RELATIONS

1.Relations & Functions

- Introduction
- Types of Relations
- Types of Functions
- Composition of Functions
- Invertible Function
- Binary Operations

Relation -

- A connection between or among things.
- E.g. Father & son is a relation , Brother & sister is a relation, student & teacher.

Note:

-Every relation has a '*pattern or property*'.

-Every relation involves '*minimum 2 identities*'.

Relation in mathematical world -

Examples –

- Number ‘p’ is greater than ‘q’.
- Line ‘m’ is perpendicular to line ‘n’
- Set A is a subset of set B.
- Relation between sides of a right triangle.

Cartesian product of set -

- Suppose we have 3 shirts(green, blue, red) & 2 pants(black , blue).
- We can pair them as {(green , black) , (green , blue) , (blue , black) ,

  (Blue , blue) , (red , black) , (red , blue)} – 6 pairs

- Given two non-empty sets P and Q.
- The Cartesian product P × Q is the set of all *ordered pairs* whose first component is a member of 'P' & second component is the member of 'Q'.

Remarks

(i) Two ordered pairs are equal, if and only if the corresponding first elements are equal and the second elements are also equal.

(ii) A x B $\neq$ B x A

(iii) $A \times A \times A = \{(a, b, c) : a, b, c \in A\}$. Here (a, b, c) is called an *ordered triplet*.

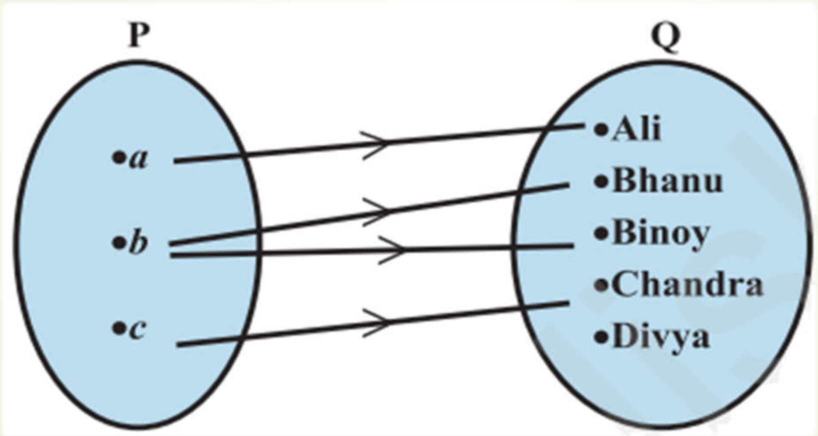
(iv) n(A x B) = n(A)*n(B) ; n(A x B x C) = n(A)*n(B)*n(C)

(v) If A x {infinite set} = {infinite set} where A is *non-empty* set.

E.g's

Relation – Some new terms

- Consider the two sets P = {*a*, *b*, *c*} and Q = {Ali, Bhanu, Binoy, Chandra, Divya}.
- The Cartesian product of P and Q has 15 ordered pairs.
- We now define a relation R,
- R= { (x,y): x is the first letter of the name y, $x \in P$, $y \in Q$}.
- R = {(*a*, Ali), (*b*, Bhanu), (*b*, Binoy), (*c*, Chandra)}
- A visual representation of this relation R is called an *arrow diagram*



- Image -

  The second element in the ordered pair is called the *image* of the first element.

  E.g. Ali, bhanu, binoy, Chandra; **not** divya

- Domain –

  The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the *domain* of the relation R.

  E.g. a , b , c

- Range-

  The set of all second elements in a relation R from a set A to a set B is called the *range* of the relation R.

  E.g. Ali, bhanu, binoy, Chandra; **not** divya

- Co-domain –

  The whole set B is called the *codomain* of the relation R.

  E.g. Ali, bhanu, binoy, Chandra **and** divya

Note - range $\subseteq$ codomain

Types of Relations -

- Empty relation
- Universal relation
- Trivial relation

- Reflexive relation
- Symmetric relation
- Transitive relation
- Equivalence relation

Empty Relation -

- A relation R in a set A is called *empty relation*, if no element of A is related to any element of A, i.e., R $= \varphi \subset A \times A$.

E.g. Let A be the set of all students of a boys school. Show that the relation R in A given by R = {(*a*, *b*) : *a* is sister of *b*} is the empty relation .

Universal Relation -

- A relation R in a set A is called *universal relation*, if each element of A is related to every element of A, i.e., $R = A \times A$.

E.g. R = {(*a*, *b*) : the difference between heights of *a* and *b* is less than 3 meters} is the universal relation.

