Functional Analysis

Banach Spaces

Vector space—A vector space over the field F, consists a set X, a mapping $(x, y) \to x + y$ of X. X into X and a mapping $(a, x) \to ax$ of F. X into X, such that :

(a) X is an Abelian group with group operation $(x, y) \to x + y$

(b) $$a(bx) = (ab)x,$$

$x \in$ X, $a, b \in$ F (associativity)

(c) $$(a + b)x = ax + bx$$
$$a(x + y) = ax + ay,$$

$x, y \in$ X, $a, b \in$ F (distributivity)

(d) $1 \cdot x = x$ (1, and identity with respect to multiplication)

Direct product—Let X_α, $\alpha \in$ A be a set of vector spaces. The direct product $X = \pi$ (Xα : $\alpha \in$ A] has an elements, the functions $[x_\alpha]$, where $x_\alpha \in$ X for each $\alpha \in$ A, and their scalar multiplication and addition are given by a $[x_\alpha] = [ax_\alpha]$ and $[x_\alpha + y_\alpha] = [x_\alpha] + [y_\alpha]$ respectively.

Direct sum—Let X_α, $\alpha \in$ A be a set of vector spaces whose identity elements are all designated as 0. The direct sum of vector spaces X_α, $\alpha \in$ A is $X = \sum [X_\alpha : \alpha \in A]$, consists of those $[x_\alpha] \in \pi [x_\alpha : \alpha \in A]$ for which $x_\alpha = 0$ for all but finite number of the $\alpha \in$ A.

1. $\sum [X_\alpha : \alpha \in A]$ is a vector subspace of $\pi [X_\alpha : \alpha \in A]$
2. Two spaces are identical iff A is finite.

Linear transformation—A linear transformation f is a mapping $f : X \to Y$ on X into Y, such that $f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2)$ for all $x_1, x_2 \in \times$ and $\alpha, \beta \in$ R.

L (X, Y)—The set of linear transformations from vector space X into Y and it is also a vector space.

L (X, R)—The set of linear transformations from vector space X into R, the elements of it are called linear functional. It is also called Algebraic dual denoted by X*.

Sub Spaces

Sub space—Let X be a vector space. A subset $Y \subset X$ is a subspace of X, if $x_1, x_2 \in$ X and $\alpha, \beta \in$ R the elements $\alpha x_1 + \beta x_2 \in$ Y.

Improper subspace—X and null vector space {0} of X.

Proper subspace—$Y \subset X$ and $Y \neq X$ and Y = {0}.

Intersection of sub spaces : $Y = \cap \{Y_\alpha : \alpha \in A\}$ where Y_α, $\alpha \in$ A is a set of subspaces of X.

Subspace generated by S, (S) : $S \subset X$, the sub space (S) generated by S is a smallest subspace of X that contains S.

Disjoint subspaces—Subspaces Y, Z of X are disjoint if $Y \cap Z = \{0\}$.

Hahn Banach Theorem

Hahn Banach Theorem—If S is a convex set which contains an interior point and $0 \notin$ S, there is a hyperplane H, passing through 0, such that all of S is on the same side of H.

1. If X is a vector space and S is a convex set in X, which contains an interior point, and if Y is a subspace of X which has no points in common with S, then there is a hyperplane H such that $Y \subset H$ and all points of S are on the same side of H.
2. If X is a vector space and S is a convex set in X and Y is a translation of a subspace of X which has no points in common with S, then there is a hyper plane H (not necessarily a sub space) containing Y, such that all points in S are on the same side of H.

Extension form : Let X be a vector space, Y is a subspace of X and P a real function on X such that

$$P(x) \geq 0, P(x + y) \leq P(x) + P(y)$$
and $$P(ax) = |a| P(x)$$

If f is a linear functional on Y such that $|f(x)| \leq P(x)$ for every $x \in Y$, there is a linear F on X such that $F(x) = f(x)$ on Y and $|F(x)| \leq P(x)$ on X.

Extension form—Let X be a (real) vector space, P a real valued function on X such that

$$P(x+y) \leq P(x) + P(y)$$

and $\qquad P(ax) = aP(x)$

for $a \geq 0$ and Ya sub space of X.

If f is linear on Y and $f(x) \leq P(x)$ for every $x \in Y$, there is a linear F on X such that

$$F(x) = f(x) \text{ on Y}$$

and $\qquad F(x) \leq P(x)$ on X.

Banach Limits Let m be the vector space of all bounded sequences of real numbers. C the sub-space of all convergent sequences. If $x \in C$, then $l(x) = \lim_n x_n$ is defined and l is a linear functional on C.

Banach limit—A Banach limit is any linear functional L defined on m such that :

(a) $L(x) \geq 0$ if $x_n \geq 0$ for all n

(b) $L(x) = L(\sigma x)$, where $\sigma x = \sigma(x_1, x_2, \ldots\ldots) = (x_2, x_3, \ldots\ldots)$

(c) $L(x) = 1$ if $x = (1, 1, \ldots\ldots)$

Almost convergent and F-limit—$x \in m$ is called almost convergent and the number S is called F-limit of x if $L(x) = s$ for all Banach limits L.

P (x) and P′ (x) : $P(x) = \inf \overline{\lim_{j \to \infty}} \frac{1}{k} \sum_{i=1}^{k} x_{n_i} + j$

and $P'(x) = -P(-x), x \in m$.

Theorems

1. If L is a Banach limit, then $\underline{\lim}\, x_n \leq L(x) \leq \overline{\lim}\, x_n$ for all $x \in m$.
2. For any $x \in m$, $\lim x_n \leq P'(x) \leq P(x) \leq \lim x_n$. In particular, $P(x) = l(x)$ for $x \in c$.
3. For any Banach limit L and $x \in m$, $P'(x) \leq L(x) \leq P(x)$.
4. $P(x)$ is such that $P(x+y) \leq P(x) + P(y)$ and $P(ax) = aP(x)$ if $a \geq 0$.
5. x is almost convergent iff $P'(x) = P(x)$.
6. x is almost convergent and F-limit of x is x iff $\lim_{P \to \infty} \frac{1}{P}(x_n + x_{n+1} + \ldots\ldots + x_{n+P-1}) = S$ holds uniformly in n.

Banach Spaces Dual Space

Norm—Let X be a vector space and $x \in X$. A real valued function on a vector space X, $\|x\|$ is a norm on X if,

(a) $x \neq 0 \Rightarrow \|x\| > 0$

(b) $\|\alpha x\| = \alpha \|x\|$

(c) $\|x + y\| \leq \|x\| + \|y\|$

Semi-norm—Let X be a vector space, Y is a sub space of X and P a real valued function on X :

(a) $x \neq 0 \Rightarrow P(x) > 0$

(b) $P(x+y) \leq P(x) + P(y)$

(c) $P(\alpha x) = |\alpha| P(x)$

Banach space—A complete normed vector space is Banach space.

Bounded linear transform—A linear transform T is bounded if there is an integer M such that

$$\|T_x\| \leq M \|x\| \text{ for every } x \in X.$$

Norm (bound) or T—Let T be a bounded linear transform and

$$\|T\| = \text{Inf}\{M : \|T_n\| \leq M \|x\|, x \in X\}$$

Then $\|T\|$ is norm (bound) of T.

Here, $\|T_x\| \leq \|T\| \|x\|$ for every $x \in X$.

Dual of X—If X is a normed vector space, then space X′ of continuous linear functionals on X, with norm.

$$\begin{aligned} \|x'\| &= \sup\{|x'(x)| : \|x\| = 1\} \\ &= \inf\{M : |x'(x)| \\ &\leq M\|x\|, x \in X \end{aligned}$$

is called dual of X.

Theorems

1. **Hahn Banach theorem**—If X is a normed vector space, Y is a sub space of X and f is a bounded linear functional on Y with bound $\|f\|$, relative to Y, then f has a continuous linear extension to an $x' \in X'$ with $\|x'\| = \|f\|$.
2. For any $x \neq 0$ in X there is an $x' \in X'$, such that $x'(x) = \|x\|$ and $\|x'\| = 1$.
3. If X is a complete normed vector space and f is a continuous linear functional on Y, a sub space of X, then f can be extended to a linear functional F on X such that

$$\|F\| = \|f\|$$

4. For every normed vector space X there is a set A such that X is isomorphic with a sub space of the Banach space of bounded functions f on A with

$$\|f\| = \sup \{|f(t)| : t \in A\}$$

Uniform Boundedness Principle

1. Let X be a Banach space $\{f_n\}$ is a sequence of continuous linear functional on X and for every $x \in X$, the sequence $\{|f_n(x)|\}$ is bounded, then the sequence of $\{\|f_n\|\}$ of norms is bounded.

   If X is a Banach space, Y is a normed vector space and $\{T_n\}$ a sequence of continuous linear transforms on X into Y such that for every $x \in X$, then sequence $\{\|T_n(x)\|\}$ is bounded, then the sequence $\{\|T_n\|\}$ of norm is bounded.

2. Let X be a Banach space and Y a normed vector space for every $m = 1, 2, \ldots\ldots$, Let $\{T_n(x, m)\}$ be a sequence of continuous linear operators on X with values in Y such that for every m there is an $x_m \in X$ for which

   $\overline{\lim_n} \|T_n(x_m, m)\| = +\infty$. Then there is an $x \in X$ for which $\overline{\lim_n} \|T(x, m)\| = +\infty$, $m = 1, 2\ldots\ldots$

3. Let X be a Banach space and Y a normed vector space. For every $m = 1, 2, \ldots\ldots$ let $\{T_n(x, m)\}$ be a sequence of continuous linear operators on X with values in Y such that for every m, there is an $x_m \in X$ for which $\{T_n(x_m, m)\}$ does not converge. Then there is an $x \in X$ for which $\{T_n(x, m)\}$ does not converge, for any $m = 1, 2, \ldots\ldots$

4. If X is a Banach space and Y is a subspace of X which is a Borel set, then Y is either of first category in X or is identical with X.

Lemma of Riesz

1. If Y is a closed proper subspace of X and $\in > 0$, there is an $x_\in$ on the unit sphere such that

$$\text{Inf}\{\|x_0 - x_\in\| : x_0 \in Y\} > -\in$$

2. A normed vector space is finite dimensional iff the closed bounded sets are compact.

3. There is a uniform convergent sequence of functions of finite Baire type whose limit is not of finite Baire type.

Compact Transformation

Compact transformation—Let X and Y be Banach spaces. A linear transformation T on X into Y is called compact if every bounded set in X is taken by T into a set whose closure is compact.

B (X, Y)—Let X and Y be Banach spaces then B (X, Y) is the set of linear transformation from X into Y. It is also a Banach space.

Adjoint of transformation—Let X and Y be Banach spaces. Let X, ∞′ and Y′ be the dual of X and Y respectively. Then X′ and Y′ are Banach spaces.

Let $T \in B(X, Y)$, then adjoint of T, T^* is a transformation $T^* \in B(Y', X')$ defined as : for $y' \in X'$ for each $x \in X$ has the value.

$$(T^* y')(x) = y'(Tx)$$

Characteristic number—Given the equation $x - \lambda Tx = 0$.

The number $\lambda_0 \in R$ is called characteristic number of this equation if $x - \lambda_0 Tx = 0$ has a non zero solution.

Theorems

1. The subset $C(X, Y) \subset B(X, Y)$ of compact transformation is a Banach space.
2. If $T \in C(X, X)$, X a Banach space, then vector space of solution of $x - Tx = 0$ is finite dimensional.
3. If $T \in C(X, X)$, then set of characteristic number of $x - \lambda Tx = 0$ is an isolated set.

Weak Convergence

Weak convergence—If X is a normed vector space and $x'_n \in X'$, $n = 1, 2, \ldots\ldots$ we say x'_n convergence weakly to $x' \in X'$ if for each $x \in X$, $\lim_{n \to \infty} x'_n(x) = x'(x)$.

Total set—If X is a normed vector space, $Y \subset X$ is total if its closed linear space is X.

A-limit—A sequence $x = (x_1, x_2, \ldots\ldots)$ is said to have A-limit, A (x) if each $i = 1, 2, 3\ldots\ldots$ the sum $A_i(x) = \sum_{k=1}^{\infty} a_{ik} x_k$ is convergent and lim $A_i(x) = A(x)$.

A-regular—A is regular if for each convergent sequence x, the A-limit exists and equals the ordinary limit of the sequence x.

