Laplace Transforms of Derivatives

Laplace Transforms of the Derivative of f(t)

Theorem: If $L\{f(t)\}=\bar{f}(s)$ then $L\{f'(t)\}=sL\{f(t)\}-f(0)=s\bar{f}(s)-f(0)$

Proof: By the definition $L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt=\bar{f}(s)\rightarrow(1)$

Now $L\{f'(t)\}=\int_0^\infty e^{-st}f'(t)dt$

$L\{f'(t)\}=\left[e^{-st}f(t)\right]_0^\infty-\int_0^\infty e^{-st}(-s)f(t)dt$ $\quad\because \int_a^b u(t)v'(t)dt=\left[u(t)v(t)\right]_a^b-\int_a^b u'(t)v(t)dt$

$L\{f'(t)\}=\left[0-f(0)\right]+s\int_0^\infty e^{-st}f(t)dt=sL\{f(t)\}-f(0)$

Laplace Transforms of the nth order of Derivative of f(t)

Theorem: Let $f(t)$ and its derivatives $f'(t),f''(t)....f^n(t)$ are continuous functions for all $t\geq 0$ then

$L\{f^n(t)\}=s^nL\{f(t)\}-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-...-f^{n-1}(0)$

Proof: Let $f(t)$ and its derivatives $f'(t),f''(t)....f^n(t)$ are continuous functions for all $t\geq 0$

We prove this by mathematical induction method

We know that $L\{f'(t)\}=sL\{f(t)\}-f(0)=s\bar{f}(s)-f(0)$ i.e. theorem is true for $n=1$

Now $L\{f''(t)\}=L\{(f'(t))'\}=sL\{f'(t)\}-f'(0)==s(sL\{f(t)\}-f(0))-f'(0)=s^2L\{f(t)\}-sf(0)-f'(0)$

$L\{f''(t)\}=s^2L\{f(t)\}-sf(0)-f'(0)$

Theorem is true for $n=2$

$L\{f'''(t)\}=L\{(f''(t))'\}=sL\{f''(t)\}-f''(0)=s(s^2L\{f(t)\}-sf(0)-f'(0))-f''(0)=s^3L\{f(t)\}-s^2f(0)-sf'(0)-f''(0)$

$L\{f'''(t)\}=s^3L\{f(t)\}-s^2f(0)-sf'(0)-f''(0)$

Theorem is true for $n=3$

By induction the theorem is true for $\forall n\in N$.

i.e. $L\{f^n(t)\}=s^nL\{f(t)\}-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-...-f^{n-1}(0)$

Problem: Using Laplace transforms of derivatives, find the Laplace transforms of (i) $t\text{Cos}at$

(ii) $\frac{1}{\sqrt{\pi t}}$ (iii) Find $L\left\{\frac{\text{Cos}\sqrt{t}}{\sqrt{t}}\right\}$, if $L\{\text{Sin}\sqrt{t}\}=\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}e^{\frac{-1}{4s}}$

Solution: (i) $t\text{Cos}at$

Let $f(t)=t\text{Cos}at \Rightarrow f(0)=0$

$f'(t)=-at\text{Sin}at+\text{Cos}at \Rightarrow f'(0)=1$

$f''(t)=-a(at\text{Cos}at+\text{Sin}at)-a\text{Sin}at=-2a\text{Sin}at-a^2f(t)$

Now $L\{f''(t)\}=L\{-2a\text{Sin}at-a^2f(t)\}$

$=-2a\frac{a}{s^2+a^2}-a^2L\{f(t)\}$

$=\frac{-2a^2}{s^2+a^2}-a^2L\{f(t)\}$

By Laplace transforms of derivatives $L\{f''(t)\}=s^2L\{f(t)\}-sf(0)-f'(0)$

$\frac{-2a^2}{s^2+a^2}-a^2L\{f(t)\}=s^2L\{f(t)\}-s(0)-1$

$\Rightarrow (s^2+a^2)L\{f(t)\}=1-\frac{2a^2}{s^2+a^2}=\frac{s^2-a^2}{s^2+a^2}$

$\Rightarrow L\{f(t)\}=\frac{s^2-a^2}{(s^2+a^2)^2}$

(ii) $\frac{1}{\sqrt{\pi t}}$

Let $f(t)=\sqrt{t} \Rightarrow f(0)=0,\ f'(t)=\frac{1}{2\sqrt{t}}$

By Laplace transforms of derivatives $L\{f'(t)\}=sL\{f(t)\}-f(0)$

$L\left\{\left(\frac{1}{2\sqrt{t}}\right)\right\}=sL\{\sqrt{t}\}-f(0) \Rightarrow \frac{1}{2}L\left\{\left(\frac{1}{\sqrt{t}}\right)\right\}=s\frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}}-0$