Note - Both the *empty relation* and the *universal relation* are some times called *'Trivial relations'*

Reflexive relation -

- A relation R in a set A is called R*eflexive*,

    if (*a*, *a*) ∈ R, for every *a* ∈ A.

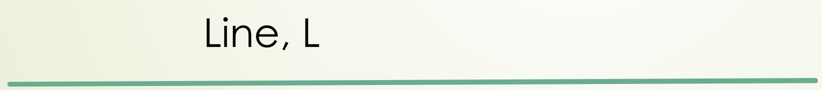
E.g. Let L be the set of all lines in XY plane and R be the relation in L.

We define a Relation, R = Two Lines are parallel.

→ If the relation, R, of lines being parallel hold true with itself i.e. (a, a) ∈ R , then it is Reflexive relation.

Here , (L, L) ∈ R, for every line, L ∈ A.

Hence it is a *'Reflexive relation'.*



E.g. Relation, R = Triangles are congruent .

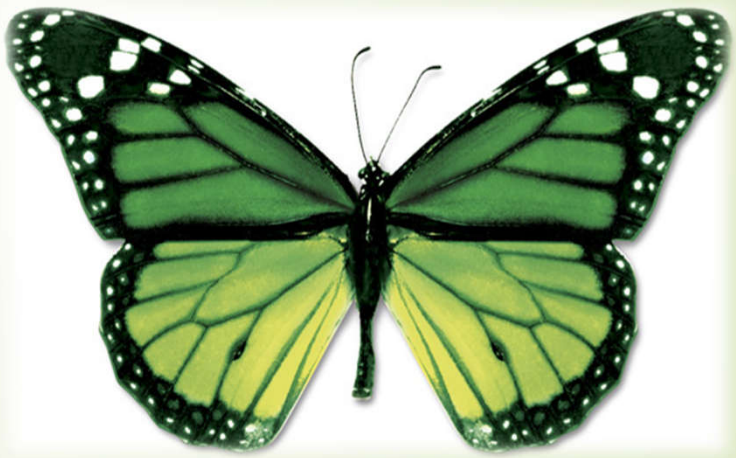
Symmetric relation -

- A relation R in a set A is called S*ymmetric*,

  if (a_1, a_2) ∈ R implies that (a_2, a_1) ∈ R, for all a_1, a_2 ∈ A.

E.g. Relation, R = Line 'l' is perpendicular to line 'm'.

→ If the relation, R, of lines being perpendicular holds true between (l, m) & also holds true between (m, l) then it is *'Symmetric relation'.*



E.g. Relation, R = Triangles are congruent .

Transitive relation -

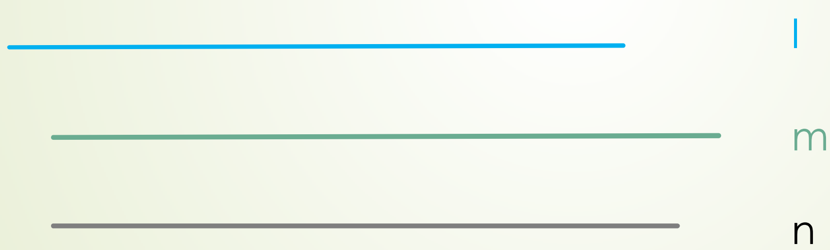
- A relation R in a set A is called T*ransitive*,

  if $(a_1, a_2) \in$ R and $(a_2, a_3) \in$ R implies that $(a_1, a_3) \in$ R, for all $a_1, a_2, a_3 \in$ R.

E.g. We define a Relation, R = Two lines are parallel.

Given: Line 'l' is parallel to line 'm', and 'm' is parallel to 'n'.

→ If the relation ,R, of lines being parallel holds true between (l, m), holds true between (m, n) & compulsorily between (l, n) , then it is *'Transitive relation'*.

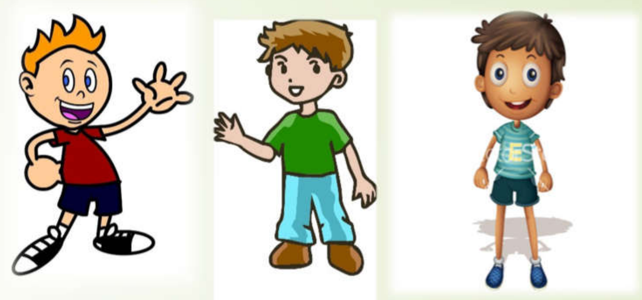


E.g. Relation, R = Triangles are congruent .

Equivalence relation

- A relation R in a set A is said to be an '*equivalence' relation* if R is reflexive, symmetric and transitive.

