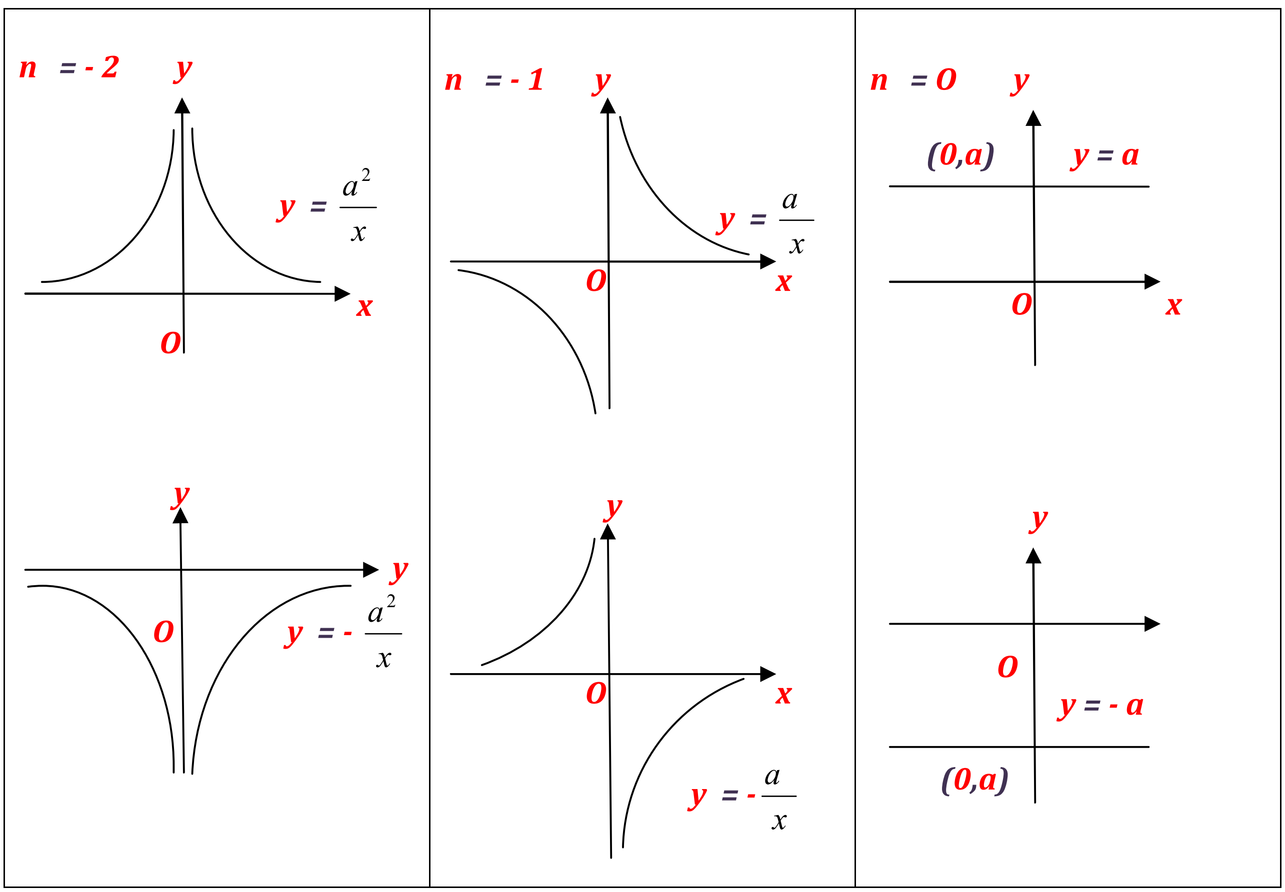
GRAPHS AND GRAPHICAL SOLUTION OF EQUATIONS

