22 SAMPLE PROBLEMS WITH SOLUTIONS

FROM

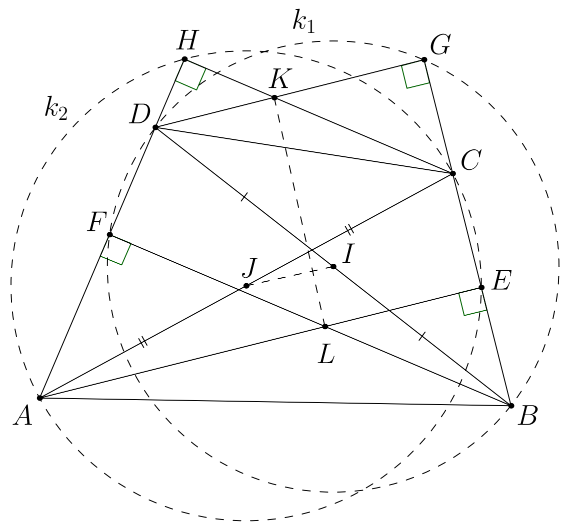
555 GEOMETRY PROBLEMS

SOLUTIONS BASED ON "GEOMETRY IN FIGURES" BY A. V. AKOPYAN

Stanislav Chobanov
Stanislav Dimitrov
Lyuben Lichev

Problem 3.9. Let $ABCD$ be a quadrilateral. Let J and I be the midpoints of the diagonals AC and BD, respectively. Let the perpendicular DG to BC ($G \in BC$) intersect the perpendicular CH to AD ($H \in AD$) at the point K. The perpendicular BF to AD ($F \in AD$) intersects the perpendicular AE to BC ($E \in BC$) at the point L. Prove that $KL \perp JI$.

Solution. Consider the circle k_1 with diameter AC and the circle k_2 with diameter BD.



Their centers are I and J, respectively. We have that $KG.KD = KC.HK$ because the quadrilateral $DCGH$ is cyclic. Therefore, the point K lies on the radical axis of k_1 and k_2.

But the same statement is also true for L because the quadrilateral $ABEF$ is cyclic, leading to $LB.LF = LA.LE$.

We know that the radical axis of two circles is always perpendicular to the line that connects their centers, and thus $KL \perp JI$.

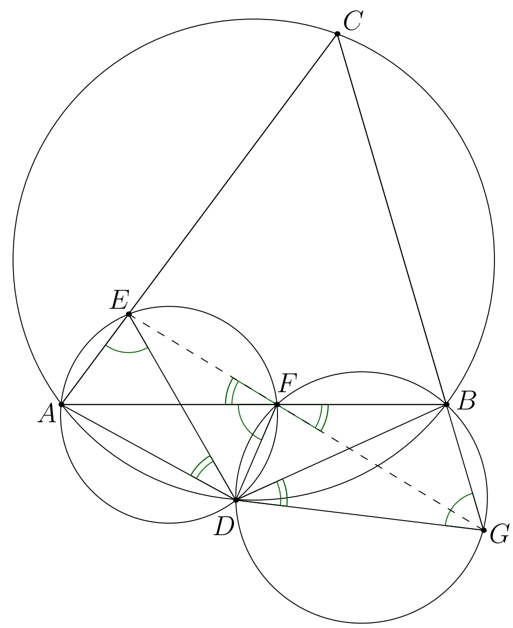
Problems 3.11 and 3.12. *(Simson line)* Let ABC be a triangle with circumcircle k. An arbitrary point D is chosen on the arc $\widehat{AB}$ of k which does not contain C. The points E, F and G lie on CA, AB and BC, respectively, and are chosen such that $\angle AED = \angle AFD = \angle BGD = \varphi$. Prove that the points E, F and G are collinear.

Solution. Using the equal angles, it is easy to show that the quadrilaterals $ADBC$, $ADFE$, $BFDG$ and $EDGC$ are cyclic.

Now we have

$$\begin{aligned}\angle EFG =& \angle EFD + \angle DFG \\ =& 180^\circ - \angle DAE + \angle DBG \\ =& \angle DBC + \angle DBG \\ =& 180^\circ.\end{aligned}$$

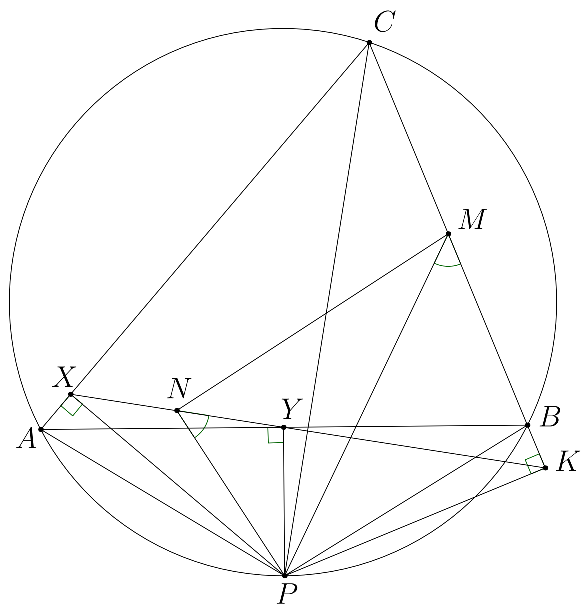
Therefore, the points E, F and G are collinear.



Note. When $\varphi = 90^\circ$, the construction represents the classic Simson line. We proved the generalization of 3.11, which is 3.12.

Problem 4.1.4. Let ABC be a triangle. Let P be an arbitrary point on the smaller arc $\widehat{AB}$ of the circumcircle of $\triangle ABC$. The projections of P onto AC and AB are X and Y, respectively. The points M and N are the midpoints of BC and XY, respectively. Prove that $\angle PNM = 90°$.

Solution. Let K be the projection of P onto BC. Then the points X, Y and K lie on the Simson line of P (Problem 3.11). It suffices to prove that the quadrilateral $PNMK$ is cyclic.