A-almost regular—A is called almost regular if for every convergent sequence $x = (x_1,$

$x_2, \ldots\ldots$) there is an N such that for every $m > N$, $\sum_{n=1}^{\infty} a_{mn} x_n$ convergence and

$$\lim_{m \to \infty} \sum_{n=1}^{\infty} a_{mn} x_n = \lim_{n \to \infty} x_n$$

Theorems

1. If X is a Banach space, then (x'_n) converges weakly to $x' \in X'$ iff the sequence $\{\| x'_n \|\}$ of norms is bounded and $\lim_n x'_n (x) = x' (x)$ for every x in some total subset of X.
2. A is regular iff
   (a) $\sum_{k=1}^{\infty} | a_{ik} | \le M, i = 1, 2, \ldots\ldots$for some M.
   (b) $\lim_i a_{ik} = 0, k = 1, 2, \ldots\ldots$ and
   (c) $\lim_i \sum_{k=1}^{\infty} a_{ik} = 1$.
3. A is almost regular iff there is a fixed m_0 such that :
   (a) $\sum_{m=1}^{\infty} | a_{mn} | < M$ for fixed M and every $m > m_0$, (b) $\lim_n a_{mn} = 0$ for every n, and
   (c) $\lim_m \sum_{n=1}^{\infty} a_{mn} = 1$.
4. If $\{S_n^{(m)}\}$ is a sequence of sequences such that $\sum_{n=1}^{\infty} | S_n^{(m)} | = \infty$. $m = 1, 2, \ldots\ldots$, there is a sequence $\{t_n\}$ converging to zero, such that all the series $\sum_{n=1}^{\infty} S_n^{(m)} t_n, m = 1, 2, \ldots$ diverge.

Dual of l_P—If $l \le P \le \infty$, and q is conjugate to P, then $(l_p) = l_q$.

Second dual space—Let X be normed vector space and X′ be the dual of X.

For every $x_0 \ne 0 \in X$ consider the functional x_0 on X′ defined as $x_0 (x') = x' (x_0)$.

Dual of C (*a, b*) and Riesz Representation Theorem

BV (*a, b*)—Class of all functions of bounded variation on $[a, b]$ *i.e.*, for all f which the total variation $V (f) = \sup \sum_{i=1}^{\infty} | f (x_i) - f (x_{i-1}) |$ is finite, where the supremum is taken over all partition $a = x_0 < x_1 < \ldots\ldots < x_n = b$.

BVN [*a, b*]—BV $[a, b]$ which satisfies normalizing condition C $[a, b]$: metric space of all continuous functions.

Riesz representation theorem—If $F \in C'$ $[a, b]$, there exist $g \in$ BVN $[a, b]$ such that

$$F (f) = \int_t^b f dg,$$

where $f \in C [a, b]$ and $\| g \| = V (g) = \| F \|$.

Open Mapping and Closed Graph Theorem

Closed linear transformation—A linear transformation T from Banach space X into a Banach space Y with dom (T) $\subset$ X is closed if $x_n \in$ dom (T),

$$\lim_n x_n = x_n$$

and $\lim_n Tx_n = y \Rightarrow x \in$ dom (T)

and $y = Tx$

Graph—A graph G (T) of mapping $T : X \to Y$ is the set of points $(x, Tx) \in$ X. Y, with $x \in$ dom T.

Theorems

1. If dom (T) is closed in X and T is bounded then T is closed.
2. T is a closed linear transformation iff G (T) is a closed vector subspace of X, Y.

Closed graph theorem—If X and Y are Banach spaces and T is a linear transformation from X and Y, then dom (T) is closed and graph G (T) closed $\Rightarrow$ T is bounded.

Open mapping theorem—If X and Y are Banach spaces and T is bounded linear transformation which maps X on to all of Y, then T is an open mapping (*i.e.*, the image of open set in X under T, is open set in Y)

1. Let X and Y are complete metric space and F is continuous mapping of X onto Y such that for every $r > 0$, there is $k > 0$ such that for every $x \in X$, the closure of the image of the sphere, $\sigma (x, r)$ in X, of centre x and radius r, contains the sphere $\sigma [F (x), k]$ in Y, then for every $\rho > r$ the image of $\sigma (x, \rho)$ contains $\sigma [F (x), k]$.

Hilbert Spaces

Inner product—Let X be a vector space over the field F (real or complex). A mapping of X. X into F, which takes each ordered pair $(x, y) \in$ X. X into the number $(x, y) \in F$ is called inner product in X if,

(i) $(x, y) = \overline{(y, x)}$

(ii) $(\alpha x_1 + \beta x_2, y) = \alpha (x_1, y) + \beta (x_2, y)$ (linearity)

(iii) $(x, x) \geq 0$ and $(x, x) = 0$ iff $x = 0$

1. If X is a real vector space, then $(x, y) = (y, x)$
2. $(x, \alpha y) = \bar{\alpha} (x, y)$

Norm—Let $x \in X$, then norm in the space X is $\| X \| = \sqrt{(x, x)} = (x, x)^{1/2}$.

Inner product space—A vector space X, together with an inner product in X, is called **inner product space.**

Hilbert space—If X is complete under the norm obtained from its inner product, then X is called **Hilbert space.**

Theorems

Cauchy-Schwarz inequality—For evey

$$x, y \in X, |(x, y)| \leq \|x\| \|y\|$$

The Cauchy-Schwarz inequality states that inner product space is a continuous function on X. X into F, where the norm of $\{x, y\} \in X \cdot X$ is $\|x\| + \|y\|$.

i.e., $\lim x_n = x, \lim y_n = y$

$\Rightarrow$ $\lim (x_n, y_n) = (x, y)$

Pythagorean theorem—

$$x \perp y$$

$\Rightarrow$ $\|x\|^2 + \|y\|^2 = \|x + y\|^2$

is valid in inner product space.

Parallelogram law—

$$\|x + y\|^2 + \|x - y\|^2 = 2 \{\|x\|^2 + \|y\|^2\}$$

holds in inner product space.

Banach space is Hilbert space iff the parallelogram law holds.

Projection theorem and dual—If k is a closed convex set in X and $x_0 \in X \sim k$, then there is a unique $y_0 \in k$, such that

$$\|x_0 - y_0\| = \inf \{\|x_0 - y\| : y \in k\}$$

Projection theorem—If M is a closed subspace of a Hilbert space X, then every $x \in X$ has a unique decomposition $x = y + z$, $y \in M$, $z \in M^\perp$, where $M^\perp = \{z \in X : z \perp y \text{ for all } y \in M\}$.

The operator $x \rightarrow P_x = y$ is a bounded linear idempotent (a-projection) such that $M = PX$ and $M^\perp = (1 - P) X$ are mutually orthogonal closed subspaces with $X = M + M^\perp$.

For each $x \in X$, P_x is the unique element of M which minimizes the distance from x to M.

If x' is a bounded linear functional on a Hilbert space X, then there is a unique $y_0 \in X$ such that

$$x'(x) = (x, y_0)\ x \in X$$

1. Let V be a bounded linear operator on a Hilbert space X with

$$\|V\| \leq 1$$

and $f \in X$

Put $\phi_n = \frac{1}{n} [f + Vf + \ldots + V^{n-1} f]$, $n = 1, 2 \ldots$

Then $\phi \in X$, is such that

$$\lim \phi_n = \phi \text{ in } X$$

Orthonormal Sets and Fourier Expansion

Orthogonal set—A set S of elements of an inner product space X is an orthogonal set if $x \perp y$ whenever $x, y \in S$ and $x \neq y$.

Orthonormal set—An orthonormal set S such that for each $x \in S$,

$$(x, y) = \begin{cases} 0, & \text{if } x \neq y \\ 1, & \text{if } x = y \end{cases} \quad x, y \in S$$

Complete set—A set S is complete in X if no non-zero $x \in X$ is orthogonal to S.

Fourier coefficients—Let $x \in X$ and $\{x_\alpha\}$ an orthonormal family in X, possibly uncountable.

Then the fourier coefficients of x with respect to $\{x_\alpha\}$ are the numbers.

$$(x, x_\beta) = \sum a_\alpha (x_\alpha, x_\beta) = a_\beta$$

Orthonormal basis—Let X be a Hilbert space and $S = \{x_\alpha : \alpha \in A\}$ is an orthonormal set in X, then S is a orthonormal basis orthonormal closed set) in X if the fourier series representation $x = \sum_{\alpha \in A} (x, x_\alpha) x_\alpha$ is valid for each $x \in X$.

Theorems—If $\{x_1, x_2, \ldots\ldots x_n\}$ is an orthonormal set, then for any $x \in X$,

$$\sum_{i=1}^{n} |x, x_i|^2 \leq \|x\|^2$$

Bessel's inquality—If X is a Hilbert space $S = \{x_\alpha : \alpha \in A\}$ is an orthonormal set in X and $x \in X$, then

$$\sum_{\alpha \in A} |(x, x_\alpha)|^2 \leq \|x\|^2$$

Let X be an Hilbert space and S = $\{x_\alpha : \alpha \in A\}$ is an orthonormal set in X, then S is a basis iff it is complete in X.

Parseval's formula—Let X be an Hilbert space, S = $\{x_\alpha : \alpha \in A\}$ is an orthonormal set. Then S is an basis iff equality holds in Bessel's inequality.

$$\| x \|^2 = \sum_{\alpha \in A} |(x, x_\alpha)|^2, x \in X$$

or, $$(x, y) = \sum_{\alpha \in A} (x, x_\alpha) \overline{(y, x_\alpha)},$$

where $x, y \in X$.

Parseval's formula (for real trigonometric system)—

$$a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \int_0^{2\pi} |f(t)|^2 \, dt,$$

where $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt = b_n \sin nt)$ is the fourier series of $f \in L_2 (0, 2\pi)$.

Isoperimetric theorem—Among all single, closed piecewise smooth curves of length L in the plane the circle encloses the maximum area.

Muntz Theorem

Gramm determinant—Let $x_1, x_2, \ldots\ldots x_n \in L_2 [0 \cdot 1]$.

The gramm determinant of $x_1, x_2, \ldots\ldots x_n$ is

$$G = G(x_1, x_2, \ldots\ldots x_n)$$

$$= \begin{vmatrix} (x_1, x_1) & (x_1, x_2) & \ldots\ldots & (x_1, x_n) \\ (x_2, x_1) & (x_2, x_2) & \ldots\ldots & (x_2, x_n) \\ \vdots & & & \\ (x_n, x_1) & (x_n, x_2) & \ldots\ldots & (x_n, x_n) \end{vmatrix}$$

Theorems—A necessary and sufficient condition for $(x_1, x_2, \ldots\ldots x_n)$ to be linearly independent is that $G \neq 0$.

Muntz theorem—A necessary and sufficient condition for the set t^n 1.

t^n 2, ……, $1 \leq n_1 < n_2 < \ldots\ldots$ to be complete in L_2 is that $\sum_{i=1}^{\infty} \frac{1}{n_i} = \infty$.

Dimension and Riesz-Fischer Theorem

Dimension—Orthogonal dimension of a Hilbert space is the cardinality of its bases.

Theorems

1. The cardinality is the same for each basis of a given Hilbert space.
2. The Hilbert space X is separable iff there is a countable basis for X.
3. Two Hilbert spaces are isomorphic if they have the same dimension.
4. **Riesz-fischer theorem**—If $\sum_{n=\infty}^{\infty} |C_n|^2 < \infty$, then there is $f \in L_2 (0, 2\pi)$ such that $\sum_{-\infty}^{\infty} C_n e^{inx}$ is the fourier series of f and coverges to f in the sense of L_2.

Reproducing Kernel

Proper functional space—A vector space X of functions on a set S, together with a norm in X, is called a proper functional space if for every $s \in S$, the evaluation functional at S is continuous, *i.e.*, there exist M such that $|x(S)| \leq M_S \| x \|, x \in X$.

Proper functional completion—A proper functional completion of X is a proper functional space $\overline{X}$ on the same basic set S such that $\overline{X}$ is complete and X is a dense subspace of $\overline{X}$.

Reproducing Kernel—A Hilbert space of functions on a set S is said to have a reproducing Kernel if there is a function $k(S, t)$ on S.

S has the propeties :

(i) $k(\cdot, t) \in X, t \in S$

(ii) $x(t) = [x, k(\cdot, t)], x \in X, t \in S$ (reproducing property).