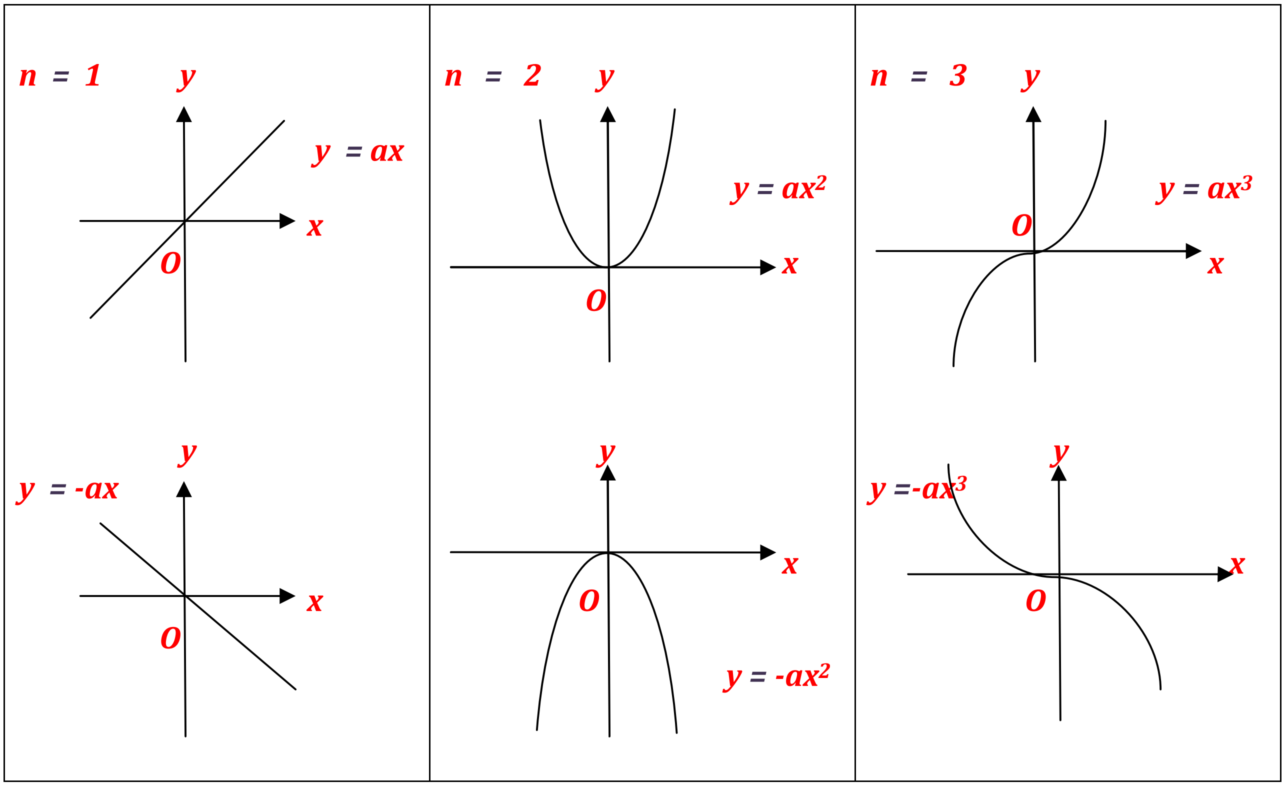
1.1 DIFFERENT TYPES AND SHAPES OF GRAPHS:

A graph can be drawn to represent are equation connecting two variables.