We show that $\triangle PYX \sim \triangle PBC$. We have $\angle PXY = \angle PAY = \angle PCB$ and $\angle XPY = \angle XAY = \angle CPB$.

Now the similarity follows and hence $\angle PNK = \angle PMK$ because PN and PM are respective medians in these triangles.

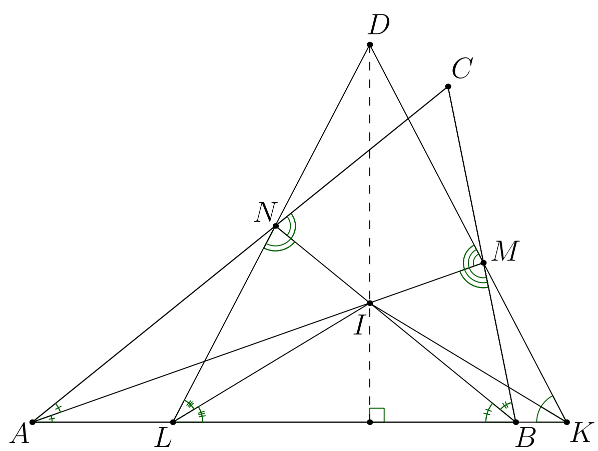
Therefore, the quadrilateral $PNMK$ is cyclic.

Problem 4.3.21. Let ABC be a triangle with angle bisectors AM and BN, intersecting at the point I. Points L and K are chosen on the line AB, such that LN and CN are symmetric with respect to BN, and such that CM and KM are symmetric with respect to AM. Let $D = LN \cap KM$. Prove that $DI \perp AB$.

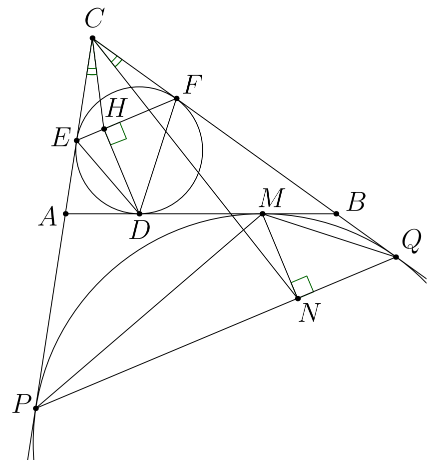
Solution. The symmetry with respect to BN gives $\angle BLN = \angle BCN$, and the symmetry with respect to AM gives $\angle AKM = \angle ACM$. Therefore, $\triangle LKD$ is isosceles.

Since I is the K-excenter of $\triangle BKM$, we deduce that KI is the angle bisector of $\angle LKD$.

Analogously, LI is the angle bisector of $\angle KLD$. Therefore, I is the incenter of the isosceles $\triangle LKD$, which implies that $DI \perp AB$.



Problem 4.5.7. Let ABC be a triangle. Let its incircle touch the sides BC, CA and AB at the points F, E and D, respectively. Let its C-excircle touch the lines BC, CA and AB at the points Q, P and M, respectively. Let MN ($N \in PQ$) be an altitude in $\triangle PMQ$, and let DH ($D \in EF$) be an altitude in $\triangle EDF$. Prove that $\angle ACN = \angle BCH$.



Solution. It is enough to prove that

$$\frac{\sin\angle PCN}{\sin\angle NCQ} = \frac{\sin\angle FCH}{\sin\angle HCE}.$$

We have that

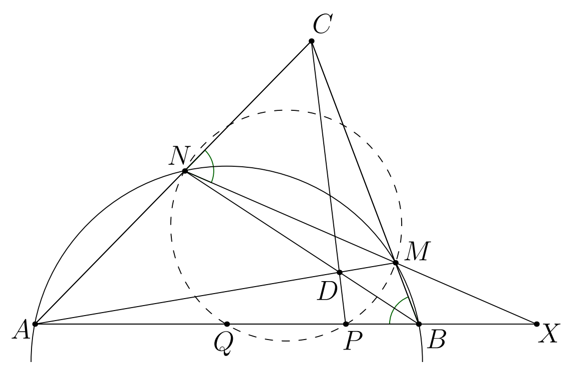
$$\angle DEF = 90^\circ - \frac{\beta}{2},\ \angle DFE = 90^\circ - \frac{\alpha}{2}.$$

Hence, $\angle EDH = \dfrac{\beta}{2}$ and $\angle HDF = \dfrac{\alpha}{2}$.

Observe that $\angle MPN = \dfrac{\beta}{2}$ and $\angle MQN = \dfrac{\alpha}{2}$. Hence, $\angle PMN = 90^\circ - \dfrac{\beta}{2}$, $\angle NMQ = 90^\circ - \dfrac{\alpha}{2}$ and by the law of sines we get

$$\frac{\sin\angle PCN}{\sin\angle NCQ} = \frac{PN}{NQ}\cdot\frac{CQ}{CP} = \frac{PN}{NM}\cdot\frac{MN}{NQ}$$
$$= \cot\frac{\beta}{2}\operatorname{tg}\frac{\alpha}{2} = \frac{DH}{HE}\cdot\frac{HF}{DH} = \frac{HF}{HE}\cdot\frac{EC}{FC} = \frac{\sin\angle FCH}{\sin\angle HCE}.$$

Problem 4.8.14 Let ABC be a triangle. The points N and M are chosen on the sides AC and BC, respectively, so that $ABMN$ is cyclic. Let $AM \cap BN = D$ and $CD \cap AB = P$. Denote the midpoint of the segment AB by Q. Prove that the quadrilateral $MNQP$ is cyclic.



Solution. Without loss of generality, let $AC > BC$. Denote $AB \cap MN = X$. Then B is between X and A.