E.g. Relation, R = Heights of boys are equal.



E.g. Relation, R = Triangles are congruent.

*** Examples

Function -

- Function is a set of action or activity.
- Visualize a function as a rule, which produces new elements out of some given elements.

E.g. Let police is function.

E.g. Teacher

Function in mathematical world-

- $F(x) = X^2$

- $F(x) = 2X$

Function (more) -

- A special type of relation called *function.*
- Visualize a function as a rule, which produces new elements out of some given elements.
- There are many terms such as 'map' or 'mapping' used to denote a function.
- A relation *f* from a set A to a set B is said to be a *function* if every element of set A has 1 and only 1 image in set B.
- If *f* is a function from A to B and $(x,y) \in f$, then $f(x) = y$, where 'y' is called the *image* of *a* under *f* and 'x' is called the *pre-image* of 'y' under function , *f.*

E.g. Test whether relation is a function or not?

- (i) R = {(2,1),(3,1), (4,2)},
- (ii) R = {(2,2),(2,4),(3,3), (4,4)}
- (iii) R = {(1,2),(2,3),(3,4), (4,5), (5,6), (6,7)}

E.g. Let **N** be the set of natural numbers and the relation R be defined on N such that R = {(x, y) : $y = 2x$, $(x, y) \in$ **N**}. What is the -

1.Domain,

2.Codomain and

3.Range of R?

Is this relation a function?

Real function & Real-valued function-

Real valued function -

- A function which has either R or one of its subsets as its range is called a *real valued function*.

Real function -

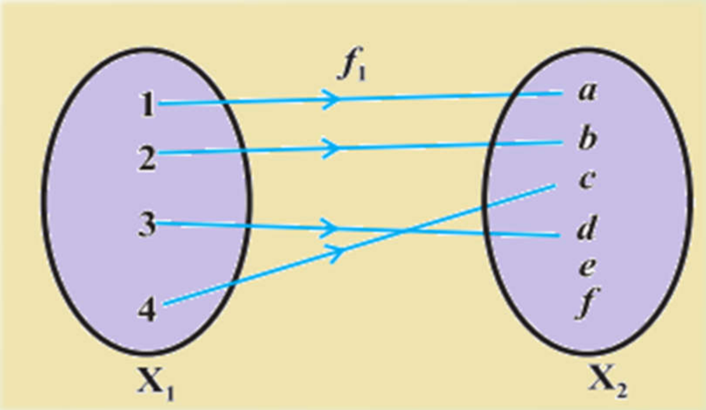
- If its domain is also either R or a subset of R, it is called a *real function*

Functions -

- One-One (or Injective) & Many- one
- Onto (or Surjective)
- One- one and Onto (or bijective)

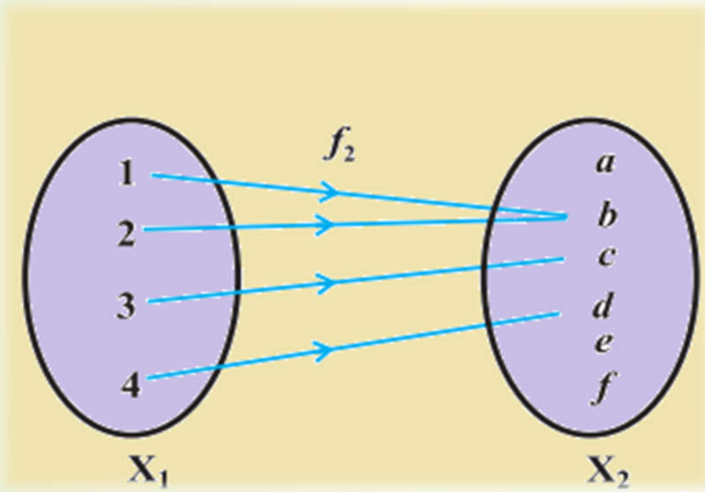
One-One (or Injective) :

- A function $f: X \rightarrow Y$ is defined to be '*one-one*' (or *injective*),
- if the images of distinct elements of X under function '*f*' are distinct.
- For every $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.



Many-One function (Not injective)

- Function which is not One-One.
- Two or more elements have same image.