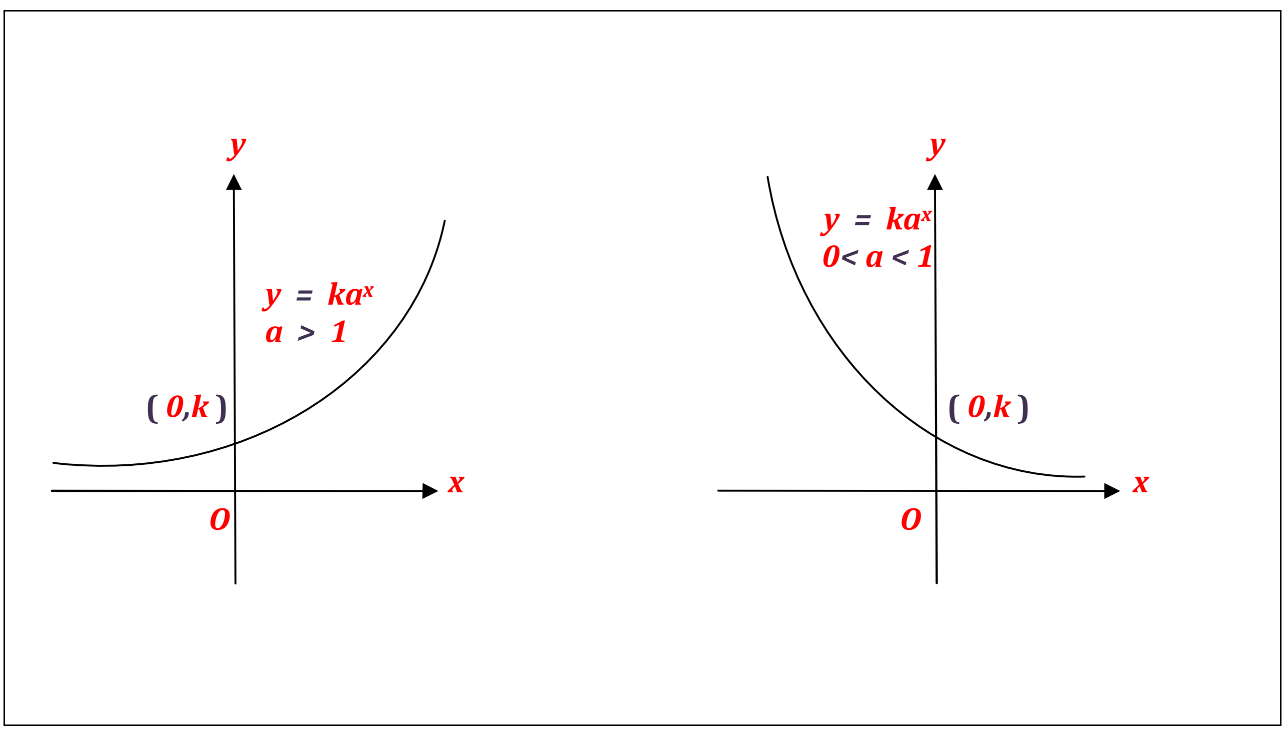
There are different types of equations which give different types and shapes of graphs .

Below is a summary of graphs in the form of $y = ax^n$, -1, 0, 1, 2 and 3 .





Graphs of the form $y = ka^x$, where $a > 0$ and $a \neq 1$ and k is a constant are shown below .



1.2 LINEAR GRAPHS:

A linear graphs is of the form $y = mx + c$, where m and c are constants .

All linear graphs are straight lines, e.g. $y = x$, $2x - 1$, $y = -3x + 2$ and $2x + 3y = 8$.

To draw a linear graph, you will need only three points.

EXAMPLE 1:

Draw the graphs of $y = 2x - 1$. From your graphs, find

a) The value of y when $x = 1.3$.

b) The value of x when $y = -5.8$,

c) The value of k given that $(k, 3.4)$ lies on the graphs of $y = 2x - 1$

SOLUTION:

To draw the graphs of $y = 2x - 1$:

STEP ⇨ 1 : Locate **3** points .

$y = 2x - 1$

x	-2	0	2
y	-5	-1	3

STEP ⇨ 2 : Plot these points and draw a straight line joining them to obtain the graph of $y = 2x - 1$.

STEP ⇨ 3 : Label your graph with the equation of the line .

GRAPH DRAWING

From the graph .

a) When $x = 1.3$, $y = 1.6$

b) When $y = 5.8$, $x = 2.4$

c) When $y = 3.4$, $k = 2.2$ ←

> The corresponding value of x when $y = 3.4$ is the value of k .

TIP FOR STUDENTS:

Only 2 points are needed to draw a straight line graphs. The third point is to check that you have not made a mistake in the substitution .