Positive matrix—A function $k(S, t)$ on S. S is a positive matrix if for each $n = 1, 2, \ldots\ldots$ and each choice of points $t_1, t_2, \ldots\ldots t_n$ the quardratic form $\sum_{i,j=1}^{n} k(t_i, t_j) \xi_i \overline{\xi}_j$ in $\xi_1, \xi_2, \ldots\ldots \xi_n$ is non negative.

Theorems

1. A proper functional space X has a proper functional completion iff for each Cauchy sequence $\{x_n\}$ in X, $\lim_n x_n (S) = 0$ for all $s \in S$

$$\Rightarrow \quad \lim_n \| x \| = 0$$

2. If X has a reproducing Kernel, then $|x(t)| = |(x, k(\cdot, t)| \leq \| k(\cdot, t) \| \| x \|, x \in X$, $t \in S$ and X is proper functional space.

3. Each reproducing Kernel is a positive matrix.
4. If X is a proper functional Hilbert space with $k(S, t)$ as reproducing Kernel and if $[x_\alpha : \alpha \in A]$ is a complete orthonormal set in X, then

$$k(\cdot, t) = \sum_{\alpha \in A} (k(\cdot, t), x_\alpha) x_\alpha$$
$$= \sum_{\alpha \in A} x_\alpha, t \in S$$

or, $$k(S, t) = \sum_{\alpha \in A} x_\alpha(S)\, \overline{x_\alpha(t)}$$

5. The mapping function f can be expressed in terms of k by the formula

$$f(z) = \sqrt{\frac{\pi}{k(\xi, \xi)}}\, k(z, \xi)$$

Adjoint Operator

Adjoint—Let T be a bounded linear operator of Hilbert space H, then there exist an unique operator on H called adjoint of T, T^* such that $(T_x, y) = (x, T^* y)$ $x, y \in H$ and also

$$\| T^* \| = \| T \|$$

1. $(T + S)^* = T^* + S^*$
2. $(\alpha T)^* = \bar{\alpha} T^*$
3. $(ST)^* = T^* S^*$
4. $T^{**} = T$
5. $\| T^* T \| = \| T \|^2$

Bounded Operator

Bounded linear operator—Let T be a bounded linear operator on Hilbert space H, then

(a) T is normal if $T^* T = TT^*$

(b) T is self adjoint if $T^* = T$

(c) T is unitary if $T^* T = 1 = TT^*$, 1 being and identity operator on H

(d) T is a projection if $T^2 = T$.

Theorems

1. If T is a bounded linear operator on Hilbert space H, then

   (a) T is normal iff $\| T_x \| = \| T^*_x \|$ for every $x \in H$.

   (b) If T is normal and $Tx = \alpha x$ for $x \in H$, $\alpha \in c$, then

   $$T^* x = \bar{\alpha} x$$

   (c) If T is normal and if α, β are distinct eigen values of T, then the corresponding eigen spaces are orthogonal to each other.

2. If U is an unitary operator on H then $(U_x, U_y) = (x, y)$ and $\| Ux \| = \| x \|$ for every $x \in H$.
3. If P is a projection on H, then

   (a) P is self adjoint

   (b) P is normal

   (c) $(Px, x) = \| Px \|^2$ for every $x \in H$.
4. If M, N, T are bounded linear operators. M, N are normal operators and MT = NT, then

$$M^* T = TN^*$$

Some Solved Examples

Example 1. If x and y are any two vectors in a Hilbert space, then prove that

(i) $\| x + y \|^2 + \| x - y \|^2 = 2 \| x \|^2 + 2 \| y \|^2$

(ii) $4(x, y) = \| x + y \|^2 - \| x - y \|^2 + i \| x + iy \|^2 - i \| x - iy \|^2$.

Solution : (i) We have

$$\| x + y \|^2 = (x + y, x + y)$$
$$= (x, x + y) + (y, x + y)$$
$$= (x, x) + (x, y) + (y, x) + (y, y) \quad \ldots(1)$$

Also

$$\| x - y \|^2 = (x - y, x - y)$$
$$= (x, x - y) - (y, x - y)$$
$$= (x, x) - (x, y) - (y, x) + (y, y)$$
$$= \| x \|^2 - (x, y) - (y, x) + \| y \|^2 \quad \ldots(2)$$

Adding (1) and (2), we get

$$\| x + y \|^2 + \| x - y \|^2 = 2 \| x \|^2 + 2 \| y \|^2$$

(ii) Subtracting (2) from (1), we get

$$\| x + y \|^2 - \| x - y \|^2 = 2(x, y) + 2(y, x) \quad \ldots(3)$$

Replacing by iy in (3), we get

$$\| x + iy \|^2 - \| x - iy \|^2 = 2(x, iy) + 2(iy, x)$$
$$= 2i(x, y) + 2i(y, x)$$
$$= -2i(x, y) + 2i(y, x) \quad \ldots(4)$$

Multiplying both sides of (4) by i, we get

$$i \| x + iy \|^2 - i \| x - iy \|^2 = -2i^2(x, y) + 2i^2(y, x)$$
$$= 2(x, y) - 2(y, x) \quad \ldots(5)$$

Adding (3) and (5), we get

$$\| x + y \|^2 - \| x - y \|^2 + i \| x + iy \|^2 - i \| x - iy \|^2 = 4(x, y)$$

Example 2. For the special Hilbert space l_2^n, by using Cauchy inequality prove Schwarz's inequality.

Solution : Let $x = (x_1, x_2, \ldots\ldots x_n)$, $y = (y_1, y_2, \ldots\ldots y_n$ be any two members of the Hilbert space L_2^n by Cauchy's inequality.

We have

$$\sum_{i=1}^{n} |x_i y_i| \le \left[\sum_{i=1}^{n} |x_i|^2\right]^{1/2} \left[\sum_{i=1}^{n} |y_i|^2\right]^{1/2}$$

For inner product on l_2^n, we have

$$(x, y) = \sum_{i=1}^{n} x_i y_i$$

$$\therefore |(x \cdot y)| = \sum_{i=1}^{n} |x_i y_i|$$

$$= |x_1 y_1 + x_2 y_2 + \ldots\ldots + x_n y_n|$$

$$= |x_1 y_1| + |x_2 y_2| + \ldots\ldots + |x_n y_n|$$

$$= |x_1 y_1| + |x_2 y_2| + \ldots\ldots + |x_n y_n|$$

$$= \sum_{i=1}^{n} |x_i y_i|$$

$$\le \left[\sum_{i=1}^{n} |x_i|^2\right]^{1/2} \left[\sum_{i=1}^{n} |y_i|^2\right]^{1/2}$$

[by Cauchy's inequality]

$$= \sqrt{(x \cdot y)}\sqrt{(y \cdot y)} \qquad \text{[by (1)]}$$

$$= \|x\| \|y\| \qquad [\because (x, x) = \|x\|^2]$$

Which is the required Schwarz's inequality.

Example 3. If P and Q are the projections on closed linear subspaces M and N of Hilbert space H. Prove that the following statements are all equivalent to each other.

(i) $P \le Q$,

(ii) $\|P_x\| \le \|Q_x\|$ for every x,

(iii) $M \subset N$,

(iv) $PQ = P$,

(v) $QP = P$

Solution : In order to prove that the five given statements are all equivalent to one another. We shall prove that (i) $\to$ (ii) $\to$ (iii), (iii) $\to$ (v), (v) $\to$ (iv), (iv) $\to$ (i)

It should be recalled that if P is any projection on H, then

$$(Px, x) = \|Px\|^2 \ \forall x \in H$$

(i) $\Rightarrow$ (ii), we have

$$P \le Q$$

$$\Rightarrow \quad (Px, x) \le (Qx \cdot x) \ \forall x \in H$$

$$\Rightarrow \quad \|Px\|^2 \le \|Qx\|^2 \ \forall x \in H$$

$$\Rightarrow \quad \|Px\| \le \|Qx\| \ \forall x \in H$$

(iii) $\to$ (iii). It is given that

$$\|Px\| \le \|Qx\| \ \forall x \in H$$

We are to prove that

$$M \subset N$$

Let $x \in M$ since M is the range of of P, therefore

$$x \in M \to Px = x \to \|Px\| = \|x\|$$

$$\Rightarrow \quad \|x\| \le \|Qx\|$$

$[\because \|Px\| \le \|Qx\|$ by hypothesis$]$

$$\Rightarrow \quad \|x\| = \|Qx\|$$

$[\because \|Qx\| \le \|x\| \ \forall x \in H]$

$$\Rightarrow \quad Qx = x$$

$\Rightarrow x \in N$ *i.e.*, the range of Q.

Thus $x \in M \to x \in N$. Therefore $M \subset N$.

Example 4. Prove that if T is an arbitrary operator on a Hilbert space H, and if α and β are Scalars such that

$$|\alpha| = |\beta| \text{ then } \alpha T + \beta T^*$$

is normal.

Solution : We know that

$$(\alpha T + \beta T^*)^* = (\alpha T)^* + (\beta T^*)^*$$

$$= \bar{\alpha} T^* + \bar{\beta} T^{**} = \bar{\alpha} T^* + \bar{\beta} T$$

Now $(\alpha T + \beta T^*)(\alpha T + \beta T^*)^*$

$$= (\alpha T + \beta T^*)(\alpha T^* + \beta T)$$

$$= \alpha\alpha TT^* \ \alpha\beta T^2 + \beta\alpha T^{*2} + \beta\beta T^* T$$

$$= |\alpha|^2 TT^* + \alpha\beta T^2 + \beta\alpha T^{**} + |\beta|^2 T^* T \quad \ldots(1)$$

Also $(\alpha T + \beta T)^* (\alpha T + \beta T^*)$

$$= (\alpha T^* + \beta T)(\alpha T + \beta T^*)$$

$$= (\alpha T^* + \beta T)(\alpha T + \beta T^*)$$

$$= \alpha\alpha T^* T + \alpha\beta T^{*2} + \beta\alpha T^2 + \beta\beta TT^*$$

$$= |\alpha|^2 T^* T + \alpha\beta T^{*2} + \beta\bar{\alpha} T^2 + |\beta|^2 TT^* \quad \ldots(2)$$

Since $|\alpha| = |\beta|$ therefore, we see that the right hand sides of (1) and (2) are equal. Hence, the left hand sides (1) and (2) are also equal therefore, we have

$$(\alpha T + \beta T^*)(\alpha T + \beta T^*)^* = (\alpha T + \beta T^*)^* (\alpha T + \beta T^*)$$

Hence $\alpha T + \beta T^*$ is normal.

Example 5. An operator T on a Hilbert space H is normal iff $\| T^* x \| = \| Tx \|$ for every x.

Solution : We have T is normal

$\Rightarrow \quad TT^* = T^* T$

$\Rightarrow \quad TT^* - T^* T = 0$

$\Rightarrow \quad [(TT^* - T^* T)\, x, x) = 0 \ \forall\, x$

$\Rightarrow \quad (TT^* x, x) = T^* Tx, x) \ \forall\, x$

$\Rightarrow \quad (T^* x, T^* x) = (Tx, T^{**} x) \ \forall\, x$

$\Rightarrow \quad (T^* x, T^* x) = (Tx, Tx) \ \forall\, x$

$\Rightarrow \quad \| T^* x \|^2 = \| Tx \|^2 \ \forall\, x$

$\Rightarrow \quad \| T^* x \| = \| Tx \| \ \forall\, x$

Example 6. Prove that if T is an operator on H, then the following are equivalent :

(1) $\quad T^* T = I$,

(2) $\quad (Tf, Tg) = (f, g)$, for all $f, g \in H$,

(3) $\quad \| Tf \| = \| f \|$, for all $f \in H$.

Solution : Here, (1) $\Rightarrow$ (2), if $T^* T = I$, then

$(Tf, Tg) = (f, T^* Tg)$

$\quad = (f, g)$, for all $f, g \in H$

(2) $\Rightarrow$ (3) if (Tf, Tg)

$\quad = (f, g)$, for all $f, g \in H$,

then in particular

$(Tf, Tf) = (f, f)$

i.e., $\quad \| Tf \| = \| f \|$, for all $f \in H$

(3) $\Rightarrow$ (1), if $\| Tf \|$

$\quad = \| f \|$, for all $f \in H$, then

i.e., $\quad \| Tf \|^2 = \| f \|^2$

$(Tf, Tf) = (f, f)$

i.e., $(T^* Tf, f) - (f, f) = 0$

Which is equivalent to $(T^* T - I)\,(f, f) = 0$ for all $f \in H$. Therefore,

$T^* T = I$

Example 7. Prove that in a Hilbert space the inner product is jointly continuous, *i.e.,* $x_n \to x$, $y_n \to y \Rightarrow (x_n, y_n) \to (x, y)$.