Menelaus' Theorem, applied to $\triangle ABC$ and the points N, M and X, gives

$$\frac{AX}{BX} = \frac{AN}{NC} . \frac{CM}{MB}.$$

Ceva's Theorem, applied to $\triangle ABC$ and the points P, M and N, gives $\dfrac{AP}{BP} = \dfrac{AN}{NC} . \dfrac{CM}{MB} = \dfrac{AX}{BX}$.

Note that since $\dfrac{AX}{BX} > 1$, the inequality $\dfrac{AP}{BP} > 1$ holds, and so P is between Q and B.

We have that $XA.XB = XM.XN$, and so it suffices to show that $XA.XB = XP.XQ$.

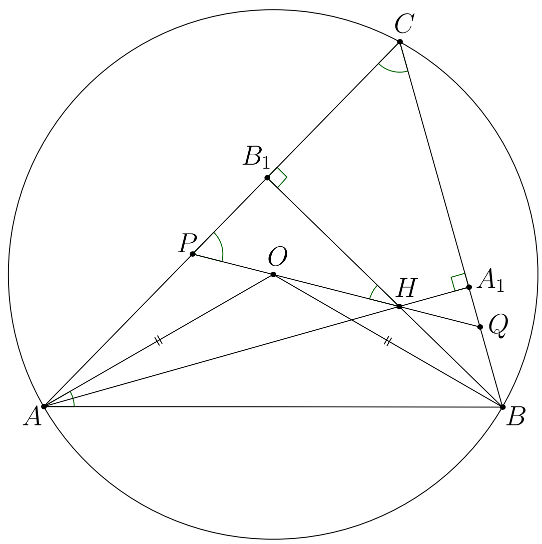
Let $AQ = a + b$, $QP = a$, $PB = b$ and $XB = c$. Then we have $\dfrac{2a + b}{b} = \dfrac{2a + 2b + c}{c}$, which implies $ac = ab + b^2$.

Equivalently, $(b + c)(a + b + c) = c(2a + 2b + c)$. Therefore, $XP.XQ = XA.XB$.

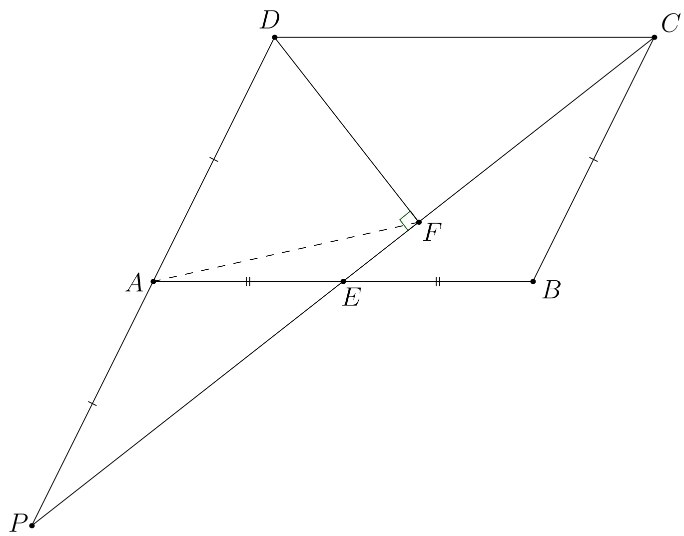
Problem 4.11.4. Let ABC be a triangle with circumcenter O and $\angle ACB = 60°$. Let AA_1 ($A_1 \in BC$) and BB_1 ($B_1 \in AC$) be altitudes in the triangle, intersecting at the point H. The line OH intersects the lines AC and BC at the points P and Q. Prove that $\triangle PQC$ is equilateral.

Solution. We have that $\angle AOB = \angle AHB = 120°$, so the quadrilateral $ABHO$ is cyclic. Also, $\angle BAO = 30°$, thus $\angle AHO = 30°$.

Note that $\angle AHB_1 = 60°$. Therefore, $\angle B_1HP = 30°$, which implies that $\angle QPC = 60°$ and that $\triangle PQC$ is equilateral.

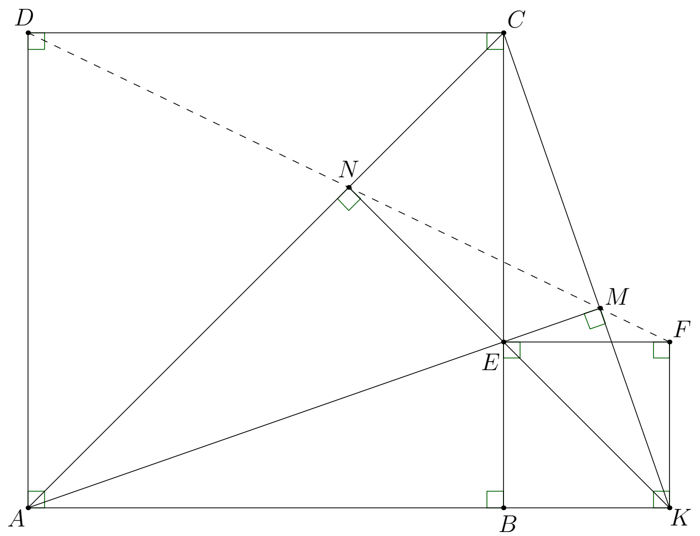


Problem 5.1.1. Let $ABCD$ be a parallelogram. Let E be the midpoint of AB and let F be the foot of the perpendicular from D to the line EC. Prove that $AF = AD$.



Solution. Let $CE \cap DA = P$. Since $DC = 2AE$ and $DC \parallel AE$, we have that AE is a midsegment in $\triangle PCD$. Hence, $AP = AD$ and thus AF is the median to the hypotenuse in the right-angled $\triangle PFD$. We deduce that $AF = AP = AD$.

Problem 5.3.1. Let $ABCD$ be a square and let E be a point on the segment BC. The square $BKFE$ is constructed, such that it is external to $ABCD$. Let $AE \cap CK = M$ and $KE \cap AC = N$. Prove that the points D, N, M and F are collinear.