EXAMPLE2:

a) Draw the graphs of each of the following equation on the same axes .

i) $x + 3 = 0$

ii) $y = \frac{1}{2}x$

iii) $5y + 3x = 11$

b) Find the area of the triangle bounded by these three lines .

SOLUTION:

a) i) $x + 3 = 0$

$x = 3$ ←

> The graphs of $x = -3$ is a vertical line parallel to the y – axis and crosses the x- axis at $x = -3$.

ii) $y = \frac{1}{2}x$ ←

The graphs of $y = mx$ is a straight line passing through the origin .

x	-2	0	2
y	-1	0	1

iii) $5y + 3x = 11$

$5y = 3x + 11$

$y = -\frac{3}{5}x + 2\frac{1}{5}$

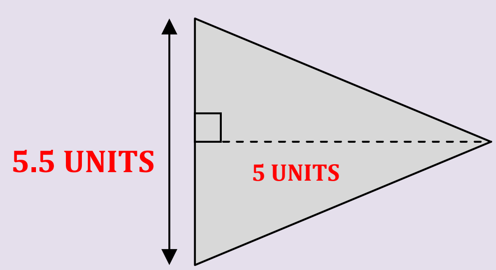
x	-3	0	3
y	4	2.2	0.4

b) Area of triangle bounded by these three lines

$= \frac{1}{2}$ X Base X Height

$= \frac{1}{2}$ X 5.5 X 5 ←

$= 13.75$ units^2



1.3 SOLVING SIMULTANEOUS LINEAR EQUATIONS BY USING THE GRAPHICAL METHOD:

Given a pair of simultaneous linear equations, we can solve them algebraically as learned or we can also use the graphical method to solve them

EXAMPLE:

Solve the following simultaneous equations graphically .

$x + 2y = 5$

$y = x + 1$

SOLUTION :

We have learned to solve the pair of simultaneous equations either by using the substitution or the elimination method . Here, we will use a graphical method to solve them .

METHOD : Draw the two lines on the same axes . The solution to the simultaneous equations is the x – and y - coordinates of the point of intersection of both lines .

$x + 2y = 5$

$2y = -x + 5$

$y = -\frac{1}{2}x + 2\frac{1}{2}$

x	-3	1	3

$y = x + 1$

x	-3	0	3
y	-2	1	4

y	4	2	1

GRAPH DRAWING

From the graphs, the coordinates of the point of intersection are (1, 2) .

∴ The solution to the simultaneous equation is $x = 1$ and $y = 2$.

TIP FOR STUDENTS:

Choose a scale as large as possible to increase the accuracy of the solution.

1.4 GRAPHS AND GRAPHICAL SOLUTION OF QUADRATIC EQUATIONS:

1. **Quadratic graph are graphs are whose equations are of the form $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$. e.g. $y = 2x^2$, $y = x^2 + x - 5$ and $y = -\frac{1}{2}x^2 + 8$.**

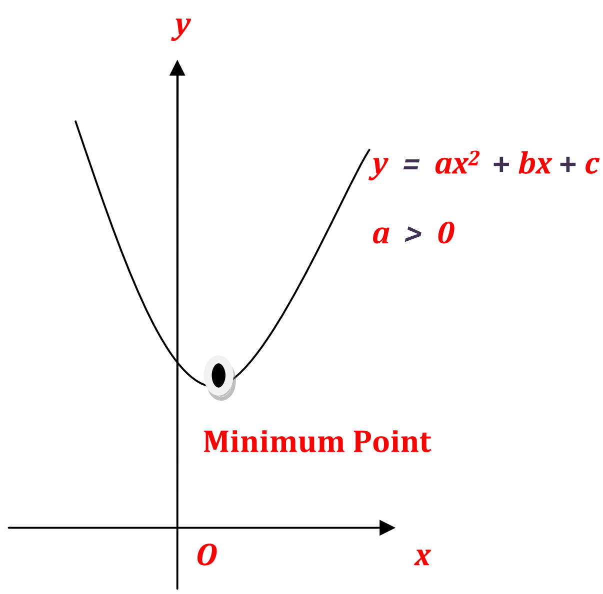
TIPS FOR STUDENTS:

