B.STAT + B.MATH for ISI

Selected Problems from "Test of Mathematics at 10+2 Level" - published by ISI

Ankur Roy Choudhury

20/7/18.

VOL-1

Scanned with CamScanner

CONTENTS

Scanned with CamScanner

NUMBER THEORY

CONTENTS

Ankur Roy Choudhury

[redacted]

14/10/17.

NUMBER THEORY

A) Finding no. of divisors of a composite no.

$N = a^p b^q c^r$ where a, b, c are different prime nos and p, q, r are positive integers

$\therefore$ No. of divisors is $(p+1)(q+1)(r+1)$ [Note: 1 and N included]

B) No. of ways a composite no. can be resolved into two factors is $\frac{1}{2}(p+1)(q+1)(r+1)$ if N is not a perfect square

If N is a perfect square, then $\frac{1}{2}[(p+1)(q+1)(r+1)+1]$

C) No. of ways in which a composite no. can be resolved into 2 factors which are coprime is 2^n-1 where n is the no. of different prime factors.

Illustration: a) In how many ways the number 10,800 can be resolved into

i) product of 2 factors

ii) Total no of divisors.

$N = 10800 = 2^4 3^3 5^2$

$\therefore$ i) $1/2(4+1)(3+1)(2+1) = 30$

ii) $(4+1)(3+1)(2+1) = 60$.

b) In how many ways the number 18900 can be split into 2 factors which are co-prime.

$N = 18900 = 2^2 \cdot 3^3 \cdot 5^2 \cdot 7^1 \Rightarrow n=4$

No of ways $= 2^4-1 = 15$.

D) To find Exponent of Prime P in n!

b) i) Find the exponent of 2 and 3 in 100!

ii) Find how many zeroes are there in 100! ?

Sol: $E_2(100!) = \left[\frac{100}{2}\right] + \left[\frac{100}{2^2}\right] + \left[\frac{100}{2^3}\right] + \left[\frac{100}{2^4}\right] + \left[\frac{100}{2^5}\right] + \left[\frac{100}{2^6}\right]$

$= 50 + 25 + 12 + 6 + 3 + 1 = 27$.

$E_3(100!) = \left[\frac{100}{3}\right] + \left[\frac{100}{3^2}\right] + \left[\frac{100}{3^3}\right] + \left[\frac{100}{3^4}\right]$

$= 33 + 11 + 3 + 1 = 68$

$E_5(100!) = \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] + \left[\frac{100}{5^3}\right] = 20+4 = 24$

No of zeroes $= (2\times5)^{24} = 24$ zeroes.

Scanned with CamScanner

E) To find the sum of divisors of a number

Let $N = a^p b^q c^r$

$$\text{Sum} = \frac{a^{p+1}-1}{a-1} \times \frac{b^{q+1}-1}{b-1} \times \frac{c^{r+1}-1}{c-1}$$

Illustration : c) $21600 = 6^3 10^2 = 2^3 . 3^3 . 2^2 . 5^2 = 2^5 . 3^3 . 5^2$

No of divisors $= (5+1)(3+1)(2+1) = 72$

Sum of divisors $= \frac{2^6-1}{2-1} \times \frac{3^4-1}{3-1} \times \frac{5^3-1}{5-1} = 78120$.

d) If m is odd show $m(m^2-1)$ is divisible by 24.

Sol : We have $m(m^2-1) = m(m-1)(m+1)$

$\because$ m is odd, $(m-1)$ & $(m+1)$ are two consecutive even nos. Hence one of them is divisible by 2 & other by 4.

Again $(m-1), m, (m+1)$ are 3 consecutive nos., hence one of them is divisible by 3. Thus given expression is divisible by product of 2, 3 and 4 i.e 24.

F) Euler's ϕ function (Euler's totient function)

If n is a no. >1, then the no. of positive integers less than n and prime to it is denoted by $\phi(n)$. Thus the nos less 12 and prime to it are 1, 5, 7, 11.

$\phi(12) = 4$.

$\phi(10) = 4$ (as 1, 3, 7, 9 are coprime to 10).

$\phi(7) = 6$ (as 1, 2, 3, 4, 5, 6 are coprime to 7).

$\phi(mn) = \phi(m) \cdot \phi(n)$; $\phi(11.13) = \phi(11)\phi(13)$
$= (11-1)(13-1) = 120$.