$\Rightarrow \frac{1}{2}L\left\{\left(\frac{1}{\sqrt{t}}\right)\right\}=s\frac{\frac{1}{2}\sqrt{\pi}}{s^{\frac{3}{2}}} \Rightarrow L\left\{\left(\frac{1}{\sqrt{\pi t}}\right)\right\}=\frac{1}{s^{\frac{1}{2}}}=\frac{1}{\sqrt{s}}$

(iii) Find $L\left\{\frac{Cos\sqrt{t}}{\sqrt{t}}\right\}$, if $L\left\{Sin\sqrt{t}\right\}=\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}e^{\frac{-1}{4s}}$

Given $L\left\{Sin\sqrt{t}\right\}=\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}e^{\frac{-1}{4s}} \rightarrow (1)$

Let $f(t)=Sin\sqrt{t}$

$\Rightarrow f'(t)=\frac{Cos\sqrt{t}}{2\sqrt{t}}$ and $f(0)=0$

By Laplace transforms of derivatives $L\left\{f'(t)\right\}=sL\left\{f(t)\right\}-f(0)$

$$L\left\{\frac{Cos\sqrt{t}}{2\sqrt{t}}\right\}=sL\left\{Sin\sqrt{t}\right\}-0$$

$$\frac{1}{2}L\left\{\frac{Cos\sqrt{t}}{\sqrt{t}}\right\}=s\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}e^{\frac{-1}{4s}}$$

$$\Rightarrow L\left\{\frac{Cos\sqrt{t}}{\sqrt{t}}\right\}=\frac{\sqrt{\pi}}{s^{\frac{1}{2}}}e^{\frac{-1}{4s}}=\sqrt{\frac{\pi}{s}}e^{\frac{-1}{4s}}$$

$$\Rightarrow L\left\{\frac{Cos\sqrt{t}}{\sqrt{t}}\right\}=\sqrt{\frac{\pi}{s}}e^{\frac{-1}{4s}}$$

Multiplication Theorem

Laplace Transform of t.f(t)

Theorem: If $L\left\{f(t)\right\}=\bar{f}(s)$ then $L\left\{t.f(t)\right\}=\frac{-d}{ds}L\left\{f(t)\right\}=\frac{-d}{ds}\bar{f}(s)$

Proof: By the definition $L\left\{f(t)\right\}=\bar{f}(s)=\int_0^\infty e^{-st}f(t)dt \rightarrow (1)$

Differentiating (1) w.r.t. s, we get

$$\frac{d}{ds}\bar{f}(s)=\frac{d}{ds}\int_0^\infty e^{-st}f(t)dt=\int_0^\infty f(t)\frac{d}{ds}e^{-st}\,dt=\int_0^\infty f(t)(-t)e^{-st}\,dt=-\int_0^\infty e^{-st}(t)f(t)dt$$

$$\frac{d}{ds}\bar{f}(s)=-L\left\{t.f(t)\right\}\Rightarrow L\left\{t.f(t)\right\}=-\frac{d}{ds}\bar{f}(s)$$

$$L\left\{t.f(t)\right\}=\frac{-d}{ds}L\left\{f(t)\right\}=\frac{-d}{ds}\bar{f}(s)$$

Note: In general, $L\left\{t^n.f(t)\right\}=(-1)^n\frac{d^n}{ds^n}\bar{f}(s)$

Problem: Find i) $L\{tSin3tCos2t\}$ ii) $L\{t^2Sin2t\}$ iii) $L\{te^{-t}Cosht\}$

iv) $L\{t^2e^{-2t}Cost\}$ v) $L\{te^{2t}Sin3t\}$ iv) $L\{t^2e^{-t}Cos^2t\}$

Solution: i) Let $f(t)=Sin3tCos2t$

$$f(t)=\frac{1}{2}(2Sin3tCos2t)=\frac{1}{2}(Sin5t+Sint)$$

$$\text{Now } L\{tSin3tCos2t\}=\frac{1}{2}L\{t(Sin5t+Sint)\}$$

$$=\frac{1}{2}[L\{tSin5t\}+L\{tSint\}]$$

$$=\frac{1}{2}\left[\frac{-d}{ds}L\{Sin5t\}-\frac{d}{ds}L\{Sint\}\right] \qquad \because L\{t.f(t)\}=\frac{-d}{ds}L\{f(t)\}=\frac{-d}{ds}\bar{f}(s)$$

$$=-\frac{1}{2}\left[\frac{d}{ds}\left(\frac{5}{s^2+25}\right)+\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right]$$

$$=-\frac{1}{2}\left[\left(\frac{-10s}{(s^2+25)^2}\right)+\left(\frac{-2s}{(s^2+1)^2}\right)\right]$$