Solution : We have

$| (x_n, y_n) - (x, y) |$

$= | (x_n, y_n) - (x_n, y) + (x_n, y) - (x, y) |$

$= | (x_n, y_n - y) + (x_n - x, y) |$

[by linearity property of inner product]

$\leq | (x_n, y_n - y) | + | (x_n - x, y) |$

$[\because | \alpha + \beta | \leq | \alpha | + | \beta |]$

$\leq \| x_n \| \, \| y_n - y \| + | (x_n - x \| \, \| y \|$

[by Schwarz inequality]

Now $x_n \to x$ and $y_n \to y$ for $n \to \infty$

Therefore, $\| y_n - y \| \to 0$ and $\| x_n - x \| \to 0$, for $n \to \infty$

Therefore, $| (x_n, y_n) - (x, y) | \to 0$ for $n \to \infty$

Hence, $(x_n, y_n) \to (x, y)$

Example 8. Prove that an operator T on a Hilbert space H is unitary iff it is an isometric isomorphism of H onto itself.

Solution : Suppose T is a unitary operator on H, then T is invertible and, therefore, T is onto also $T^* T = I$. Therefore we have

$$\| Tx \| = \| x \| \ \forall\, x \in H$$

Thus T preserves norms and so T is an isometric isomorphism of H onto itself.

Conversely suppose that T is an isometric isomorphism of H onto itself. Then T is one-one and onto. Therefore T – 1 exists. Also T is an isometric isomorphism

$\Rightarrow \quad \| Tx \| = \| x \| \ \forall x, \Rightarrow T^* T = I$

$\Rightarrow \quad (T^* T)\, T^{-1} = IT^{-1}$

$\Rightarrow \quad T^* (TT^{-1}) = T^{-1}$

$\Rightarrow \quad T^* I = T^{-1} \Rightarrow T^* = T^{-1}$

$\therefore TT^* = I = T^* T$ and so T is unitary.

Example 9. Prove that if $P_1, P_2, \ldots\ldots P_n$ are projections on closed linear sub spaces $M_1, M_2, \ldots\ldots, M_n$ of Hilbert space H, then $P = P_1 + P_2 + \ldots\ldots + P_n$ is a projection if and only if the P_i S are pairwise orthogonal, and in this case P is the projection on $M = M_1 + M_2 + \ldots\ldots + M_n$.

Solution : Suppose that $P_1, P_2, P_3, \ldots\ldots, P_n$ are the perpendicular projections and P_i S are pairwise orthogonal. Then we have (a) P is self-adjoint since each P_i is a projection.

$P_j^* = P_j = P_j^2$, for each $j = 1, 2, \ldots\ldots, n$

Therefore,

$P^* = (P_1 + P_2 + \ldots\ldots + P_n)^*$

$= P_1^* + P_2^* + \ldots\ldots + P_n^*$

$= P_1 + P_2 + \ldots\ldots + P_n = P$

(b) P is idempotent. If P_i's are pairwise orthogonal, then

$$P^2 = (P_1 + P_2 + \ldots\ldots + P_n)^2$$

$$= \sum_{j=1}^{n} P_i^2 + 2 \sum_{j=1} P_i P_f = \sum_{i=1}^{n} P_i = P$$

Thus P is idempotent.

(c) If P is idempotent, then P_i's are pairwise orthogonal, if $f \in$ range of P_i, then $P_i(f) = f$ and

$$\| f \|^2 = \| P_i f \|^2 \leq \sum_{i=1}^{n} \| P_j f \|^2$$

$$= \sum_{j=1}^{n} (P_j f, P_j f)$$
$$= \sum_{j=1}^{n} (P_j, f, P_j^* f)$$
$$= \sum_{j=1}^{n} (P_j f, f) = (Pf, f) = (P^2 f, f)$$
$$= (Pf, Pf) = \| Pf \|^2 \leq \| f \|^2$$

Hence, $\sum_{j=1}^{n} \| P_j f \|^2 = \| P_i f \|^2$. Therefore, $\| P_j f \| = 0$, for $j \neq i$, and hence $P_j f = 0$, for $j \neq i$, thus $f \in$ null space of P_j, for $j \neq i$ and hence range of $P_i \subseteq$ null space of P_j for $j \neq i$.

i.e., $M_i \subseteq M_j^{\perp}$, for $j \neq i$.

Consequently, $M_j + M_j$, for $j \neq i$ and so P_i's are pairwise orthogonal *i.e.,* $P_i P_j = 0$, for $i \neq j$.

(d) P is a projection on M. Since $\| P f \| = \| f \|$, for all $f \in M_i$, each M_i is contained in the range of P.

Therefore, $M = \sum_{j=1}^{n} M_j$ is also contained in the range of P. Further if $f \in$ range of P, then

$$f = Pf = (P_1 + P_2 + \ldots\ldots + P_n) f$$
$$= P_1 f + P_2 f + \ldots\ldots + P_n f$$

i.e., $f \in M = \sum_{i=1}^{n} M_i$, since $P_i f \in M_i$

Example 10. Let f be a bounded linear functional on the Hilbert space H which is separable, prove that there is a unique $y \in H$ such that $f(x) = (x, y)$ for all x, moreover, $\| f \| = \| y \|$.

Solution : Let $< \phi_v >$ be a complete orthonormal system for H, and set $b_v = f(\phi_v)$. Then for each n, we have

$$\sum_{v=1}^{n} b_v^2 = f\left(\sum_{v=1}^{n} b_v \phi_v\right)$$
$$= \| f \| \cdot \| \sum_{v=1}^{n} b_v \phi_v \|$$
$$= \| f \| \left[\sum_{v=1}^{n} b_v^2\right]^{1/2}$$

Thus $\sum_{v=1}^{n} b_v^2 \leq \| f \|^2$ and so $\sum_{v=1}^{n} b_v^2 \leq \| f \|^2 < \infty$.

Hence, there is an element $y = \sum_{v=1}^{n} b_v \phi_v$. we have $\| y \| \leq \| f \|$. Let x be any element of H. Then $\sum_{v=1}^{n} a_v \phi_v \to x$ and so

$$f(x) = \lim f\left(\sum_{v=1}^{n} a_v \phi_v\right)$$
$$= \lim \sum_{v=1}^{n} a_v b_v$$
$$= \sum_{v=1}^{\infty} a_v b_v = (x, y)$$

By the Schwarz inequality $\| f \| \leq \| y \|$.

Example 11. Prove that the set of all normal operators on H is closed under scalar multiplication.

Solution : If T is normal operator and a is any scalar, then

$$(aT)(aT)^* = (aT)(\bar{a}T^*) = a\bar{a}(TT^*)$$
$$= a\bar{a}(T^*T) = (\bar{a}T^*)(aT)$$
$$= (aT)^*(aT)$$

Therefore aT is normal.

Example 12. Let H be a Hilbert space and let $S = (l_1, l_2, \ldots\ldots, l_n, \ldots\ldots\}$ be a countably infinite orthonormal set in H. Prove that a series of the form $\sum_{n=1}^{\infty} \alpha_n l_n$ is convergent if and only if $\sum_{n=1}^{\infty} | \alpha_n |^2 < \infty$. Further if $\sum_{n=1}^{\infty} \alpha_n l_n$ converges to x, then $\alpha_n = (x, l_n)$.

Solution : Consider the partial sum

$$S_n = \sum_{j=1}^{n} \alpha_i l_i$$

For $m > n$, we have

$$\| S_m - S_n \|^2 = \| \sum_{j=n+1}^{m} \alpha_i l_i \|^2$$
$$= \| \sum_{i=n+1}^{m} | \alpha_i |^2$$

If the series $\sum_{n=1}^{\infty} \alpha_n l_n$ is convergent, then the sequence $< S_n >$ of partial sums is convergent and every convergent sequence is a Cauchy sequence.

Therefore as $m, n \to \infty$, we have

$$\| S_m - S_n \|^2 \to 0$$
$$\Rightarrow \quad \sum_{j=n+1}^{\infty} | \alpha_i |^2 \to 0$$

The series $\sum_{n=1}^{\infty} \Sigma | \alpha_n |^2$ is convergent *i.e.,* conversely suppose that the series $\sum_{n=1}^{\infty} | \alpha_n |^2$ is convergent, *i.e.,*

$$\sum_{n=1}^{\infty} | \alpha_n |^2 < \infty$$

Then as $m, n \to \infty$, we have

$$\sum_{i=n+1}^{m} |\alpha_i|^2 \to 0$$

$$\| S_m - S_n \|^2 \to 0$$

Therefore, the sequence $< S_n >$ is a Cauchy sequence in H. But H is complete. So $< S_n >$ is a convergent sequence in H and thus the series $\sum_{n=1}^{\infty} \alpha_n l_n$ is convergent.

Now suppose that the series $\sum_{n=1}^{\infty} \alpha_n L \pm 1 l_n$ is convergent and let $x = \sum_{n=1}^{\infty} \alpha_n l_n$. If $S_n = \sum_{i=1}^{\infty} \alpha_i l_i$ then for $n > j$, we have

$$(S_n, l_j) = \left(\sum_{n=1}^{\infty} \alpha_i, l_i, l_j\right) = \alpha_i$$

Since this relationship is true for any $n > j$ therefore, it must also be true in the limit. So we have $\alpha_j = \lim (S_n, l_j) = (\lim S_n, l_j)$, by continuity of inner product $= (x, l_j)$. Thus $\alpha_j = (x, l_j)$ for each j.

Example 13. Prove that B (N, N′) is a normal linear space.

Solution : Define a function

$$\| \ \| : B(N, N') \to R \text{ by}$$

$$\| T \| = \text{Sup} \| T(f) \| \, \| f \| \leq 1,$$

for all B (N, N′), since || || satisfies.

(i) $\| T \| = \sup \| T(f) \|,$

which is non-negative

$\| f | \leq \|$

(ii) $\| T \| = 0$

$\Rightarrow \quad \text{sup.} \| T \| (f) = 0$

$\Rightarrow \quad \sup_{j \neq 0} \left[\frac{\| T(f) \|}{\| f \|} \right] = 0$

$\Rightarrow \frac{\| T(f) \|}{\| f \|} = 0$, for all $f \in N$ for which $f \neq 0$

$\Rightarrow \| T(f) \| = 0$, where $f \neq 0$ and $f \in N$,

$\Rightarrow T(f) = 0$, even when $f = 0$,

$\Rightarrow T = 0$ (Null operator),

(iii) $\sup_{\| f \| \leq 1} \| (at)(f) \| = \sup_{\| f \| \leq 1} \| aT(f) \|$

$= |a| \sup_{\| f \| \leq 1} \| T(f) \|$

$= |a| \, \| T \|,$

for all scalars a.

(iv) $\| T_1 + T_2 \| = \sup_{\| f \| \leq 1} \| (T_1 + T_2)(f) \|$

$\leq \sup_{\| f \| \leq 1} \| T_1(f) \| + \sup_{\| f \| \leq 1} \| T_2(f) \|$

$\leq \| T_1 \| + \| T_2 \|$

Thus B (N, N′) is a normed linear space.

Example 14. Prove that if T_1 and T_2 are self-adjoint operators on H, then their product T_1T_2 is self-adjoint if and only if.

$T_1T_2 = T_2T_1$

Solution : Let T_1 and T_2 be self-adjoint operators on H,

then, $T_1^* = T_1$ and $T_2^* = T_2$

Suppose that

$T_1T_2 = T_2T_1$

Then $(T_1T_2)^*$

$= T_2^* \, T_1^*$

$= T_2T_1$

$= T_1T_2$

Thus T_1T_2 is self-adjoint

Let T_1T_2 be self-adjoint

Then $T_1T_2 = (T_1T_2)^*$

$= T_2^*T_1^*$

$= T_2T_1$.

Example 15. State and prove that Cauchy-Schwarz inequality.

Solution : Cauchy-Schwarz inequality : For any two vectors f and g of vector space X.