Formula for $\phi(N)$: $N = p_1^a \times p_2^b \times p_3^c$ where p_1, p_2, p_3 are primes, then $\phi(N) = N(1-1/p_1)(1-1/p_2)(1-1/p_3)$

$\phi(1024) = 2^{10}(1-1/2) = 2^9 = 512$

$\phi(1025) = \phi(5^2 \times 41) = \phi(5^2)\phi(41) = 5^2(1-1/5) \times 41(1-1/41)$
$= 1025 \times 4/5 \times \frac{40}{41} = 800$

$\phi(1026) = \phi(2 \cdot 3^3 \cdot 19) = 1026(1-1/2)(1-1/3)(1-1/19) = ~~1~~ 324$

Bezout's identity: If gcd (a,b) is d, then $d = ar + bs$ for some integers r and s. If a and b are prime, then $d=1$ so $ar+bs = 1$

Consider 365 & 1876. Both are coprime so $365x + 1876y = 1$.

Finding x, y:

$1876 = 365 \cdot 5 + 51$	Now: $1 = 3 - 2 \cdot 1$
$365 = 51 \times 7 + 8$	$2 = 8 - 3 \cdot 2$
$51 = 8 \times 6 + 3$	$3 = 51 - 8 \cdot 6$
$8 = 3 \times 2 + 2$	$8 = 365 - 51 \cdot 7$
$3 = 2 \times 1 + 1$	$51 = 1876 - 5 \cdot 365$

$$\therefore 1 = 3 - (8 - 3 \cdot 2) \cdot 1 = 3 \cdot 3 - 8 = 3(51 - 8 \cdot 6) - 8$$
$$= 3 \cdot 51 - 8 \cdot 19 = 3 \cdot 51 - (365 - 51 \cdot 7) \times 19$$
$$= 51 \times 136 - 365 \cdot 19 = (1876 - 5 \cdot 365) \times 136 - 365 \cdot 19$$
$$= 1876 \times 136 - 365 \times 699 = 1876(136) + (-699)365$$

$\therefore x = -699, \; y = 136.$

Congruence

1. Congruence: Let m be any positive whole number, which we shall call modulus. If a, b are any numbers, positive or negative such that $(a-b)$ is divisible by m, then a, and b are congruent with respect to modulus m, and each is said to be a residue of the other.

This is expressed by writing $a \equiv b \pmod m$

or $a - b \equiv 0 \pmod m$, where the symbol $\equiv$ is an abbreviation for "is congruent with".

Thus $a \equiv b \bmod m \Leftrightarrow m \mid (a-b) \Leftrightarrow a - b = \alpha m \Leftrightarrow a = b + \alpha m$

$\Leftrightarrow$ a and b have same remainder upon division by m.

Ex: $5 \equiv 1 \bmod 2$, $13 \equiv 1 \bmod 12$, $11 \equiv 11 \bmod 12$, $11 \equiv (-1) \bmod 12$

Congruences can be added, substracted and multiplied

Suppose $a \equiv b \bmod m$ and $c \equiv d \bmod m$

Then $a \pm c \equiv b \pm d \bmod m$

and $ac \equiv bd \bmod m$.

This have several consequences: $a \equiv b \bmod m \Rightarrow a^k \equiv b^k \bmod m$

and $a \equiv b \bmod m \Rightarrow f(a) \equiv f(b) \bmod m$

where $f(x) = a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0$. $a_i \in \mathbb{Z}$

In general we cannot divide, but we have the following

[cancellation rule]:

A-3

Scanned with CamScanner

$\gcd(c,m)=1$, $ca \equiv cb \mod m \Rightarrow a \equiv b \mod m$

Ex 1) Give a test of divisibility of a number by 7, 11 or 13.

Sol: Any Number N may be written as

$N = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$ where $t = 1000$

and $0 \leq a_r < t$

Let $s = a_0 - a_1 + a_2 - \cdots + (-1)^m a_m$, then we have

$t+1 = 1001 = 7 \times 11 \times 13 \Rightarrow t \equiv -1 \Rightarrow t^r \equiv (-1)^r$

Consequently $N \equiv s \pmod{7, 11 \text{ or } 13}$

Thus if $N = 12345671$

$s = 671 - 345 + 123 = 338$.

Now $13 | 338$, but 338 is not divisible by 7 or 11

$\therefore$ 12345671 is divisible by 13 but not by 7 or 11.

explained clearly in later sections.