$$L\{tSin3tCos2t\}=\frac{1}{2}\left[\frac{10s}{(s^2+25)^2}+\frac{2s}{(s^2+1)^2}\right]$$

ii) Let $f(t)=Sin2t$

$$\text{Now } L\{t^2Sin2t\}=(-1)^2\frac{d^2}{ds^2}L\{(Sin2t)\} \qquad \because L\{t^n.f(t)\}=(-1)^n\frac{d^n}{ds^n}L\{f(t)\}$$

$$=\frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right)=2\frac{d}{ds}\left(\frac{-2s}{(s^2+4)^2}\right)$$

$$=2\left(\frac{(s^2+4)^2(-2)-(-2s)2(s^2+4)2s}{(s^2+4)^4}\right)$$

$$=4(s^2+4)\left(\frac{-s^2-4+4s^2}{(s^2+4)^4}\right)=4\left(\frac{3s^2-4}{(s^2+4)^3}\right)$$

$$L\{t^2Sin2t\}=4\left(\frac{3s^2-4}{(s^2+4)^3}\right)$$

iv) $L\left\{e^{-2t}t^2 Cost\right\}$

Let $f(t)=t^2 Cost$

$$L\left\{f(t)\right\}=L\left\{t^2 Cost\right\}=(-1)^2\frac{d^2}{ds^2}L\left\{(Cost)\right\}$$

$$=\frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right)$$

$$=\frac{d}{ds}\left(\frac{(s^2+1)(1)-s(2s)}{(s^2+1)^2}\right)$$

$$=\frac{d}{ds}\left(\frac{s^2+1-2s^2}{(s^2+1)^2}\right)$$

$$=\frac{d}{ds}\left(\frac{1-s^2}{(s^2+1)^2}\right)$$

$$=\left(\frac{(s^2+1)^2(-2s)-(1-s^2)2(s^2+1)2s}{(s^2+1)^2}\right)$$

$$=(s^2+1)\left(\frac{(s^2+1)(-2s)-(1-s^2)4s}{(s^2+1)^4}\right)$$

$$=\left(\frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3}\right)$$

$$=\left(\frac{2s^3-6s}{(s^2+1)^3}\right)$$

$$L\left\{f(t)\right\}=\overline{f}(s)=\left(\frac{2s^3-6s}{(s^2+1)^3}\right)$$

$$L\left\{e^{-2t}t^2 Cost\right\}=L\left\{e^{-2t}f(t)\right\}=\overline{f}(s+2)=\left(\frac{2(s+2)^3-6(s+2)}{\left((s+2)^2+1\right)^3}\right)$$

By First Shifting Theorem $L\left\{e^{-at}f(t)\right\}=\overline{f}(s+a)$

Division Theorem

Laplace Transform of f(t) by t

Theorem: If $L\left\{f(t)\right\}=\overline{f}(s)$ then $L\left\{\frac{f(t)}{t}\right\}=\int_s^{\infty}L\left\{f(t)\right\}ds=\int_s^{\infty}\overline{f}(s)ds$

Proof: By the definition $L\left\{f(t)\right\}=\overline{f}(s)=\int_0^{\infty}e^{-st}f(t)dt\rightarrow(1)$

Integrating (1) w.r.t. s from s to ∞ we get

$$\int_s^{\infty}\overline{f}(s)ds=\int_s^{\infty}\left(\int_0^{\infty}e^{-st}f(t)dt\right)ds=\int_0^{\infty}\left(\int_s^{\infty}e^{-st}ds\right)f(t)dt$$

$$=\int_0^{\infty}\left(\frac{e^{-st}}{-t}\right)_s^{\infty}f(t)dt=\int_0^{\infty}\left(0+\frac{e^{-st}}{t}\right)f(t)dt$$

$$\int_s^{\infty}\bar{f}(s)ds=\int_0^{\infty}e^{-st}\left(\frac{f(t)}{t}\right)dt \Rightarrow L\left\{\left(\frac{f(t)}{t}\right)\right\}=\int_s^{\infty}\bar{f}(s)ds$$

$$L\left\{\frac{f(t)}{t}\right\}=\int_s^{\infty}L\{f(t)\}ds=\int_s^{\infty}\bar{f}(s)ds$$

Problem: Find (i) $L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}$ (ii) $L\left\{\frac{Sin3tCost}{t}\right\}$ (iii) $L\left\{\frac{1-Cosat}{t}\right\}$ (iv) $L\left\{\frac{1-e^{t}}{t}\right\}$ (vi) $L\left\{\frac{Cos4tSin2t}{t}\right\}$