$|f,g \ | \leq \| f \| \cdot \| g \|$

Proof : When $g = 0$,

In this laye both sides are zero and euqality holds.,

when $g \neq 0$,

since $g \neq 0 \Rightarrow \| g | > 0$,

we can write $0 \leq \| \, \| g \|^2 f - (f, g) g \, \|^2$

$= (\| g \|^2 f - (f, g) g \, \| g \|^2 \, f - (f, g) g)$

$= \| g \|^4 (f, f) - (f, g) \| g \|^2 \, (g, f) - \| g \|^2 (\| f \|^2 \| g \|^2 - |(f, g)|^2 - |(f,g)|^2 + |(f, g)|^2)$

$= \| g \|^2 (\| f \|^2 \| g \|^2 - |(f, g)|^2)$

Dividing by $\| g \|^2$ so, we have

$$0 \leq (\| f \|^2 \| g \|^2 - | (f, g) |^2)$$

$$\Rightarrow \quad | (f, g) \leq \| f \| \| g \|.$$

Example 16. Prove that if x and y are any two vectors in a Hibert space then :

(i) $\| x + y \|^2 - \| x - y \|^2 = 4 \text{ Re } (x, y)$

(ii) $(x, y) = \text{Re } (x, y) + i \text{ Re } (x, iy)$

Solution : If $\alpha = \beta + i\gamma$ is a complex number where β, γ are real numbers, then we write

$\beta = \text{Re } \alpha$ and $\gamma = \text{I}_m \alpha$.

(i) We have

$$\| x + y \|^2 = \| x \|^2 + \| y \|^2 + (x, y) + (y, x) \quad \ldots(1)$$

and $\| x - y \|^2$

$$= \| x \|^2 + \| y \|^2 - (x, y) - (y, x) \quad \ldots(2)$$

subtracting (2) from (1), we get

$$\begin{aligned} \| x + y \|^2 - \| x - y \|^2 &= 2(x, y) + 2(y, x) \\ &= 2[(x, y) + \overline{(x, y)}] \\ &= 2[2 \text{ Re } (x, y)] \end{aligned}$$

$[\because \ \alpha + \overline{\alpha} = 2 \text{ Re } \alpha$, if α is a complex number$]$

$$= 4 \text{ Re } (x, y)$$

(ii) We have $(x, y) = \text{Re } (x, y) + i \text{ I}_m (x, y)$

If $\alpha = \beta = iy$ is a complex number, then

$$\begin{aligned} y &= \text{I}_m \alpha \\ &= \text{R}_e \{ -i (\beta + iy) \} \\ &= \text{R}_e (-i\alpha) \end{aligned}$$

$\therefore \text{I}_m (x, y)$

$$\begin{aligned} &= \text{R}_e \{ -i (x, y) \} \\ &= \text{R}_e (x, iy) \end{aligned}$$

$[\because \ (x, iy) = -i (x, y)]$

$$\therefore \quad (x, y) = \text{R}_e (x, y) + i\text{R}_e (x, iy)$$

Example 17. Prove that an operator T on H is normal if and only if $\| \text{T}^* f \| = \| \text{T}_f \|$ for all $f \in \text{H}$.

Solution : We see that

$$\begin{aligned} \| \text{T}^* f \| &= \| \text{T} f \| \\ \Leftrightarrow \quad \| \text{T}^* f \|^2 &= \| \text{T} f \|^2 \\ \Leftrightarrow \quad (\text{T}^* f, \text{T}^* f) &= (\text{T}f, \text{T}f) \\ \Leftrightarrow \quad (\text{TT}^* f, f) &= (\text{T}^* \text{T}f, f) \\ \Leftrightarrow ((\text{TT}^* - \text{T}^*\text{T}) f, f) &= 0 \end{aligned}$$

for all $f \in \text{H}$,

$$\Rightarrow \ (\text{TT}^* - \text{T}^*\text{T}) f = 0$$

or $\quad \text{TT}^* = \text{T}^* \text{T},$

Thus T is normal.

Example 18. Prove that if T_1 and T_2 are normal operators on H with the property that either commutes with the adjoint of the other, than $\text{T}_1 + \text{T}_2$ and $\text{T}_1 \text{T}_2$ are normal.

Solution : (a) $\text{T}_1 + \text{T}_2$ is normal

Since T_1 and T_2 commutes with the adjoint of others

$$\therefore \ \text{T}_1\text{T}_2^* = \text{T}_2^*\text{T}_1 \text{ and } \text{T}_1^*\text{T}_2 = \text{T}_2\text{T}_1^*.$$

Therefore,

$$\begin{aligned} (\text{T}_1 + \text{T}_2)(\text{T}_1 + \text{T}_2)^* &= (\text{T}_1 + \text{T}_2)(\text{T}_1^* + \text{T}_2^*) \\ &= \text{T}_1\text{T}_1^* + \text{T}_1\text{T}_2^* + \text{T}_2\text{T}_1^* + \text{T}_2\text{T}_2^* \\ &= \text{T}_1^*\text{T}_1 + \text{T}_2^*\text{T}_1 + \text{T}_1^*\text{T}_2 + \text{T}_2^*\text{T}_2 \\ &= (\text{T}_1^* + \text{T}_2^*)(\text{T}_1 + \text{T}_2) \\ &= (\text{T}_1 + \text{T}_2)^* (\text{T}_1 + \text{T}_2) \end{aligned}$$

Thus $\text{T}_1 + \text{T}_2$ is normal.

(b) T_1T_2 is normal

Since T_1 and T_2 commutes with the adjoint of others.

$\therefore (\text{T}_1\text{T}_2)(\text{T}_1\text{T}_2)^*$

$$\begin{aligned} &= (\text{T}_1\text{T}_2)(\text{T}_2^*\text{T}_1^*) \\ &= \text{T}_1 (\text{T}_2\text{T}_2^*) \text{T}_1^* \\ &= (\text{T}_1\text{T}_2^*)(\text{T}_2\text{T}_1^*) \\ &= (\text{T}_2^*\text{T}_1)(\text{T}_1^*\text{T}_2) \\ &= \text{T}_2^*(\text{T}_2\text{T}_1^*)\text{T}_2 \\ &= \text{T}_2^*(\text{T}_1^*\text{T}_1)\text{T}_2 \\ &= (\text{T}_2^*\text{T}_1^*)(\text{T}_1\text{T}_2) \\ &= (\text{T}_1\text{T}_2)^* (\text{T}_1\text{T}_2) \end{aligned}$$

Thus T_1T_2 is normal.

Example 19. Prove that if the real and imaginary parts commute, then T is normal.

Solution : Let T_1 and T_2 be the real and imaginary parts of T, then

$$\text{T} = \text{T}_1 + i\text{T}_2$$

and $\quad \text{T}^* = \text{T}_1 - i\text{T}_2$

Further, assume that

$$\text{T}_1\text{T}_2 = \text{T}_2\text{T}_1,$$

then

$$\begin{aligned} \text{TT}^* &= (\text{T}_1 + i\text{T}_2)(\text{T}_1 - i\text{T}_2) \\ &= \text{T}_1^2 + \text{T}_2^2 + i (\text{T}_2\text{T}_1 - \text{T}_1\text{T}_2) \end{aligned}$$

and $T^*T = (T_1 - iT_2)(T_1 + iT_2)$

$= T_1^2 + T_2^2 + i(T_1T_2 - T_2T_1)$

Since T_1 and T_2 commute, we obtain

$TT^* = T^*T$, *i.e.*, T is normal conversely, suppose that T is normal, then we have.

$TT^* = T^*T$

$\Rightarrow T_2T_1 - T_1T_2$

$= T_1T_2 - T_2T_1,$

$\Rightarrow \quad 2T_1T_2 = 2T_2T_1$

$\Rightarrow \quad T_1T_2 = T_2T_1$

i.e., T_1 and T_2 commute.

Example 20. If T is an arbitrary operator on a Hilbert space H, then T is a zero operator iff $(T_x, y) = 0$ for all x and y.

Solution : Suppose T = 0 (*i.e.*, zero operator), than for all x and y

We have $(Tx, y) = (ox, y)$

$= (o, y)$

$= 0$

Conversely $(Tx, y) = 0 \ \forall x, y \in H$

$\Rightarrow \quad (T_x, T_x) = 0 \ \forall_x \in H$

(taking $y = Tx$)

$\Rightarrow \quad Tx = 0 \ \forall_x \in H$

$\Rightarrow \quad T = 0$, *i.e.*, zero operator.

Example 21. If A_1 and A_2 are self-adjoint operators on Hilbert space H, then their product A_1A_2 is self adjoint iff $A_1 A_2 = A_2 A_1$.

Solution : Let A_1 and A_2 be two self-adjoint operators on a Hilbert space H. Then $A_1^* = A_1$ and $A_2^* = A_2$.

Suppose A_1 and A_2 commute, *i.e.*, $A_1A_2 = A_2A_1$. Then to prove that A_1A_2 is self-adjoint, we have.

$$(A_1A_2)^* = A_2^* A_1^*$$
$$= A_2A_1$$
$$= A_1A_2$$

Threefore A_1A_2 is self-adjoint.

Conversely, suppose that A_1A_2 is self-adjoint then,

$$(A_1A_2)^* = A_1A_2$$
$$\Rightarrow \quad A_2^*A_1^* = A_1A_2$$
$$\Rightarrow \quad A_2A_1 = A_1A_2$$

$\Rightarrow$ A, and A_2 commute.

Example 22. Let N be a normal linear space and let $x, y \in N$. Then

prove that $| \|x\| - \|y\| | \leq \|x - y\|$

Solution : Since N is a normal linear space

$\therefore \quad \|x\| = \|(x - y) + y\|$

$\leq \|x - y\| + \|y\|$

or $\quad \|x\| - \|y\| \leq \|x - y\| \quad \ldots(1)$

Similarly $\|y\| - \|x\| \leq \|y - x\| \quad \ldots(2)$

But $\quad \|y - x\| = \|(-1)(x - y)\|$

$= |-1| \, |x - y\|$

$= \|x - y\| \quad \ldots(3)$

From (2) and (3), we have

$\|y\| - \|x\| \leq \|x - y\| \quad \ldots(4)$

Then from (1) and (4), we obtain

$| \|x\| - \|y\| | \leq \|x - y\|$

Example 23. Let H be a Hilbert space and let $\{e_i\}$ be an orthonormal set in H. Then prove that the following conditions are equivalent to one another.

(1) $\{e_i\}$ is complete.

(2) $f \perp \{e_i\} \Rightarrow f = 0$

(3) if f is an arbitrary vector in H, then $f = \in (f, e)\, e_i$,

(4) if f is an arbitrary vector in H. then $\|f\|^2$. $\Sigma\, |(f, e_i|^2$

Solution : (1) $\Rightarrow$ (2). If (2) is not true, then we can obtain a vector $f \neq 0$ such that $f \perp \{e_i\}$ taking $e = \frac{f}{\|f\|}$, we observe that $\{e, e_i\}$ is an orthonormal set which properly contains $\{e_i\}$. This controdicts the completeness of $\{e_i\}$. Thus (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). As $f - \Sigma\,(f, e_i)\, e_i$ is orthogonal to $\{e_i\}$, so it follows from (2) that $f - \Sigma\,(f, e_i)\, e_i = 0$ or equivalently, that

$$f = \Sigma\,(f, e_i)\, e_i$$

(3) $\Rightarrow$ (4) As the inner product is jointly continuous, we have

$$\|f\|^2 = (f, f) = (\Sigma\,(f, e_i)\, e_i\, \Sigma\,(f, e_i)\, e_i)$$
$$= \Sigma\,(f, e_i)\left(\overline{fe_i}\right)$$
$$= \Sigma\,|(f, e_i)|^2$$

(4) $\Rightarrow$ (1). If $\{e_i\}$ is not complete, then it is a proper subset of an orthonormal set $\{e, e_i\}$. Since e is orthogonal to all the $e_i s$ (4) gives $\|e\|^2$

$= \Sigma \mid (e, e_i) \mid^2 = 0$. This contradicts the fact that e in a unit vector.

Example 24. Prove that a bounded linear operator is uniformly continuous. If a linear operator is continuous at one point, it is bounded.

Solution : Suppose A is bounded linear operator. Then

$\| Ax_1 - Ax_2 \| \leq \| A \| \cdot \| x_1 - x_2 \| < \varepsilon$. for all x_1 and x_2 in X with $\| x_1 - x_2 \| < \varepsilon \| A \|$. Thus is uniformly continous.

