}$$

$$\Rightarrow \qquad \tan\alpha = \frac{\frac{x+y}{c} - \frac{y}{c}}{1+\frac{xy+y^2}{c^2}} = \frac{ac}{c^2+xy+y^2}$$

$$\Rightarrow \qquad \tan\beta = \frac{\frac{x+y}{2c} - \frac{y}{2c}}{1+\frac{xy+y^2}{4c^2}} = \frac{2xc}{4c^2+xy+y^2}$$

By eliminating $(xy + y^2)$, we can find x.
Equate values of $(xy + y^2)$ from the two equations.

$$\Rightarrow \quad \frac{xc}{\tan\alpha} - c^2 = \frac{2xc}{\tan\beta} - 4c^2$$

$$\Rightarrow \quad xc(\cot\alpha - 2\cot\beta) = -3c^2$$

$$\Rightarrow \quad x = \frac{3c}{2\cot\beta - \cot\alpha}$$

Example : 28

A right circular cylinder tower of height h and radius r stands on a horizontal lane. Let A be a point in the horizontal plane and PQR be the semi-circular edge of the two of the tower such that Q is the point in it nearest A. The angles of elevation of the point P and Q from A are 45º and 60º respectively. Show that :

$$\frac{h}{r} = \frac{\sqrt{3}(1+\sqrt{5})}{2}$$

Solution

Let P′, Q′, R′ be the projection of P, Q, R in the base of the lower. Hence PP′, QQ′, RR′ are vertical lines.

From $\Delta AQQ'$ $\quad AQ' = h\cot 60^\circ$

From $\Delta APP'$ $\quad AP' = h\cot 45^\circ$

If O is the centre of the circular base of the lower, triangle $\Delta AOP'$ is right angled

$$(h\cot 60^\circ + t)^2 + r^2 = (h\cot 45^\circ)^2$$

$$\Rightarrow \quad \frac{h^2}{3} + r^2 + \frac{2hr}{\sqrt{3}} + r^2 = h^2$$

$$\Rightarrow \quad 2h^2 - 2\sqrt{3}\,hr - 6r^2 = 0$$

$$\Rightarrow \quad \frac{h}{r} = \frac{2\sqrt{3} + \sqrt{12+48}}{4} \qquad \text{(taking only position values of } \frac{h}{r}\text{)}$$

$$\Rightarrow \quad \frac{h}{r} = \frac{\sqrt{3}(1+\sqrt{3})}{2}$$

Example : 29

From a point on the horizontal plane, the elevation of the top of the hill is α. After walking a metres towards the summit up a slope inclined at an angle β to the horizontal, the angle of elevation is γ. Find the height of the hill.

Solution

Let PQ = h = height of the hill.

P is the top of the hill (summit)

At A, on the ground level, elevation of P is α

at B(AB = a) elevation of P is γ . AB is inclined at β to the horizontal

Let NQ = y

from ΔPAQ : $\quad AQ = h\cot\alpha$

from ΔPBN : $\quad BN = (h - y)\cot\gamma$

from ΔBAM : $\quad AM = a\cos\beta$

$BM = y = a\sin\beta$

But $\quad AQ = AM = BM$

$$\Rightarrow \quad h\cot\alpha = a\cos\beta + (h + y)\cot\gamma$$

$$\Rightarrow \quad h\cot\alpha = a\cos\beta + (h - a\sin\alpha)\cot\gamma$$

$$\Rightarrow \quad h = \frac{a\cos\beta - a\sin\beta\cot\gamma}{\cot\alpha - \cot\gamma}$$

$$\Rightarrow \quad h = \frac{2[\cos\beta\sin\alpha\sin\gamma - \sin\beta\cos\gamma\sin\alpha]}{\sin\gamma\cos\alpha - \cos\gamma\sin\alpha}$$

$$\Rightarrow \quad h = \frac{a\sin\alpha\sin(\gamma-\beta)}{\sin(\gamma-\alpha)}$$

Example : 30

Due south of a tower which is leaving towards north, there are two stations at distances x, y respectively from its foot. If α and β are the angles of elevation of the top of the tower at these station respectively, show that the inclination of the tower to the horizontal is given by : $\cot^{-1}\left(\frac{y\cot\alpha - x\cot\beta}{y-x}\right)$

Solution

Let PQ be the tower and θ be its inclination with the horizontal. At A, elevation of the top is α and at B, the elevation is β

Let PM is perpendicular to the ground and PM = h

from ΔPQM : MQ = h cot θ

from ΔPAM : AM = h cot α

from ΔPBM : BM = h cot β

$\Rightarrow$ AM – QM = x $\Rightarrow$ h cot α – h cot = x ....(i)

$\Rightarrow$ BM – QM = y $\Rightarrow$ h cot β – h cot θ = y ....(ii)

dividing (i) by (ii), we get

$$\Rightarrow \quad \frac{\cot\alpha - \cot\theta}{\cot\beta - \cot\theta} = \frac{x}{y}$$

$$\Rightarrow \quad \cot\theta = \frac{y\cot\alpha - x\cot\beta}{y-x}$$

$$\Rightarrow \quad \theta = \cot^{-1}\left[\frac{y\cot\alpha - x\cot\beta}{y-x}\right]$$

Example : 31

The width of a road is b feet. On one side of the road, there is a pole h feet high. On the other side, there is a building which subtends an angle θ at the top of the pole. Show that the height of the building is $\frac{(b^2+h^2)\sin\theta}{b\cos\theta + h\sin\theta}$.