Solution. We have $AB = BC$ and $BE = BK$. Hence, $\triangle ABE \cong \triangle CBK$. Thus, $\angle BCK = \angle BAM$. This implies that the quadrilateral $ABMC$ is cyclic, which yields $\angle AMK = \angle CBA = 90°$.

Also, $\angle EFK = \angle EMK = 90°$, so the quadrilateral $EMFK$ is cyclic. Since $CE \perp AK$ and $AE \perp CK$, the point E is the orthocenter of $\triangle AKC$, which yields $\angle ANK = 90°$. Consequently, the quadrilateral $AKMN$ is cyclic.

Now, $\angle NMA = \angle NKA = 45° = \angle EKF = 180° - \angle EMF$ implies that the points N, M and F are collinear. We have that NM is the radical axis of the circumcircles of $AKMN$ and $MENC$.

It suffices to show that D also lies on this radical axis. We have $\angle DCN = 45° = \angle BEK = \angle NEC = \angle CMN$. Thus, DC touches the circumcircle of $\triangle CNM$.

Also, $\angle DAN = 45° = \angle NKA$ and DA touches the circumcircle of $\triangle NKA$. Therefore, the point D has equal powers with respect to these two circles.

Problem 5.4.12. Let $ABCD$ be a circumscribed quadrilateral with incircle k with center I. Let $ABCD$ be also inscribed in a circle k_1. Let k touch AB, BC, CD and DA at the points M, N, P and Q, respectively. Prove that $MP \perp NQ$.

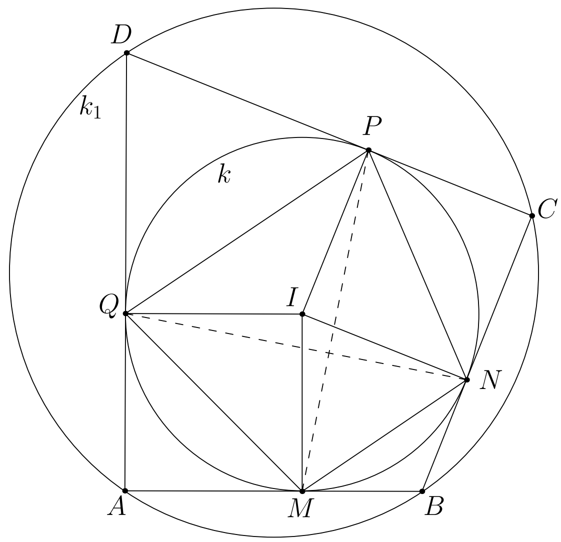
Solution. We have

$$\angle(MP, NQ) = \\ = \frac{\widehat{PN} + \widehat{QM}}{2} \\ = \frac{\angle PIN + \angle QIM}{2} \\ = \frac{360^\circ - \angle DAB - \angle DCB}{2}.$$

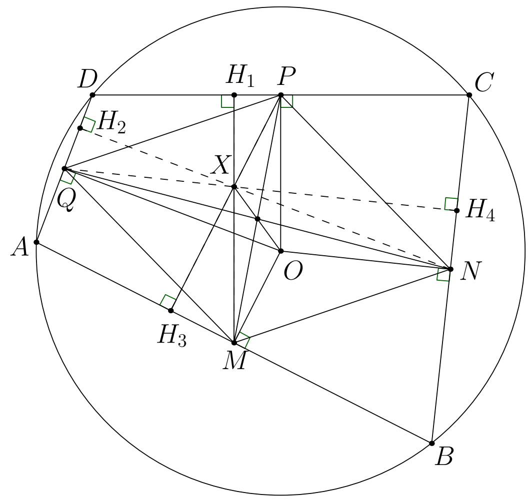
But we know that

$$\angle DCB + \angle DAB = 180^\circ.$$

Hence, $\angle(MP, NQ) = 90^\circ$.



Problem 5.7.4. Let $ABCD$ be a cyclic quadrilateral. Let M, N, P and Q be the midpoints of AB, BC, CD and DA, respectively. The points H_1, H_2, H_3 and H_4 are the feet of the perpendiculars from M, N, P and Q to CD, DA, AB and BC, respectively. Prove that the lines MH_1, NH_2, PH_3 and QH_4 are concurrent.



Solution. Midsegments give that PN is parallel and equal to MQ.

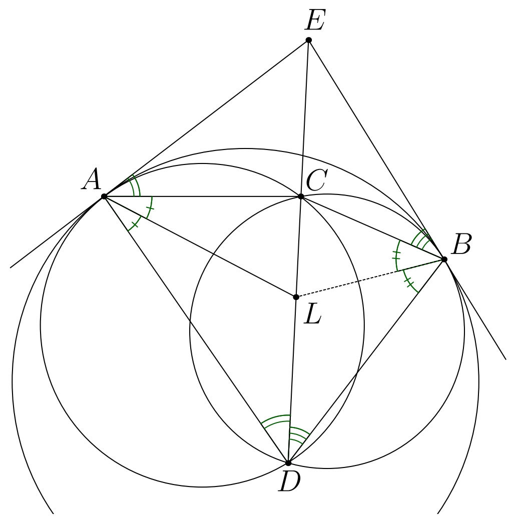
Therefore, the quadrilateral $PNMQ$ is a parallelogram. Denote the circumcenter of $ABCD$ by O.

Now we have that $OP \perp CD$, $ON \perp BC$, $OM \perp AB$ and $OQ \perp AD$.

If $MH_1 \cap PH_3 = X$, then the quadrilateral $POMX$ is a parallelogram. Thus, the midpoint of OX coincides with the midpoint of PM, which coincides with the midpoint of NQ. Hence, the quadrilateral $ONXQ$ is also a parallelogram. Consequently, $NX \perp AD$ and $QX \perp BC$, which gives $QH_4 \cap NH_2 = X$.