Suppose now that A is a linear operator that is contiuous at x_0. Then there is a $\delta \to 0$ such that $\| Ax - Ax_0 \| < 1$ for all x such that $\| x - x_0 \| < \delta$. For any z in X with $Z \neq \theta$, set $\omega = \eta z' \| z \|$. Where $0 < \eta < \delta$. Then

$$\frac{\eta}{\| z \|} Az = A\omega$$
$$= A(\omega + x_0) - A(x_0)$$

and

$$\frac{\eta}{\| z \|} \| Az \| = \| A(\omega + x_0) - A(x_0) \| < 1$$

Since

$$\| w + x_0 - x_0 \| = \| \omega \|$$
$$= \eta < \delta$$

Consequently,

$$\| Az \| \leq \eta^{-1} \| z \|,$$

and A is bounded.

Example 25. Prove that if T is normal operator on H, then

$$\| T^2 \| = \| T \|^2.$$

Solution : Since T is normal, then

$$\| T^* f \| = \| T f \|,$$

for every $f \in$ H.

Therefore

$$\| T^* \| = \text{Sup } \{\| T^2 f \| : \| f \| \leq 1\}$$
$$= \text{Sup } \{\| T (T f) \| : \| f \| \leq 1\}$$
$$= \text{Sup } \{\| T^* (T f) \| : \| f \| \leq 1\}$$
$$= \text{Sup } \{\| T^* T f \| : \| f \| \leq 1\}$$
$$= \| T^* T \|$$
$$= \| T \|^2.$$

OBJECTIVE TYPE QUESTIONS

1. Let T be a bounded linear operator on Hilbert space H, then T is projection, if—
   (A) T is unitary if $T^* T = 1 = TT^*$, I being an identity operator on H.
   (B) $T^2 = T$
   (C) $T^* T = T^* T^*$
   (D) None of these

2. Let T be a bounded linear operator on Hilbert space H, then T is unitary if—
   (A) T is unitary if $T^* T = I = TT^*$, 1 being an identity operator on H. (A)
   (B) $T^* = T$
   (C) $T^* T = T^* T^*$
   (D) None of these

3. Let T be a bounded linear operator on Hilbert space H, then T is self adjoint if—
   (A) $T^* = T$
   (B) $T^* T = T^* T^*$
   (C) $T^* T = T T^*$
   (D) None of these

4. Let T be a bounded linear operator on Hilbert space H, then T is normal if—
   (A) $T^* T = TT$
   (B) $T^* T = T^* T^*$
   (C) $T^* T = T T^*$
   (D) None of these

5. For inner product—
   (A) $(x, \alpha y) = \bar{\alpha} (x, y)$
   (B) $(x, \alpha y) = (x, y)$
   (C) $(x, \alpha y) = \alpha (x, y)$
   (D) None of these

6. For adjoint operator—
   (A) $\| T^* T \| = \| T \|$
   (B) $\| T^* T \| = \| T^* \|$
   (C) $\| T^* T \| = \| T \|^2$
   (D) None of these

7. For adjoint operator—
   (A) $T^{**} = T$
   (B) $T^{**} = T^*$
   (C) $T^{**} = T + T$
   (D) $T^{**} = T - T$

8. For adjoint operator—
   (A) $(ST)^* = T^* S^*$

(B) $(ST)^* = TS$
(C) $(ST)^* = T^* + S^*$
(D) $(ST)^* = T^* - S^*$

9. For adjoint opearator—
(A) $(\alpha T)^* = \bar{\alpha}\, T$
(B) $(\alpha T)^* = \bar{\alpha}\, T^*$
(C) $(\alpha T)^* = \bar{\alpha}$
(D) None of these

10. For adjoint operator—
(A) $(T + S)^* = T^* + S^*$
(B) $(T + S)^* = T + S$
(C) $(T + S)^* = TS$
(D) None of these

11. Let T be a bounded linear operator of Hilbert space H, then there exist an unique operator on H called adjoint of T, T* such that—
(A) $(Tx, y) = (x, T^*y)\, x, y \in H$
(B) $\| T^* \| = \| T \|$
(C) Both (A) and (B)
(D) None of these

12. If x has a reproducing Kernel, then—
(A) $| x(t) | = | (x, k(;t) | \leq \| k(;t) \| \| x \|, x \in X, t \in S$
(B) X is a proper functional space
(C) Both (A) and (B) above
(D) None of these

13. Let x be a Hilbert space and $S = \{x_\alpha : \alpha \in A\}$ is an orthonormal set in X, then S is a orthonormal basis (orthonormal closed set) in X if—
(A) The fourier series representation $x = \sum_{\alpha \in A} (x, x_\alpha)\, x_\alpha$ is valid for each $x \in X$.
(B) The fourier series representation $x = \sum_{\alpha \in A} (x, x_\alpha)\, x_\alpha$ is not valid for each $x \in X$.
(C) The fourier series representation $x = \sum_{\alpha \in A} (x, x_\alpha)\, x_\alpha$ is valid for some $x \in X$
(D) None of these

14. A set S is complete in X—
(A) If no non-zero $x \in X$ is orthogonal to S
(B) If non-zero $x \in X$ is orothogonal to S
(C) If no non-zero $x \in X$ is orthogonal to S
(D) None of these

15. Orthonormal set is—
(A) An orthonormal set S such that $(x, y) = \begin{cases} 0 & \text{if } x \neq y \\ 1, & \text{if } x = y \end{cases}, x, y \in S$
(B) An orthonormal set S such that $(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}, x, y \in S$
(C) An orthonormal set S such that $(x, y) = 0$, $x, y \in S$
(D) None of these

16. If x' is a bounded linear functional on a Hilbert space X, then—
(A) There is unique $y_0 \in X$ such that $x'(x) = (y_0), x \in X$
(B) There is unique $y_0 \in x$ such that $x'(x) = (y_0, x), x \in X$
(C) There is a unique $y_0 \in X$ such that $x'(x) = (x, y_0), x \in X$
(D) None of these

17. If k is a closed convex set in X and $x_0 \in X \sim k$, then there is a unique $y_0 \in k$, such that—
(A) $\| x_0 - y_0 \| = \sup \{ \| x_0 - y \| : y \in k \}$
(B) $\| x_0 - y_0 \| = \inf \{ \| x_0 - y \| : y \in k \}$
(C) $\| x_0 - y_0 \| = \{ \| x_0 - y \| : y \in k \}$
(D) None of these

18. A set S of elements of an inner product space. X is an orthogonal set—
(A) if $x \perp y$ whenever $x, y \in S$ and $x \neq y$
(B) If $x \perp y$ whenever $x, y \in S$ and $x = y$
(C) If $x \perp y$ whenever $x, y \in S$
(D) None of these

19. If X is a real vector space, then—
(A) Inner product $(x, y) = (y, x)$
(B) Inner product $(x, y) \neq (y, x)$
(C) Inner product $(x, y) > (y, x)$
(D) None of these

20. For inner product—
(A) $(x, x) = 0$ iff $x = 0$
(B) $(x, x) > 0$ iff $x = 0$
(C) $(x, x) < 0$ iff $x = 0$
(D) None of these

21. For inner product—
   (A) $(x, x) \geq 0$ (B) $(x, x) < 0$
   (C) $(x, x) = 0$ (D) None of these

22. For inner product—
   (A) $(x, y) = \overline{(y, x)}$
   (B) $(x, y) = (y, y)$
   (C) $(x, y) = (y, x)$
   (D) None of these

23. If M, N, T are bounded linear operators, M, N are normal operators and MT = NT, then—
   (A) M* T = T N* (B) M* T = T* N*
   (C) MT = T* N* (D) M* T* = T* N*

24. If p is a projection on H, then (i) p is self adjoint (ii) p is normal (iii) $(px, x) = \| px \|^2$ for every $x \in$ H—
   (A) (i) is true
   (B) (ii) is true
   (C) (iii) is true
   (D) (i), (ii) and (iii) are true

25. If U is an unitary operator on H, then—
   (A) $(Ux, Uy) = (x, y)$
   (B) $\| Ux \| = \| x \|$ for every $x \in$ H
   (C) Both A and B are true
   (D) None of these

26. If T is a bounded linear operator on Hilbert space H, then—
   (A) T is normal iff $\| Tx \| = \| T^* x \|$ for every $x \in$ H
   (B) T is normal iff $\| Tx \| > \| T^*x \|$ for every $x \in$ H
   (C) T is normal iff $\| Tx \| < \| T^*x \|$ for every $x \in$ H
   (D) None of these

27. If T is a bounded linear operator on Hilbert space H, then—
   (A) T is normal iff $\| Tx \| = \| T^* x \|$ for every $x \in$ H
   (B) If T is normal and $Tx = \alpha x$ for $x \in$ H, $\alpha \in$ C, then $T^*x = \bar{\alpha} x$
   (C) If T is normal and if α, β are distinct eigen values of T, then the corresponding eigen spaces are orthogonal to each other
   (D) All the above

28. If $\{S_n^{(m)}\}$ is a sequence of sequences such that $\sum_{n=1}^{\infty} | S_n^{(m)} | = \infty$. $m = 1, 2, \ldots\ldots$, then—
   (A) There is a sequence $\{t_n\}$ diverge, such that all the series $\sum_{n=1}^{\infty} S_n^{(m)} t_n, m = 1, 2, \ldots\ldots$ converge
   (B) There is a sequence $\{t_n\}$ converging to zero, such that all the series $\sum_{n=1}^{\infty} S_n^{(m)} t_n, m = 1, 2, \ldots\ldots$ converge
   (C) Three is a sequence $\{t_n\}$ converging to zero, such that all the series $\sum_{n=1}^{\infty} S_n^{(m)} t_n, m = 1, 2, \ldots\ldots$ diverge
   (D) There is a sequence $\{t_n\}$ diverging, such that all the series $\sum_{n=1}^{\infty} S_n^{(m)} t_n, m = 1, 2, \ldots\ldots$ diverge

29. A is called almost regular if, for every convergent sequence $x = (x_1, x_2 \ldots\ldots)$—
   (A) There is an N such that for every $m >$ N, $\sum_{n=1}^{\infty} a_{mn} x_n$ diverges
   (B) There is an N for every $m >$ N, $\sum_{n=1}^{\infty} a_{mn} x_n$ converges and $\lim_{m \to \infty} \sum_{n=1}^{\infty} a_{mn} x_n = \lim_{n \to \infty} x_n$
   (C) There is an N such that for every $m <$ N, $\sum_{n=1}^{\infty} a_{mn} x_n$ converges
   (D) None of these

30. If dom (T) is closed in Banach space X and linear transformation T is bounded, then—
   (A) T is closed
   (B) T is open
   (C) T is constant
   (D) T is null

31. Given the equation $x - \lambda Tx = 0$. The number $\lambda_0 \in$ R is called characteristic number of this equation if—
   (A) $x - \lambda_0 Tx = 0$ has a non-zero solution
   (B) $x - \lambda_0 Tx = 0$ has a no solution
   (C) $x - \lambda_0 Tx = 0$ has a finite solution
   (D) None of these