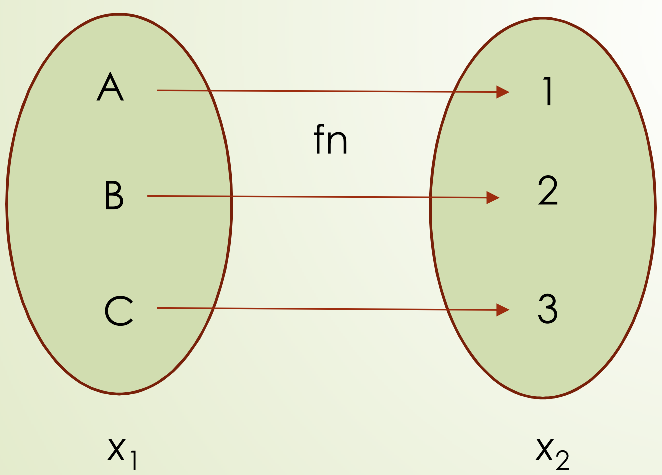


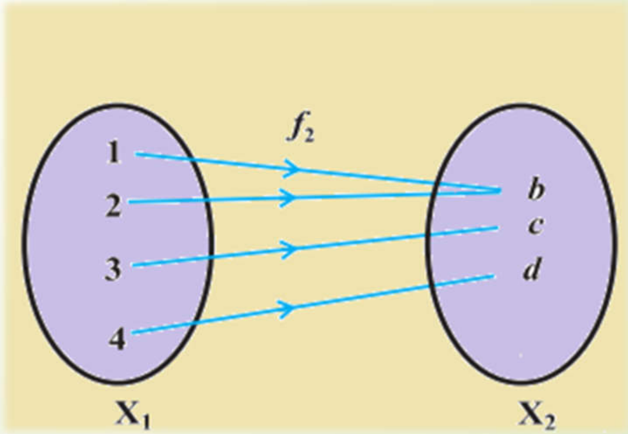
Onto (or Surjective) function -

- A function f: X → Y is said to be *onto* (or *surjective*),

  if <u>every</u> element of Y is the image of some element of X under '*f*'.

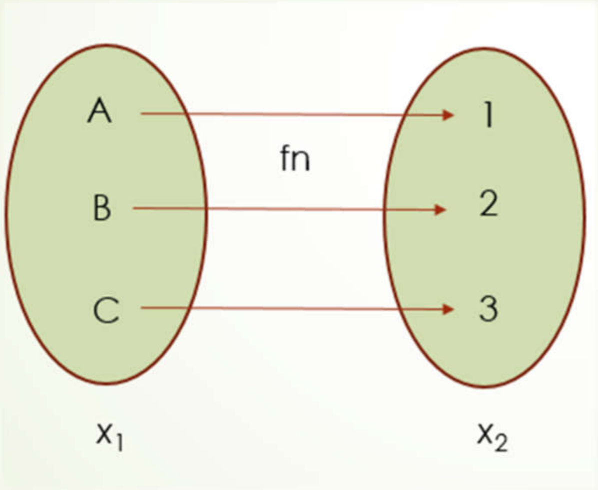
- For every $y \in$ Y, there exists an element x in X such that $f(x) = y$.
- *No orphan image left.*





Bijective(One-one & Onto) function -

- A function $f: X \to Y$ is said to be *bijective* , if 'f' is both one-one and onto.



E.g. Let A be the set of all 50 students of Class X in a school. Let $f: A \to \mathbf{N}$ be function defined by $f(x)$ = roll number of the student x. Show that f is one-one but not onto.

***E.g. 8 & 9 & exer.*

Composition of functions: Setting the stage -

- Consider the set A of all students, who appeared in Class X of a Board Examination.

12565

Marks

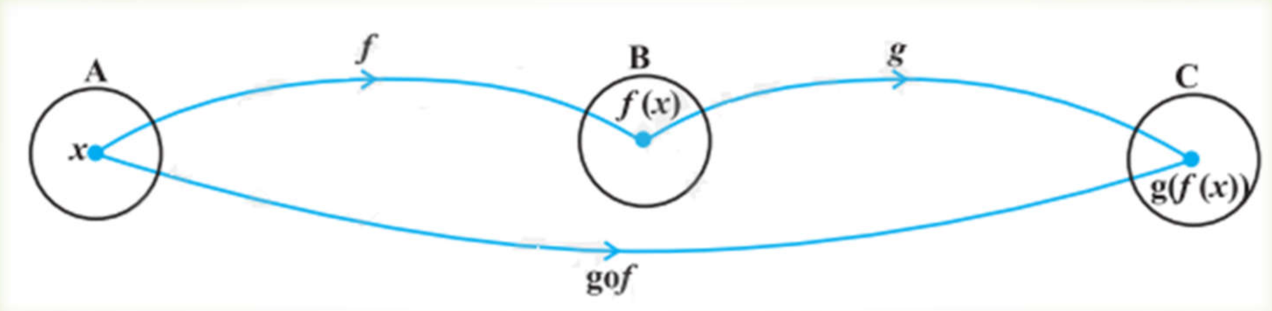
Rohit

Roll no.