Problem 6.1.8. Let k_1 and k_2 be circles that touch another circle k internally at the points A and B, respectively. Let $k_1 \cap k_2 = \{C, D\}$. Prove that the intersection point L of the angle bisectors of $\angle CAD$ and $\angle CBD$ lies on CD.

Solution. The angle bisector theorem, applied to $\triangle ACD$ and $\triangle BCD$, yields that it suffices to show that $\dfrac{AC}{AD} = \dfrac{BC}{BD}$.



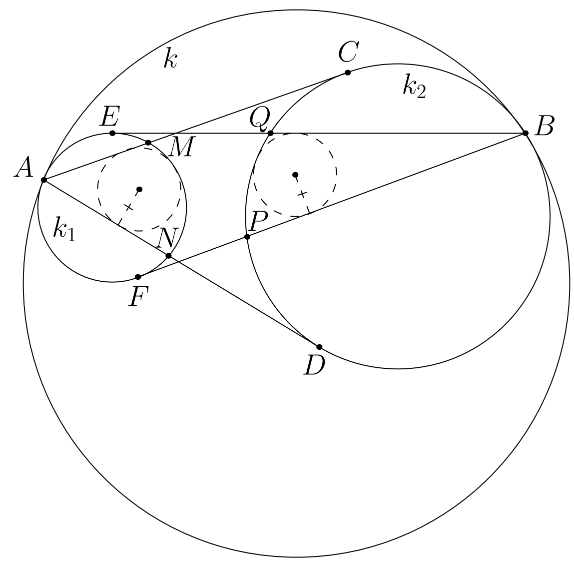
Let E be the intersection point of the tangent lines to k at A and B. Observe that E lies on the radical axis of k_1 and k_2. Hence, $E \in CD$.

Note that $\triangle EAC \sim \triangle EDA$ and $\triangle EBC \sim \triangle EDB$.

Thus,

$$\frac{AC}{AD} = \frac{AE}{ED} = \frac{BE}{ED} = \frac{BC}{BD}.$$

Problem 6.2.2. Let k, k_1 and k_2 be circles with radii r, r_1 and r_2, respectively. The circles k_1 and k_2 do not intersect each other and they touch k internally at the points A and B, respectively. The tangent lines AC and AD from A to k_2 ($C, D \in k_2$) intersect k_1 at the points M and N, respectively. The tangent lines BE and BF from B to k_1 ($E, F \in k_1$) intersect k_2 at the points Q and P, respectively. Let r' be the radius of the circle that touches the segments AM and AN, and the arc $\widehat{MN}$ of k_1. Let r'' be the radius of the circle that touches the segments BP and BQ, and the arc $\widehat{PQ}$ of k_2. Prove that $r' = r''$.

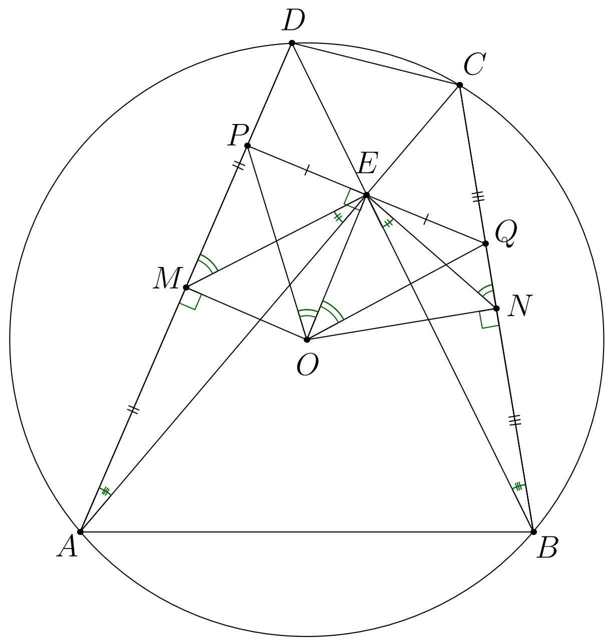


Solution. Consider the homothety centered at A that sends k_1 to k. It sends the incircle of the curvilinear triangle ANM to k_2 and hence $\frac{r'}{r_2} = \frac{r_1}{r}$, so $r' = \frac{r_1 . r_2}{r}$.

Analogously, considering the homothety centered at B that sends k_2 to k, we get

$$r'' = \frac{r_1 . r_2}{r} = r'.$$

Problem 6.4.3. *(The butterfly theorem)* Let $ABCD$ be a quadrilateral inscribed in a circle with center O. The diagonals of $ABCD$ intersect at the point E. A line perpendicular to EO at E intersects AD and BC at the points P and Q, respectively. Prove that $PE = QE$.



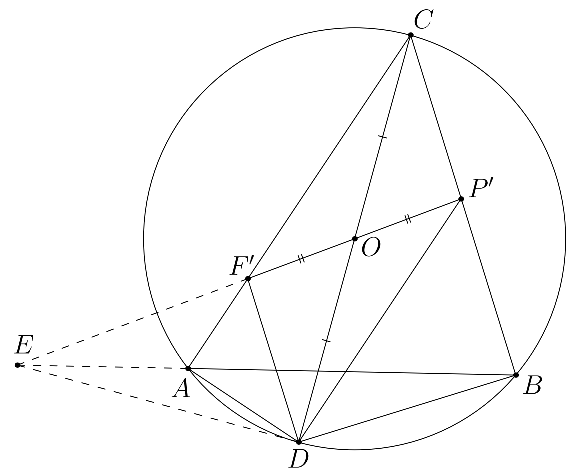
Solution. Note that $\angle DAE = \angle CBE$ and $\angle AED = \angle BEC$. Hence, $\triangle AED \sim \triangle BEC$. Denote the midpoints of AD and BC by M and N, respectively. Then EM and EN are the corresponding medians of the similar triangles $\triangle ADE$ and $\triangle BCE$. Thus, $\angle EMD = \angle ENC$.