32. Projection theorem is—
    (A) If X and Y are Banach Spaces and T is a linear trasnformation from X and Y, then dom (T) is closed and graph G(T) closed $\Rightarrow$ T is bounded.
    (B) If X and Y are Banach spaces and T is bounded linear transformation which maps X on to all of Y, then T is an open mapping (*i.e.,* the image of open set in X under T, is open set in Y).
    (C) If M is a closed subspace of a Hilbert space X, then every $x \in$ X, has a unique decomposition $x = y + z$, $y \in$ M, Z $\in$ $M^\perp$, where $M^\perp = \{z \in X : z \perp y \text{ for all } y \in M\}$.
    (D) None of these
33. Open mapping theorem is—
    (A) If X and Y are Banach spaces and T is a linear transformation from X and Y, then dom (T) is closed and graph G(T) closed $\Rightarrow$ T is bounded
    (B) If X and Y are Banach spaces and T is bounded linear transformation which maps X on to all of Y, then T is an open mapping (*i.e.,* the image of open set in X under T, is open set in Y)
    (C) If M is a closed subspace of a Hilbert space X, then every $x \in$ X has a unique decomposition $x = y + z$, $y \in$ M, $z \in M^\perp$ where $M^\perp = \{z \in X : z \perp y \text{ for all } Y \in M\}$
    (D) None of these
34. Closed graph theorem is—
    (A) If X and Y are Banach spaces and T is a linear transformation from X and Y, then dom (T) is closed and graph G(T) closed $\Rightarrow$ T is bounded
    (B) If X and Y are Banach spaces and T is bounded linear transformation which maps X on to all of Y, then T is an open mapping (*i.e.,* the image of open set in X under T, is open set in Y)
    (C) If M is a closed subspace of a Hilbert space X, then every $x \in$ X has a unique decomposition $x = y + z$, $y \in$ M, $z \in M^\perp$, where $M^\perp = \{z \in X : z \perp Y \text{ for all } y \in M\}$
    (D) None of these
35. T is closed linear transformation if graph G (T) is a—
    (A) Closed vector subspace of Banach spaces $X \cdot Y$
    (B) Open vector subspace of Banach spaces $X \cdot Y$
    (C) Null vector subspace of Banach spaces $X \cdot Y$
    (D) None of these
36. A graph G(T) of a mapping T : X $\rightarrow$ Y is the set of points—
    (A) $(x, Tx) \in X \cdot Y$, with $Tx \in$ dom T
    (B) $(x, Tx) \in X \cdot Y$ with $x \in$ dom T
    (C) $(x, Tx) \in X \cdot X$ with $x \in$ dom T
    (D) None of these
37. A linear transformation T from a Banach space X into a Banach space Y with dom (T) $\subset$ X is closed if—
    (A) $x_n \in$ dom (T)
    (B) $\lim_n x_n = x_n$
    (C) $\lim_n Tx_n = y \Rightarrow x \in$ dom (T) and $y = Tx$
    (D) All are true
38. If X is a Banach space, Y is a normed vector space and $\{T_n\}$ a sequence of continuous linear transformation on X into Y such that for every $x \in$ X, the sequence $\{\| T_n (x) \|\}$ is bounded, then—
    (A) The sequence $\{\| T_n \|\}$ of norm is bounded
    (B) The sequence $\{\| T_n \|$ of norm is unbounded
    (C) The sequence $\{\| T_n \|\}$ of norm is open
    (D) None of these
39. Let X be a Banach space, $\{f_n\}$ is a sequence of continuous linear functional on X and for every $x \in$ X, the sequence $\{| f_n (x)|\}$ is bounded, then—
    (A) The sequence of $\{\| f_n \|\}$ of norms is bounded
    (B) The sequence of $\{\| f_n \|\}$ of norms is unbounded
    (C) The sequence of $\{\| f_n \|\}$ of norms is closed
    (D) None of these

40. For every normed vector space X, these is a set A such that X is isomorphic with a subspace of the Banach space of bounded functions f on A with—
   (A) $\| f \| = \inf \{ | f(t) | : t \in A \}$
   (B) $\| f \| = \{ | f(t) | : t \in A \}$
   (C) $\| f \| = \sup \{ | f(f) | : t \in A \}$*
   (D) None of these

41. If X is a complete normed vector space and f is a continuous linear functional on Y, a subspace of X, then—
   (A) f can be extended to a linear functional F on X such that $\| F \| = \| f \|$
   (B) f can not be extended to a linear functional F on X such that $\| F \| = \| f \|$
   (C) f can be extended to a linear functional F on X such that $F = f$
   (D) None of these

42. If X is a normed vector space, Y is a subspace of X and f is a bounded linear functonal on Y with bound $\| f \|$ relative to Y then f has a continuous linear extension to an $x' \in X'$ with $\| x' \| = \| f \|$—
   (A) Hahn Banach theorem
   (B) Uniform Boundedness Principle
   (C) Muntz theorem
   (D) None of these

43. Each reproducing Kernel is—
   (A) A positive matrtix
   (B) A negative matrix
   (C) A null matrix
   (D) A identity matrix

44. Muntz theorem states necessary and sufficient condition for the set $t^{n_1}, t^{n_2}, \ldots, 1 \le n_1 < n_2 < \ldots\ldots$ to be complete in L_2 is that—
   (A) $\sum_{i=1}^{\infty} \frac{1}{n_i} > \infty$
   (B) $\sum_{i=1}^{\infty} \frac{1}{n_i} = \infty$
   (C) $\sum_{i=1}^{\infty} \frac{1}{n_i} = 0$
   (D) $\sum_{i=1}^{\infty} \frac{1}{n_i} < \infty$

45. A function $K(s, t)$ on $S \cdot S$ is a positive matrix if for each $n = 1, 2\ldots\ldots$ and each choice of points $t_1, t_2 \ldots, t_n$ the quardratic form $\sum_{i,j=1}^{n} kc\ t_i, t_j\ \xi_i \bar{\xi}_j$ in $\xi_1, \xi_2, \ldots \xi_n$—
   (A) is non-negative
   (B) is non-positive
   (C) is negative
   (D) is positive

46. A Hilbert space of functions on a set S is said to have a reproducing Kernel if there is a function $K(s, t)$ on $S \cdot S$ is such that—
   (1) $k(; t) \in X, t \in S$
   (2) $x(t) = (x, K(; t), x \in X, t \in S$
   (A) (1) is true only
   (B) (2) is true only
   (C) (1) and (2) both are true
   (D) None is true

47. A proper functional completion of X is a proper functional space $\overline{X}$ on the same basic set S such that—
   (A) $\overline{X}$ is complete
   (B) X is a dense subspace of $\overline{X}$
   (C) Both (A) and (B)
   (D) None of these

48. A vector space X of functions *a* set S, together with a norm in X, is called a proper functional space if for every $s \in S$—
   (A) There exist Ms such that $| x(S) | = Ms \| x \|, x \in X$
   (B) There exist Ms such that $| x(S) | > Ms \| x \|, x \in X$
   (C) There exist Ms such that $| x (S | \le Ms \| x \|, x \in X$
   (D) None of these

49. A vector space X of functions on a set S, together with a norm in X, is called a proper functional space—
   (A) If for evrey $s \in S$, the evaluation functional at s is discontinuous
   (B) If for every $s \in S$, the evaluation functional at s is constant
   (C) If for every $s \in S$, the evaluation functional at s is continuous*
   (D) None of these

50. Riesz-Fischer theorem states that if $\sum_{n-\infty}^{\infty} |C_n|^2 < \infty$, then there is $f \in L_2(0, 2\pi)$ such that $\sum_{-\infty}^{\infty} C_n e^{inx}$ is—
(A) the fourier series of f
(B) converges to f in the sense of L_2
(C) both (A) and (B) above
(D) None of these

51. If there is a countable basis for X, then—
(A) The Hilbert space X is separable
(B) The Hilbert space X is not separable
(C) The Hilbert space X is null
(D) None of these

52. Two Hilbert spaces are isomorphic if they have—
(A) Same dimension
(B) Different dimension
(C) Almost different dimension
(D) None of these

53. If Hilbert space X is separable, then there is a—
(A) Countable basis for X
(B) Non countable basis for X
(C) Null basis for X
(D) None of these

54. The Cardinality is……for each basis of a given Hilbert space—
(A) Same
(B) Different
(C) Almost different
(D) None of these

55. A necessary and sufficient condition for the set $t^{n^1}, t^{n^2}, \ldots\ldots 1 \le n_1 < n_2 < \ldots$ to be complete in L_2 is that $\sum_{i=1}^{\infty} \frac{1}{n_1} = \infty$. This is—
(A) Muntz theorem
(B) Isoperimetric theorem
(C) Bassel's inequality
(D) Parseval's formula

56. A necessary and sufficient condition for $(x_1, x_2 \ldots x_n)$ to be linearly independent is that—
(A) Gramm determinant $G \neq 0$
(B) Gramm determinant $G > 0$
(C) Gramm determinant $G < 0$
(D) Gramm determinant $G = 0$

57. Among all simple, closed, plecewise smooth curves of length L in the place the circle encloses the maximum area. This is—
(A) Isoperimetric theorem
(B) Bassel's inequality
(C) Cauchy-Schwarz inequality
(D) Parseval's formula

58. Let X be an Hilbert space, $S = \{x_\alpha : \alpha \in A\}$ is an orthonormal set. Then S is basis iff equality holds in Bassel's inequality this is—
(A) Bassel's inequality
(B) Cauchy-Schwarz inequality
(C) Parseval's formula
(D) None of these

59. Let X be an Hilbert space and $S = \{x_\alpha : \alpha \in A\}$ is an orthonormal set in X, then S is a basis—
(A) If it is complete in X
(B) If it is not complete in X
(C) If it is compact in X
(D) None of these

60. If X is a Hilbert space, $S = \{x_\alpha : \alpha \in A\}$ is an orthonormal set in X and $x \in X$, then $\sum_{\alpha \in A} |(x, x_\alpha)|^2 \le \|x\|^2$. This is—
(A) Bassel's inequality
(B) Cauchy-Schwarz inequality
(C) Parseval's formula
(D) None of these

61. If $\{x_1, x_2, \ldots\ldots, x_n\}$ is an orthonormal set, then for any $x \in X$—
(A) $\sum_{i=1}^{n} |x, x_i)|^2 \le \|x\|^2$
(B) $\sum_{i=1}^{n} |x, x_i|^2 \le |x|^2$
(C) $\sum_{i=1}^{n} |x, x_i|^2 \le x^2$
(D) None of these

62. Parallelogram law states that—
(A) $x \perp y \Rightarrow \|x\|^2 + \|y\|^2 = \|x + y\|^2$ is valid in inner product space

(B) $\| x + y \|^2 + \| x - y \|^2 = 2\{\| x \|^2 + \| y \|^2\}$ holds in inner products space

(C) $x, y = \begin{cases} 0, & \text{if } x \neq y \\ 1, & \text{if } x = y \end{cases}$, $x, y \in S$

(D) None of these

63. Pythagorean theorem states that—

(A) $x \perp y \Rightarrow \| x \|^2 + \| y \|^2 = \| x + y \|^2$ is valid in inner product space*

(B) $\| x + y \|^2 + \| x - y \|^2 = 2\{\| x \|^2 + \| y \|^2\}$ holds in inner product space

(C) $x, y = \begin{cases} 0, & \text{if } x \neq y \\ 1, & \text{if } x = y \end{cases}$, $x, y \in S$

(D) None of these

64. Banach space is Hilbert space if—

(A) Parallelogram law holds

(B) Pythagorean theorem holds

(C) Projection theorem holds

(D) None of these

65. $\| x + y \|^2 + \| x - y \|^2 = 2\{\| x \|^2 + \| y \|^2\}$ holds in inner product space this is—

(A) Pythagorean theorem

(B) Parallelogram law

(C) Cauchy-Schwarz inequality

(D) None of these

66. $\perp y \Rightarrow \| x \|^2 + \| y \|^2 = \| x + y \|^2$ is valid in inner product space. This is—

(A) Pythagorean theorem

(B) Parallelogram law

(C) Cauchy-Schwarz inequality

(D) None of these

67. The Cauchy-Schwarz inequality states that inner product space is—

(A) Continuous function on X. X into F, where the norm of $(x, y) \in X \cdot X$ is $\| x \| + \| y \|$

(B) Constant function on X. X into F, where the norm of $(x, y) \in X \cdot X$ is $\| x \| + \| y \|$

(C) Discontinuous function on $X \cdot X$ into F. where norm of $(x, y) \in X, X$ is $\| x \| + \| y \|$

(D) None of these

68. For a fixed m_0 such that—

(1) $\sum_{m=1}^{\infty} | a_{mn} | <$ M for fixed M and every $m > m_0$

(2) $\lim_{m} a_{mn} = 0$ for every n, and $\lim_{m} \sum_{n=1}^{\infty} a_{mn} = 1$, then