Composition of functions: Definition -

- Let f: A → B and g: B → C be two functions.

The composition of 'f' & 'g', denoted by 'gof', is defined as the function 'gof': A → C given by $gof(x) = g(f(x))$, ∀ $x \in$ A.



E.g. Let f: {2, 3, 4, 5} → {3, 4, 5, 9} and g: {3, 4, 5, 9} → {7, 11, 15} be functions defined as $f(2) = 3$, $f(3) = 4$, $f(4) = f(5) = 5$ and $g(3) = g(4) = 7$ and $g(5) = g(9) = 11$. Find gof.

Examples

Invertible function -

A function f: X → Y is defined to be *invertible*, if there exists a function g : Y → X such that $gof = I_x$ and $fog = I_y$. The function 'g' is called the *inverse of 'f'* and is denoted by f^{-1}.

Thus, if 'f' is invertible, then 'f' must be one-one and onto and conversely, if 'f' is one-one and onto, then 'f' must be invertible.

<u>E.g.</u> Let f: N → Y be a function defined as $f(x) = 4x + 3$, where, Y = {y ∈ N : $y = 4x + 3$ for some x ∈ N }. Show that f is invertible. Find the inverse.

- Theorem 1:

If $f: X \to Y$, $g: Y \to Z$ and $h: Z \to S$ are functions, then $h o (g o f) = (h o g) o f$.

E.g. Consider $f: N \to N$, $g: N \to N$ and $h: N \to R$ defined as $f(x) = 2x$, $g(y) = 3y + 4$ and $h(z) = \sin z$, $\forall$ x, y and z in N. Show that $h o (g o f) = (h o g) o f$.

- Theorem 2:

Let $f: X \to Y$ and $g: Y \to Z$ be two invertible functions. Then $g o f$ is also invertible with $(g o f)^{-1} = f^{-1} o g^{-1}$.

E.g. Consider $f: \{1, 2, 3\} \to \{a, b, c\}$ and $g: \{a, b, c\} \to$ {apple, ball, cat}
defined as $f(1) = a$, $f(2) = b$, $f(3) = c$, $g(a)$ = apple, $g(b)$ = ball and $g(c)$ = cat.
Show that f, g and $g o f$ are invertible. Find out f–1, g–1 and ($g o f$)–1 and show that $(g o f)^{-1} = f^{-1} o g^{-1}$

Exercise

Binary operations -

- Addition, multiplication, subtraction & division are examples of binary operation, as 'binary' means two.
- Generally binary operation is nothing but association of any pair of elements a, b from X to another element of X.
- A binary operation $*$ on a set A is a function $* : A \times A \rightarrow A$. We denote "$*(a, b)$" by $a * b$.

E.g. Show that addition, subtraction and multiplication are binary operations on **R**, but division is not a binary operation on **R**. Further, show that division is a binary operation on the set **R**$*$ of nonzero real numbers.

E.g. Show that subtraction and division are not binary operations on **N**.

Properties & operations -

1. Commutative property -

- A binary operation $*$ on the set X is called *commutative*, if $a * b = b * a$, for every $a, b \in X$.

E.g. Show that $* : R \times R \to R$ defined by $a * b = a + 2b$ is not commutative.

2. Associative property –

- A binary operation $* : A \times A \to A$ is said to be *associative* if $(a * b) * c = a * (b * c)$, $\forall\ a, b, c, \in A$.

E.g. Show that $* : R \times R \to R$ given by $a * b \to a + 2b$ is not associative.

3. Identity binary operation -

- Given a binary operation $*: A \times A \to A$, an element $e \in A$, if it exists, is called '*identity*' for the operation $*$, if $a * e = a = e * a$, $\forall\ a \in A$.

<u>E.g.</u> Show that zero is the identity for addition on R and 1 is the identity for multiplication on R. But there is no identity element for the operations

$$'-' : R \times R \to R \text{ and } '\div' : R \times R \to R .$$

4. Invertible binary operation -

- Given a binary operation $*: A \times A \to A$ with the identity element 'e' in A, an element $a \in A$ is said to be '*invertible*' with respect to the operation '$*$', if there exists an element 'b' in A such that $a * b = e = b * a$ and 'b' is called the *inverse of 'a'* and is denoted by a^{-1}.

<u>E.g.</u> Show that '$-a$' is the inverse of a for the addition operation '+' on R and $1/a$ is the inverse of $a \neq 0$ for the multiplication operation '×' on R.

Exercise

☺