Solution

Let PQ = y be the height of the building

Let AB = h be the height of the pole.

Let $\angle$QAB = α = $\angle$AQP

from ΔAOB :

AQ = $\sqrt{b^2+h^2}$ and $\sin\alpha = \frac{b}{\sqrt{b^2+h^2}}$, $\cos\alpha = \frac{h}{\sqrt{b^2+h^2}}$

in ΔAPQ

$\angle$APQ = π = (θ + α)

using the sine rule in this triangle

$$\frac{y}{\sin\theta} = \frac{AQ}{\sin\angle APQ}$$

$$\Rightarrow \quad \frac{y}{\sin\theta} = \frac{\sqrt{b^2+h^2}}{\sin(\pi - \overline{\theta+\alpha})}$$

$$\Rightarrow \quad y = \frac{\sqrt{b^2+h^2}\sin\theta}{\sin\theta\cos\alpha + \cos\theta\sin\alpha}$$

$$\Rightarrow \quad y = \frac{\sqrt{b^2+h^2}\sin\theta}{\frac{h}{\sqrt{b^2+h^2}}\sin\theta + \frac{b}{\sqrt{b^2+h^2}}\cos\theta}$$

$$\Rightarrow \quad y = \frac{(b^2+h^2)\sin\theta}{h\sin\theta + b\cos\theta}$$

Example : 32

The angle of elevation of a tower at a point A due south of it is 30º. AT a point b due east of A, the elevation is 18º. If AB = a, show that the height of the tower is : $\frac{a}{\sqrt{2+2\sqrt{5}}}$

Solution

Let PQ = h be the height of the tower.

At A, due to south of it, the elevation is $\angle PAQ = 30^o$

At B, due east of A, the elevation is $\angle PBQ = 18^o$

from ΔPAQ : AQ = h cot 30º

from ΔPBO : BQ = h cot 18º

Now consider the right angled triangle ΔAQB in the horizontal plane :

$AQ^2 + AB^2 = BQ^2$

$h^2 \cot^2 30^o + a^2 = h^2 \cot^2 18^o$

$$\Rightarrow \quad h = \frac{a}{\sqrt{\cot^2 18^o - \cot^2 30^o}}$$

we have $\cot^2 30^o = 3$ and $\cot 18^o = 5 + 2\sqrt{5}$ (try to calculate it yourself)

$$\Rightarrow \quad h = \frac{a}{\sqrt{2+2\sqrt{5}}}$$

Example : 33

A circular plate of radius a touches a vertical wall. The plate is fixed horizontally at a height b above the ground. A lighted candle of length c stands vertically at the centre of the plate. Prove that the breadth of the shadow thrown on the wall where it meets the horizontal ground is : $\frac{2a}{c}\sqrt{b^2+2bc}$

Solution

Let r be the radius of the circle formed by the shadow of the plate on the ground

Length of candle = PQ = c

$$\frac{r}{a} + \frac{c+b}{c}$$

$$\Rightarrow \quad r = \frac{a}{c}(c+b)$$

let AB be the shadow cut by the vertical wall.

$$\Rightarrow \quad AB = \sqrt{r^2 - a^2} = 2\sqrt{\frac{a^2}{c^2}(c+b)^2 - a^2}$$

$$\Rightarrow \quad AB = \frac{2a}{c}\sqrt{(c+b)^2 - c^2} = \frac{2a}{c}\sqrt{b^2+2bc}$$

Example : 34

A man standing south of a lamp-post observes his shadow on the horizontal plane to be 24 feet long. On walking eastward a distance of 300 feet, he finds that his shadow is now 30 feet. If his height is 6ft, find the height of the lamp above the horizontal plane.

Solution

Let PQ be the lamp-post and AB be the man in his initial position. He moves from AB to A′B′.

$\Rightarrow$ AA′ = 300ft and AX = 24ft

initial length of the shadow = AX = 24ft.

final length of the shadow = A′Y′ = 30ft

ΔQXP ~ ΔBXA

$$\Rightarrow \quad \frac{PQ}{AB} = \frac{PX}{AX} \quad \Rightarrow \quad \frac{h}{6} = \frac{24 + PA}{24}$$

$\Rightarrow$ PA = 4 h – 24

ΔQYP ~ D′YA′

$$\Rightarrow \quad \frac{PQ}{A'B'} = \frac{PY}{A'Y} \quad \Rightarrow \quad \frac{h}{6} = \frac{30 + PA'}{30}$$

$\Rightarrow$ PA′ = 5h – 30

Apply Pythagoras Theorem in ΔPAA′ :

$\Rightarrow$ $PA^2 + AA'^2 = PA^{\alpha}$

$\Rightarrow$ $(4h - 24) + 300^2 = (5h - 30)^2$

$\Rightarrow$ $9(h - 6)^2 = 300^2$

$\Rightarrow$ h = 106 ft.