(A) A is regular (B) A is almost regular

(C) A not regular (D) None of these

69. If

(1) $\sum_{k=1}^{\infty} | a_{ik} | \leq M, i = 1, 2, \ldots$ for some M

(2) $\lim_{i} a_{ik} = 0, k = 1, 2, \ldots$ and

(3) $\lim_{i} \sum_{k=1}^{\infty} a_{ik} = 1$, then—

(A) A is regular

(B) A is almost regular

(C) A not regular

(D) None of these

70. If parallelogram law holds, then—

(A) Banach space is Hilbert space

(B) Hilbert space is Banach space

(C) Banach space is vector space

(D) None of these

71. Cauchy-Schwarz inequality states—

(A) For every $x, y \in X, | (x, y) | \leq \| x \| \| y \|$

(B) For every $x, y \in X, | (x, y) | > \| x \| \| y \|$

(C) For every $x, y \in X, | (x, y) | < \| x \| \| y \|$

(D) For every $x, y \in X, | (x, y) | = \| x \| \| y \|$

72. X is called Hilbert space, then—

(A) If X is complete under the norm obtained from its inner product*

(B) If X is complete space

(C) If X is not complete under theorem obtained from its inner product

(D) None of these

73. A vector space X, together with an inner product in X, is called—

(A) Inner product space

(B) Outer product space

(C) Closed space

(D) None of these

74. Let $x \in X$, then norm in the space X is—

(A) $\| x \| = \sqrt{(x, x)} = (x, x)^{1/2}$

(B) $\| x \| = (x, x)^2$

(C) $\| x \| = 1$

(D) None of these

75. If $l \le p \le \infty$, and q is conjugate to p, then—
(A) $(l_p)' = l_q$ (B) $(l_p)' > l_q'$
(C) $(l_p)' < l_q$ (D) None of these

76. A is almost regular iff there is a fixed m_0 such that—
(A) $\sum_{m=1}^{\infty} | a_{mn} | < M$ for fixed M and every $m > m_0$
(B) $\lim_{m} a_{mn} = 0$ for every n
(C) $\lim_{m} \sum_{n=1}^{\infty} a_{mn} = 1$
(D) All the above

77. A is regular iff—
(A) $\sum_{k=1}^{\infty} | a_{ik} | \le M$, $i = 1, 2, \dots$ for some M
(B) $\lim_{i} a_{ik} = 0$, $k = 1, 2, \dots\dots$
(C) $\lim_{i} \sum_{k=1}^{\infty} a_{ik} = 1$
(D) All the above

78. A is regular if for each convergent sequence x—
(A) the A-limit exists
(B) A-limit equals the ordinary limit of the sequence x
(C) Both (A) and (B)
(D) None of these

79. If x is a normed vector space $Y \subset X$ is total if—
(A) its closed linear span is X
(B) its open linear span is X
(C) its closed linear span is not X
(D) None of these

80. If X is a normed vector space and $x'_n \in X'$, ($n = 1, 2\dots\dots$) we say x'_n converges weakly to $x' \in X'$ if—
(A) for each $x \in X$, $\lim_{n \to \infty} x'_n (x) = x' (x)$
(B) for each $x \in X$, $x'_n (x) = x' (x)$
(C) for each $x \in X$, $\lim_{n \to \infty} x'_n (x) = 0$
(D) None of these

81. A normed vector space is finite dimensional if—
(A) the closed bounded sets are compact
(B) the closed bounded sets are not compact
(C) the open bounded sets are compact
(D) the open bounded sets are not compact

82. If X is a Banach space and Y is a subspace of X, which is a Borel set, then—
(A) Y is of first category in X and is identical with X
(B) Y is of second category in X and is identical with X
(C) Y is either of first category in X or is identical with X
(D) Y is either of second category in X or is identical with X

83. If $P' (x) = P (x)$, then—
(A) x is almost convergent
(B) x is divergent
(C) x is constant
(D) None of these

84. The closed bounded sets are compact if—
(A) A normed vector space is finite dimensional
(B) A vector space is finite dimensional
(C) A normed vector space is inifinite
(D) None of these
(E) $P'(x) = P(x)$

85. A linear transformation T is bounded if—
(A) T is continuous
(B) T is discrete
(C) T is dis continuous
(D) None of these

86. If X is normed vector space. The space X of continuous linear functional on X, is referred as dual of X if—
(A) $\| x' \| = \{M : | x' (x) | \le M \| x \|, x \in X\}$
(B) $\| x' \| = \text{Sup } \{M : | x' (x) | \le M \| x \|. x \in X\}$
(C) $\| x' \| = \inf \{M : | x' (x) | \le M \| x \|, x \in X\}$
(D) None of these

87. A linear transformation T of X into Y is continuous, then—
(A) T is bounded
(B) T is not bounded
(C) T is constant
(D) None of these

88. If X is a normed vector space, the space X′ of continuous linear functional on X′ is refered as dual of X if—
(A) $\| x' \| = \inf \{|x'(x)| : \| x \| = 1\}$
(B) $\| x' \| = \text{Sup} \{|x'(x)| : \| x \| = 1\}$
(C) $\| x' \| = \{| x'(x)| : \| x \| = 1\}$
(D) None of these

89. If $\| T \|$ is norm of T, then—
(A) $\| T_x \| \leq \| T \| \| x \|$ for every $x \in X$
(B) $\| T_x \| < \| T \| \| x \|$ for every $x \in X$
(C) $\| T_x \| > \| T \| \| x \|$ for every $x \in X$
(D) $\| T_x \| = \| T \| \| x \|$ for every $x \in X$

90. If $\| T \|$ is norm of T if—
(A) T is a linear transform
(B) $\| T \| \inf \{M : \| T_x \| \leq M \| x \|, x \in X\}$
(C) T is bounded
(D) All the above

91. A linear transform T is bounded if there is an integer M such that—
(A) $\| T_x \| < M \| x \|$ for every $x \in X$
(B) $\| T_x \| = M \| x \|$ for every $x \in X$
(C) $\| T_x \| < M \| x \|$ for every $x \in X$
(D) None of these

92. Banach space is a—
(A) Complete normed vector space
(B) Normed vector space
(C) Complete vector space
(D) None of these

93. Let x be a vector space and p a real valued function on X is semi norm if—
(A) $x \neq 0 \Rightarrow p(x) > 0$
(B) $p(x + y) \leq p(x) + p(y)$
(C) $p(\alpha x) = |\alpha| p(x)$
(D) All the above

94. Let X be a vector space, and p a real valued function on X is semi norm then—
(A) $p(\alpha x) = \alpha\, p(x)$
(B) $p(\alpha x) = |\alpha| p(x)$
(C) $p(\alpha x) = \alpha/|p(x)|$
(D) $p(\alpha x) = \alpha + p(x)$

95. Let X be a vector space, and p a real valued function on X is semi norm then—
(A) $p(x + y) \leq p(x) + p(y)$
(B) $p(x + y) = p(x) + p(y)$
(C) $p(x + y) > p(x) + p(y)$
(D) $p(x + y) < p(x) + p(y)$

96. Let X be a vector space, and p a real valued function on X is semi norm then—
(A) $x \neq 0 \Rightarrow p(x) = 0$
(B) $x \neq 0 \Rightarrow p(x) < 0$
(C) $x \neq 0 \Rightarrow p(x) > 0$
(D) None of these

97. Let X be a vector space and $x \in X$. A real valued function on a vector space X, $\| x \|$ is a norm on X, then—
(A) $\| \alpha x \| = \alpha / \| x \|$
(B) $\| \alpha x \| = \| x \|$
(C) $\| \alpha x \| = \alpha \| x \|$
(D) None of these

98. Let X be a vector space and $x \in X$. A real valued function an a vector space X, $\| x \|$ is a norm on X, then—
(A) $\| x + y \| < \| x \| + \| y \|$
(B) $\| x + y \| > \| x \| + \| y \|$
(C) $\| x + y \| = \| x \| + \| y \|$
(D) $\| x + y \| \leq \| x \| + \| y \|$

99. Let x be a vector space and $x \in X$, A real valued function on a vector space X, $\| x \|$ is a norm on X, then—
(A) $x \neq 0 \Rightarrow \| x \| > 0$
(B) $x \neq 0 \Rightarrow \| x \| < 0$
(C) $x \neq 0 \Rightarrow \| x \| = 0$
(D) None on these

100. x is almost convergent if—
(A) $p'(x) = p(x)$ (B) $p'(x) > p(x)$
(C) $p'(x) < p(x)$ (D) None of these

101. If L is a Banach limit then—
(A) $\underline{\lim}\, x_n \leq L(x) \leq \overline{\lim}\, x_n$ for all $x \in m$
(B) $L(x) \leq \overline{\lim}\, x_n \leq \underline{\lim}\, x_n$ for all $x \in m$
(C) $\overline{\lim}\, x_n \leq \underline{\lim}\, x_n \leq L(x)$ for all $x \in m$
(D) None of these

102. The subspaces Y, Z of vector space X are disjoint if—
(A) $Y \cap Z = \{0\}$ (B) $Y \cap Z = X$
(C) $Y \cap Z = Y$ (D) $Y \cap Z = Z$

103. The subspace generated by S of vector space X is a—
(A) Smallest subspace of X that contains S
(B) Largest subspace of X that contains S
(C) Largest subspace of X that does not contains S
(D) Largest subspace of X that does not contains S

104. $Y = \cap \{Y_\alpha : \alpha \in A\}$ where $Y_\alpha, a \in A$ is a set of subspace of X, is refered as—
(A) Intersection of subspace
(B) Union of subspace
(C) Disjoint intersection of subspace
(D) Disjoint union of subspace

105. Proper subspace of vector space X is—
(A) $Y \subset X$ and $Y \neq X$ and $Y \neq \{0\}$
(B) $Y \subset X$ and $Y = X$ or $Y = \{0\}$
(C) $Y \subset X$ and $Y \neq X$ but $Y = \{0\}$
(D) None of these

106. Improper subspace of vector space X are—
(A) X only
(B) Null vector space {0} of X only
(C) X and null vector space {0} of X
(D) None of these

107. Let m be the vector space of all bounded sequences of real numbers, $x \in m$ is called almost convergent and the number's is called F-limit of x if—
(A) $L(x) = s$ for all Banach limits L
(B) $L(x) > s$ for all Banach limits L
(C) $L(x) < s$ for all Banach limits L
(D) None of these

108. A Banach limit is any linear functional L defined on m, m. be the vector space of all bounded sequences of real numbers—
(A) $L(x) = L(\sigma x)$, where $\sigma x = \sigma(x_1, x_2, \ldots) = x_2, x_3, \ldots\ldots)$
(B) $L(x) = 1$ if $x = (1, 1, \ldots\ldots)$
(C) Both (A) and (B)
(D) None of these

109. A Banach limit is any linear functional L defined on $m \cdot m$ be the vector space of all bounded sequences of real numbers, then—
(A) $L(x) \geq 0$ if $x_n \geq 0$ for all n
(B) $L(x) = 0$ if $x_n > 0$ for all n
(C) $L(x) > 0$ if $x_n = 0$ for all n
(D) None of these

110. A sequence $x = (x_1, x_2, \ldots)$ is said to have A-limit, A(x), if for each i = 1, 2, 3 …… the sum $A_i(x) = \sum_{k=1}^{\infty} a_{ik}\, x_k$—
(A) is convergent and $\lim A_i(x) = A(x)$
(B) is divergent and $\lim A_i(x) = A(x)$
(C) is oscillating
(D) None of these

111. A dual of normed vector space is—
(A) Banach space
(B) Not a Banach space
(C) Bounded space
(D) None of these

Answer

1. (C)	2. (C)	3. (A)	4. (A)	5. (A)
6. (B)	7. (B)	8. (A)	9. (B)	10. (A)
11. (C)	12. (C)	13. (B)	14. (C)	15. (A)
16. (C)	17. (B)	18. (A)	19. (A)	20. (A)
21. (A)	22. (A)	23. (A)	24. (D)	25. (A)
26. (A)	27. (D)	28. (C)	29. (B)	30. (A)
31. (A)	32. (C)	33. (B)	34. (A)	35. (D)
36. (B)	37. (D)	38. (A)	39. (A)	40. (C)
41. (A)	42. (A)	43. (A)	44. (B)	45. (A)
46. (C)	47. (C)	48. (C)	49. (C)	50. (A)
51. (D)	52. (A)	53. (A)	54. (A)	55. (A)
56. (A)	57. (A)	58. (C)	59. (A)	60. (A)
61. (A)	62. (B)	63. (A)	64. (A)	65. (B)
66. (A)	67. (A)	68. (B)	69. (A)	70. (A)
71. (A)	72. (A)	73. (A)	74. (A)	75. (A)
76. (A)	77. (B)	78. (C)	79. (A)	80. (A)
81. (A)	82. (C)	83. (A)	84. (C)	85. (A)
86. (C)	87. (A)	88. (B)	89. (A)	90. (D)
91. (C)	92. (A)	93. (D)	94. (B)	95. (A)
96. (C)	97. (C)	98. (D)	99. (A)	100. (A)
101. (A)	102. (A)	103. (A)	104. (A)	105. (A)
106. (C)	107. (A)	108. (B)	109. (A)	110. (A)
111. (A)				

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