---

a) **Addition & Substraction of Congruence**

$$\begin{array}{l|l|l} 9 \equiv 2 \mod 7 & 9 \equiv 2 \mod 7 & \therefore \text{ if } a \equiv b \mod m \\ + \; 8 \equiv 1 \mod 7 & - \; 8 \equiv 1 \mod 7 & \Rightarrow a \pm c \equiv b \pm d \mod m \\ \hline 17 \equiv 3 \mod 7 & 1 \equiv 1 \mod 7 & \end{array}$$

Ex-2) Find Remainder of $\frac{60002 - 601}{6}$ [The remainder of the Difference of 60002 & 601

Sol: $60002 - 601 \equiv (2-1) \mod 6$ when divided by 6 = Difference of remainders of $\frac{60002}{6}$ and $\frac{601}{6}$. $\Rightarrow 60002 - 601 \equiv 1 \mod 6$]

Ex-3) Find last digit of: $2403 + 791 + 688 + 4339$.

Sol: when we have to find last digit divide by 10

$\therefore (2403 + 791 + 688 + 4339) \equiv (3+1+8+9) \mod 10$

$\equiv (21) \mod 10 \equiv 1 \mod 10$.

---

b) **Multiplication in Congruence.**

$$\left.\begin{array}{l} 9 \equiv 2 \mod 7 \\ 8 \equiv 1 \mod 7 \\ \hline 72 \equiv 2 \mod 7 \end{array}\right\} \Rightarrow (9 \times 8) \mod 7 = (7 \cdot 1 + 2) \times (7 \cdot 1 + 1) = (7^2 \cdot 1^2 + 7 \cdot 1 \cdot 3 + 2)$$

$\Rightarrow$ If we divide this expression we get 2 as remainder.

Ex 4) Find Remainder of $\frac{124 \times 134 \times 23}{3}$

Sol: $124 \times 134 \times 23 \equiv (1 \times 2 \times 2) \mod 3 \equiv 4 \mod 3 \equiv 1 \mod 3$

$\therefore$ Remainder of multiplication of 3 nos is same as multiplication of individual remainders.

Scanned with CamScanner

Ex 5) There are 44 boxes of chocolates with 113 chocolates in each box. If you sell the chocolates by dozens, how many will be left over?.

Sol: $44 \times 113 \equiv (8 \times 5) \bmod 12 \equiv 40 \bmod 12 \equiv 4 \bmod 12$

$\therefore$ 4 chocolates will be left over.

c) Exponentiation in Congruence

$$9 \equiv 2 \bmod 7$$
$$\times\, 9 \equiv \times\, 2 \bmod 7$$
$$9^2 \equiv 2^2 \bmod 7$$

Thus if $a \equiv b \bmod 7$

$\Rightarrow a^K = b^K \bmod 7$.

Ex 6) Find last digit of 17^{16}.

Sol: Here we have to divide by 10.

$\therefore 17 \equiv 7 \bmod 10 \Rightarrow 17^{16} = 7^{16} \bmod 10$

$\Rightarrow 17^{16} \equiv (7 \times 7)^8 \bmod 10$

$\equiv (49)^8 \bmod 10$

$\equiv (-1)^8 \bmod 10$

$\equiv 1 \bmod 10$

$\therefore$ Remainder is 1.

Ex 7. Find Remainder of $\frac{6^{6^{6^6}}}{7}$

Sol: $6 \equiv (-1) \bmod 7$

$6^6 \equiv (-1)^6 \bmod 7$

$6^6 = 1 \bmod 7$.

$\therefore 6^{6^6} \equiv 1^6 \bmod 7 \equiv 1 \bmod 7$

$6^{6^{6^6}} \equiv (1^6) \bmod 7 = 1 \bmod 7$

$\therefore$ Remainder is 1.

d) Division of Congruences: Never Divide

$10 \equiv 2 \bmod 8$ - divide by 2

$5 \equiv 1 \bmod 8$ (wrong $\times$)

This is because, $(10 = 8 \cdot 1 + 2) \div 2$

$5 = 4 \cdot 1 + 1$ $\rightarrow$ Divisor is changing.

But $24 \equiv 4 \bmod 5$ - Divide by 2

$12 \equiv 2 \bmod 5$

This is because $(24 = 4 \times 5 + 4) \div 2$ Divisor is not changing.

$12 = 2 \times 5 + 2$

$\therefore$ If $a \equiv b \bmod m$

If $\gcd(c, m) = 1$ then only $ac \equiv bc \bmod m$.

Scanned with CamScanner