Example : 35

An object is observed from three points A, B, C in the same horizontal line passing through the base of object. The angle of elevation at B is twice and at C is thrice that at A. If AB = a, BC = b, prove that the height of the object is : $\frac{a}{2b}\sqrt{(a+b)(3b-a)}$.

Solution

Let PQ be tower of height h.

Let θ , 2θ and 3θ be the angles of elevations of Q at A, B and C respectively

ΔQAB is isosceles $\Rightarrow$ QB = a

from ΔPQC; $QC = \frac{h}{\sin 3\theta}$

Applying sine rule in ΔQBC :

$$\Rightarrow \quad \frac{a}{\sin(\pi - 3\theta)} = \frac{b}{\sin\theta} = \frac{h/\sin 3\theta}{\sin 2\theta}$$

$$\Rightarrow \quad \frac{a}{\sin 3\theta} = \frac{b}{\sin\theta} = \frac{h}{\sin 3\theta \sin 2\theta}$$

$$\Rightarrow \quad \frac{a}{\sin 3\theta} = \frac{b}{\sin\theta}$$

$\Rightarrow$ $a = b(3 - 4\sin^2\theta)$

$\Rightarrow$ $\sin^2\theta = 3b - a$

$$\Rightarrow \quad \cos^2\theta = 1 - \left(\frac{3b - a}{4b}\right) = \frac{b + a}{4b}$$

$$\Rightarrow \quad \frac{a}{\sin 3\theta} = \frac{h}{\sin 3\theta \sin 2\theta}$$

$\Rightarrow$ $h = a \sin 2\theta$

$\Rightarrow$ $h = 2a \sin\theta \cos\theta$

$\Rightarrow \quad h = 2a \sqrt{\dfrac{3b-a}{4b}} \sqrt{\dfrac{b+a}{4b}}$

$\Rightarrow \quad h = \dfrac{a}{2b} \sqrt{(3b-a)(b+a)}$

Example : 36

A flagstaff on the top of a tower is observed to subtend the same angle α at two points on a horizontal plane, which lie on a line passing through the centre of the base of tower and whose distance from one another is 2a and angle β at a point half-way between them. Prove that the height of flagstaff is :

$a \sin \alpha \sqrt{\dfrac{2\sin\beta}{\cos\alpha \sin(\beta-\alpha)}}$

Solution

Let PQ be the tower and QR be the flagstaff. Let QR = 2h and PN = y

QR subtends α at A and B (where N is the mid-point of QR)

$\Rightarrow$ Q, R, A, B are concyclic. Let O be the centre of circle passing through these points.

$\Rightarrow \quad \angle POQ = 2\alpha$ (angle subtended at centre is double)

from ΔNOR : ON = h cot α

OR = radius = h cosec α

Let M be the mid-point of AB where QR subtends β

Let PM = x = ON

$\Rightarrow \quad x = h \cot \alpha$ ....(i)

from ΔOBM : $OM^2 = OB^2 - a^2 = h^2 \operatorname{cosec}^2\alpha - a^2$

$\Rightarrow \quad y^2 = h^2 \operatorname{cosec}^2\alpha - a^2$ ....(ii)

Now $\beta = \tan^{-1} \dfrac{PR}{PM} - \tan^{-1} \dfrac{PQ}{PM} = \tan^{-1} \dfrac{y+h}{x} = \tan^{-1} \dfrac{y-h}{x}$

$\Rightarrow \quad \tan \beta = \dfrac{\dfrac{y+h}{x} - \dfrac{y-h}{x}}{1+\dfrac{y^2-h^2}{x^2}} \quad \Rightarrow \quad \tan \beta = \dfrac{2hx}{x^2+y^2-h^2}$

From (i), (ii) and (iii), we will eliminate x and y to get h.

$\Rightarrow \quad \tan \beta\, (h^2 \cot^2\alpha + h^2 \operatorname{cosec}^2\alpha - a^2 - h^2) = 2h\, (h \cot \alpha)$

$\Rightarrow \quad \tan \beta\, (2h^2 \cot^2\alpha - a^2) = 2h^2 \cot \alpha$

$\Rightarrow \quad h^2 = \dfrac{a^2 \tan\beta}{2\tan\beta\cot^2\alpha - 2\cot\alpha} = \dfrac{a^2 \sin\beta \sin^2\alpha}{2\sin\beta\cos^2\alpha - 2\cos\beta\cos\alpha\sin\alpha}$

$\Rightarrow \quad h^2 = \dfrac{a^2 \sin\beta \sin^2\alpha}{2\cos\alpha \sin(\beta-\alpha)} \quad \Rightarrow \quad h = a \sin \alpha \sqrt{\dfrac{\sin\beta}{2\cos\alpha\sin(\beta-\alpha)}}$

$\Rightarrow$ height of flagstaff = 2h = $a \sin \alpha \sqrt{\dfrac{2\sin\beta}{\cos\alpha\sin(\beta-\alpha)}}$