On the other hand, $OM \perp AD$. Hence, $OMPE$ is cyclic and $\angle DME = \angle PME = \angle POE$. Analogously, $\angle EOQ = \angle CNE$. We have $\angle DME = \angle CNE$ and thus $\angle POE = \angle QOE$.

We obtained that the line OE is both an altitude and an angle bisector in $\triangle POQ$. Hence, it is a median as well.

Problem 6.4.5. Let ABC be a triangle. Let $k(O)$ be the circumcircle of the triangle and let D be the reflection of C with respect to O. The tangent line to k at the point D intersects AB at the point E. Let OE intersect AC and BC at the points F and P, respectively. Prove that $FO = OP$.

Solution. Let $F' \in AC$ be a point, such that $F'D \parallel BC$. Let $P' \in BC$ be a point, such that $DP' \parallel AC$. Then clearly the quadrilateral $DF'CP'$ is a parallelogram and the point O bisects its diagonals.



Let $F'P' \cap AB = E'$.

We will prove that the point E coincides with the point E', which will lead us to $P \equiv P'$ and $F \equiv F'$, so the desired statement will follow.

We have that $CD = 2R$, $\angle F'CD = \angle CDP' = 90^\circ - \beta$, $\angle P'CD = \angle CDF' = 90^\circ - \alpha$ and by the law of sines,

$$CF' = \frac{2R\cos\alpha}{\sin\gamma},\ CP' = \frac{2R\cos\beta}{\sin\gamma}.$$

Therefore,

$$AF' = b - CF' = \frac{2R}{\sin\gamma}(\sin\beta\sin\gamma - \cos\alpha) = \frac{2R\cos\beta\cos\gamma}{\sin\gamma},$$

$$BP' = a - CP' = \frac{2R}{\sin\gamma}(\sin\alpha\sin\gamma - \cos\beta) = \frac{2R\cos\alpha\cos\gamma}{\sin\gamma}.$$

Menelaus' Theorem, applied to $\triangle ABC$ and the line $P'F'$, yields

$$\frac{BE'}{E'A} = \frac{\dfrac{2R\cos\alpha\cos\gamma}{\sin\gamma}}{\dfrac{2R\cos\beta}{\sin\gamma}} \cdot \frac{\dfrac{2R\cos\alpha}{\sin\gamma}}{\dfrac{2R\cos\beta\cos\gamma}{\sin\gamma}} = \frac{\cos^2\alpha}{\cos^2\beta}.$$

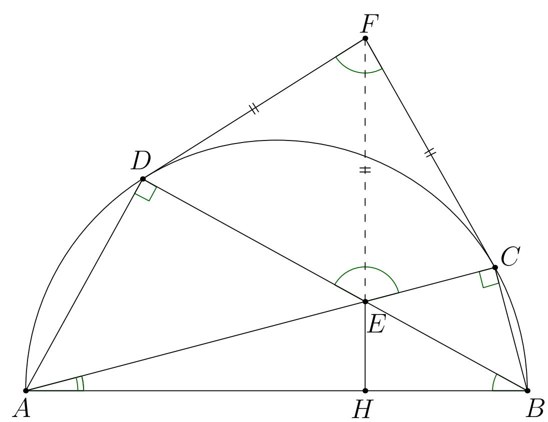
On the other hand, we have that $\angle EAD = \angle EDB = 90^\circ + \alpha$. Also, $\angle ABD = \angle ADE = 90^\circ - \beta$. The law of sines, applied to $\triangle EAD$ and $\triangle EBD$, yields

$$\frac{BE}{EA} = \frac{\sin\angle BDE}{\sin\angle ADE} \cdot \frac{\sin\angle EAD}{\sin\angle ABD} = \frac{\cos^2\alpha}{\cos^2\beta} = \frac{BE'}{E'A}.$$

Hence, $E \equiv E'$, as required.

Problem 6.7.2. The points A, B, C and D lie on a circle with diameter AB. The tangent lines to the circle at the points C and D intersect each other at the point F. Let $E = AC \cap BD$. Prove that $FE \perp AB$.

Solution. Let $H = FE \cap AB$.

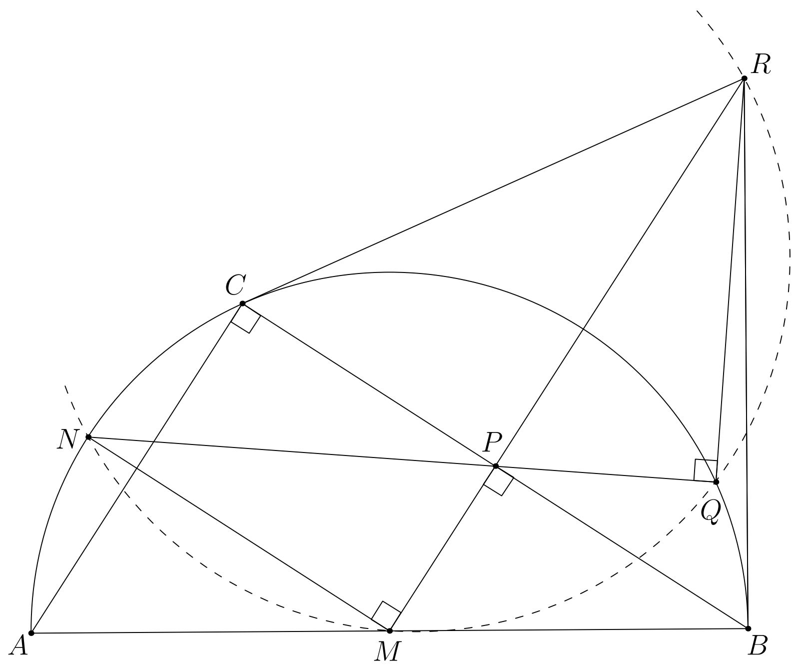


Denote $\angle ABD = \alpha$ and $\angle BAC = \beta$. Observe that $\angle ADB = \angle ACB = 90^\circ$ and $\angle BAD = 90^\circ - \alpha = \angle BDF$. Analogously, $\angle ACF = 90^\circ - \beta$. We obtain $\angle CED = \angle AEB = 180^\circ - \alpha - \beta$. It follows from the quadrilateral $DECF$ that $\angle DFC = 2\alpha + 2\beta$, which means that

$$2(180^\circ - \angle CED) = \angle DFC.$$

Hence, E lies on the circle centered at F with radius FD. Thus, $FD = FE$. Therefore, $\angle HEB = \angle DEF = \angle EDF = 90^\circ - \alpha$, $\angle HBD = \alpha$ and $FE \perp AB$.