Solution: (i) Let $f(t)=e^{-at}-e^{-bt}$

$$\Rightarrow L\{f(t)\}=L\{e^{-at}-e^{-bt}\}=\frac{1}{s-a}-\frac{1}{s+a}=\bar{f}(s)$$

By division theorem $L\left\{\frac{f(t)}{t}\right\}=\int_s^{\infty}L\{f(t)\}ds=\int_s^{\infty}\bar{f}(s)ds$

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}=\int_s^{\infty}\left(\frac{1}{s-a}-\frac{1}{s+b}\right)ds$$

$$=\left[\log(s-a)-\log(s+b)\right]_s^{\infty}=\left[\log\left(\frac{s-a}{s+b}\right)\right]_s^{\infty}$$

$$=\left[\log\left(\frac{1-\frac{a}{s}}{1+\frac{b}{s}}\right)\right]_s^{\infty}$$

$$=\left[\log\left(\frac{1-0}{1+0}\right)-\log\left(\frac{1-\frac{a}{s}}{1+\frac{b}{s}}\right)\right]=-\log\left(\frac{s-a}{s+a}\right)$$

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}=\log\left(\frac{s+a}{s-a}\right)$$

(ii) Let $f(t)=Sin3tCost=\frac{1}{2}(2Sin3tCost)=\frac{1}{2}(Sin4t+Sin2t)$

$$\Rightarrow L\{f(t)\}=\frac{1}{2}L\{(Sin4t+Sin2t)\}=\frac{1}{2}\left(\frac{4}{s^2+4^2}+\frac{2}{s^2+2^2}\right)=\left(\frac{2}{s^2+4^2}+\frac{1}{s^2+2^2}\right)=\bar{f}(s)$$

By division theorem $L\left\{\frac{f(t)}{t}\right\}=\int_s^{\infty}L\{f(t)\}ds=\int_s^{\infty}\bar{f}(s)ds$

$$L\left\{\frac{Sin3tCost}{t}\right\}=\int_s^{\infty}\left(\frac{2}{s^2+4^2}+\frac{1}{s^2+2^2}\right)ds=\left[2\frac{1}{4}Tan^{-1}\left(\tfrac{s}{4}\right)+\frac{1}{2}Tan^{-1}\left(\tfrac{s}{2}\right)\right]_s^{\infty}$$

$$=\frac{1}{2}\left[\left(Tan^{-1}(\infty)+Tan^{-1}(\infty)\right)-\left(2Tan^{-1}\left(\tfrac{s}{4}\right)+Tan^{-1}\left(\tfrac{s}{2}\right)\right)\right]=\frac{1}{2}\left[\left(\pi-Tan^{-1}\left(\tfrac{s}{4}\right)+\pi-Tan^{-1}\left(\tfrac{s}{2}\right)\right)\right]=\frac{1}{2}\left[\left(Cot^{-1}\left(\tfrac{s}{4}\right)+Cot^{-1}\left(\tfrac{s}{2}\right)\right)\right]$$

(iii) $L\left\{\frac{1-Cosat}{t}\right\}$

Let $f(t)=1-Cosat$

$$\Rightarrow L\{f(t)\}=L\{(1-Cosat)\}=\left(\frac{1}{s}-\frac{s}{s^2+a^2}\right)=\bar{f}(s)$$

By division theorem $L\left\{\frac{f(t)}{t}\right\}=\int_s^{\infty}L\{f(t)\}ds=\int_s^{\infty}\bar{f}(s)ds$

$$L\left\{\frac{1-Cosat}{t}\right\}=\int_s^{\infty}\left(\frac{1}{s}-\frac{1}{2}\frac{2s}{s^2+a^2}\right)ds=\left[\log s-\frac{1}{2}\log\left(s^2+a^2\right)\right]_s^{\infty}=\left[\frac{1}{2}.2\log s-\frac{1}{2}\log\left(s^2+a^2\right)\right]_s^{\infty}$$

$$=\frac{1}{2}\left[\log s^2-\log\left(s^2+a^2\right)\right]_s^{\infty}=\frac{1}{2}\left[\log\left(\frac{s^2}{s^2+a^2}\right)\right]_s^{\infty}=\frac{1}{2}\left[\log\left(\frac{1}{1+\frac{a^2}{s^2}}\right)\right]_s^{\infty}=\frac{1}{2}\left[\log(1)-\log\left(\frac{1}{1+\frac{a^2}{s^2}}\right)\right]$$

$$=\frac{1}{2}\left[\log(1)-\log\left(\frac{1}{1+\frac{a^2}{s^2}}\right)\right]=\frac{1}{2}\left[0-\log\left(\frac{s^2}{s^2+a^2}\right)\right]=\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2}\right)$$

$$L\left\{\frac{1-Cosat}{t}\right\}=\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2}\right)$$