Problem 6.7.9. Let ABC be a triangle with $\angle ACB = 90°$. The tangent lines to the circumcircle of $\triangle ABC$ at B and C intersect each other at the point R. Let P and N be the midpoints of BC and the arc $\widehat{AC}$, respectively. Denote the second intersection point of NP and the circumcircle of $\triangle ABC$ by Q. Prove that $\angle NQR = 90°$.



Solution. Denote the midpoint of AB by M. Hence, M is the center of the circumcircle of $\triangle ABC$, and the points M, P and R are collinear. On the other hand, MN contains the midpoint of AC.

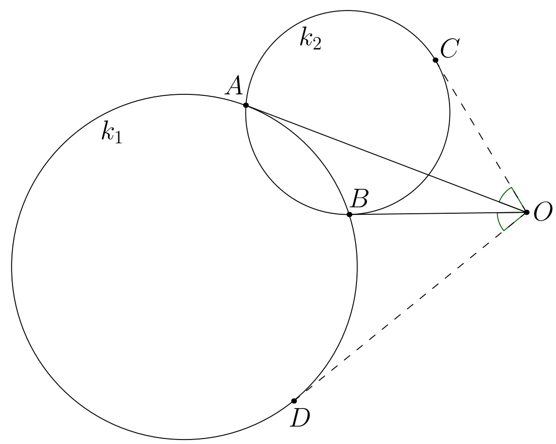
Hence, $MN \parallel BC$ and $MP \parallel AC$, which means that $\angle NMP = 90°$.

It suffices to show that the quadrilateral $NMQR$ is cyclic.

Consider the inversion $I(M, MC)$. We will prove that the images of N, Q and R are collinear. Observe that $I(N) = N$, $I(Q) = Q$ and $I(R) = P$.

Since the points N, Q and P are collinear, we conclude that the points N, Q and R lie on a circle that contains the center of the inversion M.

Problems 6.10.7. Let k_1 and k_2 be circles that intersect each other at the points A and B. The tangent lines to k_1 and k_2 at A and B, respectively, intersect at the point O. Let $D \in k_1$ and $C \in k_2$ are points, such that OD and OC are the other tangent lines to k_1 and k_2, respectively. Prove that $\angle COA = \angle DOB$.

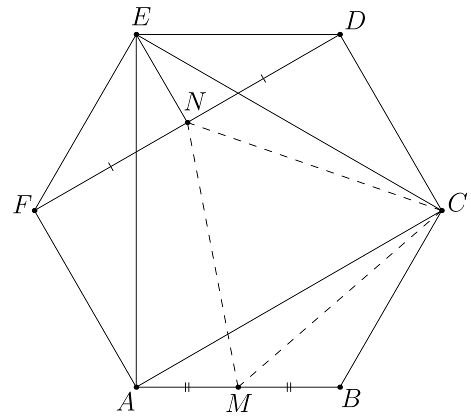


Solution. Let the mapping φ be a composition of inversion $I(O, \sqrt{OA.OB})$ and symmetry with respect to the angle bisector of $\angle AOB$.

Then $\varphi(A) = B$, $\varphi(B) = A$, $\varphi(k_1) = k_2$ and $\varphi(k_2) = k_1$, because k_1 gets transformed into a circle that touches OB at B and that passes through A.

Thus, $\varphi(OC) = OD$, and so the lines OC and OD are symmetric with respect to the angle bisector of $\angle AOB$, implying that $\angle COA = \angle DOB$.

Problem 8.12. Let $ABCDEF$ be a regular hexagon. The points M and N are the midpoints of AB and DF, respectively. Prove that $\triangle MCN$ is equilateral.



Solution. It is clear that $\triangle ACE$ is equilateral. We have

$$2\overrightarrow{MC} = \overrightarrow{AC} + \overrightarrow{BC}.$$

A $+60°$ rotation of all vectors in the above equality gives

$$2\,\overrightarrow{a} = \overrightarrow{AE} + \overrightarrow{AF},$$

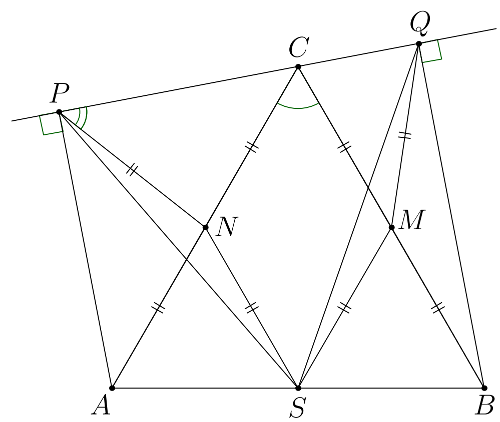
where $\overrightarrow{a}$ is the image of the vector $\overrightarrow{MC}$ under a $+60°$ rotation. Since $\overrightarrow{AE} = \overrightarrow{BD}$, we obtain $2\,\overrightarrow{a} = \overrightarrow{BD} + \overrightarrow{AF} = 2\overrightarrow{MN}$, whence $\overrightarrow{a} = \overrightarrow{MN}$.

Therefore, $\triangle CMN$ is equilateral.

Another possible approach uses the fact that $\triangle CAM \cong \triangle CEN$.

Problem 8.1.14. Let ABC be an equilateral triangle and let l be an arbitrary line through the vertex C. Let P and Q be the feet of the perpendiculars from A and B to l, respectively. If S is the midpoint of AB, prove that $\triangle PQS$ is equilateral.

Solution. Let M and N be the midpoints of BC and AC, respectively. Considering the mid-segments and the fact that $\triangle BQC$ and $\triangle APC$ are right-angled, we get that $MQ = MB = MC = MS = SN = AN = NC = NP$.



Moreover,

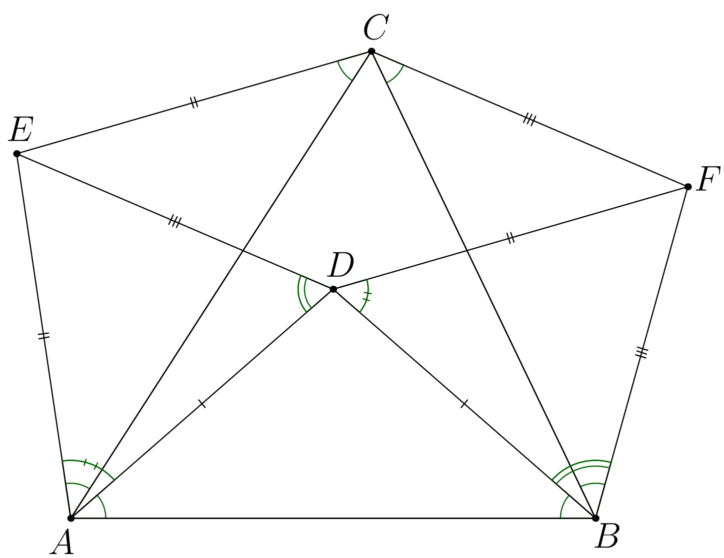
$$\begin{aligned}\angle CMQ &= 180^\circ - 2\angle MCQ \\ &= 180^\circ - 2(120^\circ - \angle PCN) \\ &= 2\angle PCN - 60^\circ \\ &= 120^\circ - \angle PNC.\end{aligned}$$

Hence, $\angle SMQ = 120^\circ + \angle CMQ = 240^\circ - \angle PNC = \angle PNS$. We deduce that $\triangle PNS \cong \triangle QMS$ and $PS = SQ$. But $\angle PSQ = \angle NSM = \angle NCM = 60^\circ$ and therefore $\triangle PQS$ is equilateral.

Problem 9.21. Let ABC be a triangle. The isosceles triangles BCF ($BF = CF$) and AEC ($AE = CE$) are constructed externally, such that $\angle ACE = \angle BCF$. The point D lies inside of $\triangle ABC$, such that $AD = BD$ and $\angle BAD = \angle CAE$. Prove that $DF = EA$ and $DE = BF$.

Solution. The problem conditions imply that $\triangle ABD \sim \triangle BCF \sim \triangle ACE$. Then $\dfrac{AE}{AD} = \dfrac{AC}{AB}$.

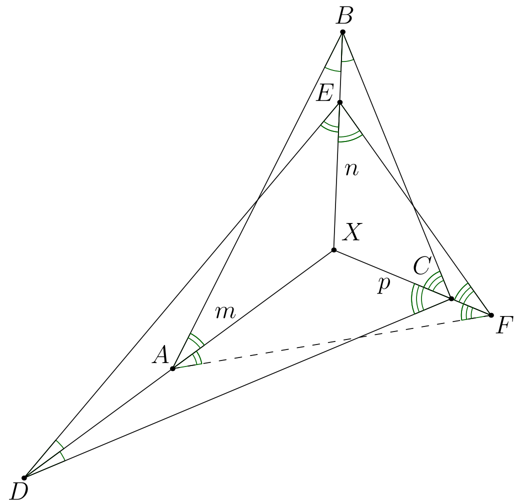
On the other hand, we have $\angle DAE = \angle BAC$, which implies that $\triangle ABC \sim \triangle ADE$.



Analogously, $\triangle DBF \sim \triangle ABC$. Hence, $\triangle DBF \sim \triangle ADE$. This and the equality $AD = BD$ imply that actually $\triangle DBF \cong \triangle ADE$. Therefore, $DF = EA$ and $DE = BF$.

Problem 10.6. Let $m^{\rightarrow}$, $n^{\rightarrow}$ and $p^{\rightarrow}$ be three rays with a common origin X. Let $A \in m$ and $B \in n$. The line symmetric to AB with respect to n intersects p at the point C. The line symmetric to BC with respect to p intersects m at the point D. The line symmetric to CD with respect to m intersects n at the point E. Finally, the line symmetric to DE with respect to n intersects p at the point F. Prove that the line AF is symmetric to both EF with respect to p and to AB with respect to m.

Solution. Let the line symmetric to AB with respect to m intersect p at the point F'. It suffices to show that $F' \equiv F$.



Recall that a point on an angle bisector is equidistant from the arms of the angle. We have

$$\begin{aligned} dist(X, AF') &= dist(X, AB) \\ = dist(X, BC) &= dist(X, CD) \\ = dist(X, DE) &= dist(X, EF) \end{aligned}$$

Let the projections of X onto the lines EF and AF' be H_1 and H_2, respectively. Then $XH_2 = XH_1$. We have

$$\begin{aligned} \angle H_1FX &= \angle XCB + \angle XBC - \angle XEF \\ &= \angle XCD + \angle XBA - \angle XED \\ &= \angle XCD + \angle XDE - \angle XAB \\ &= \angle XCD + \angle XDC - \angle XAF' \\ &= \angle H_2F'X. \end{aligned}$$

Therefore, $\triangle XH_1F \cong \triangle XH_2F'$, which means that $XF' = XF$, or $F \equiv F'$.