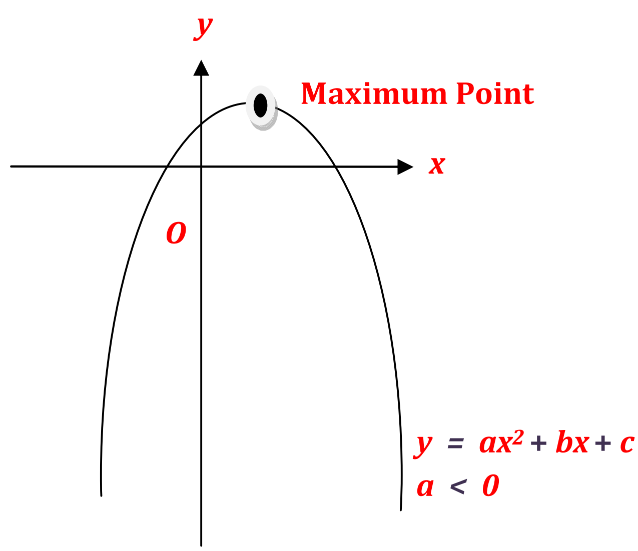
If a = 0, then equation becomes $y = bx + c$. This is a linear graph !

2. The graph of a quadratic equation is a smooth **U–shaped** curve called a **parabola** .

3. The curved of a quadratic graph is symmetrical about the **line of symmetry** .

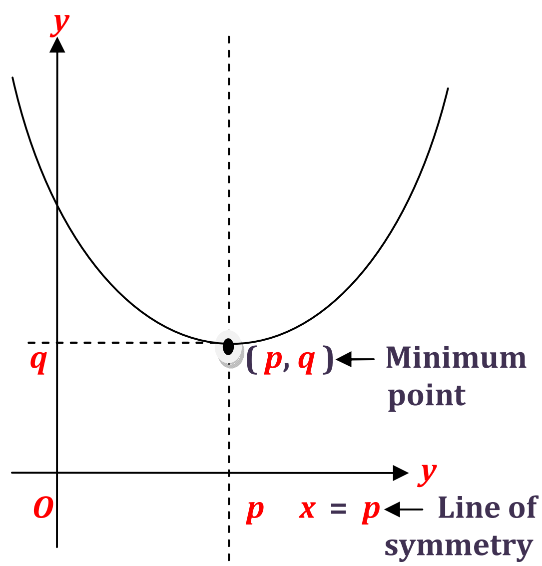
4. The curved of a quadratic graph has either a **maximum** or **minimum point** . The line of symmetry passes through the maximum/minimum point of the parabola .



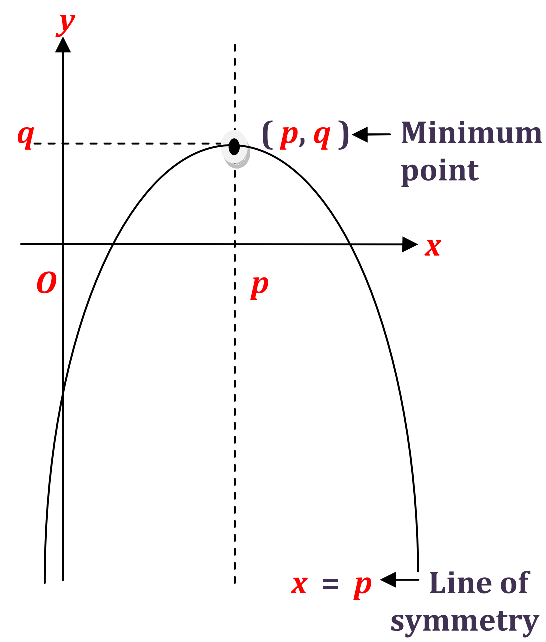


5. To sketch the graph of a quadratic equation . rewrite the equation in the form $y = \pm (x - p)^2 + q$ or $y = (x - a) (x - b)$ where a, b, p and q are constants .

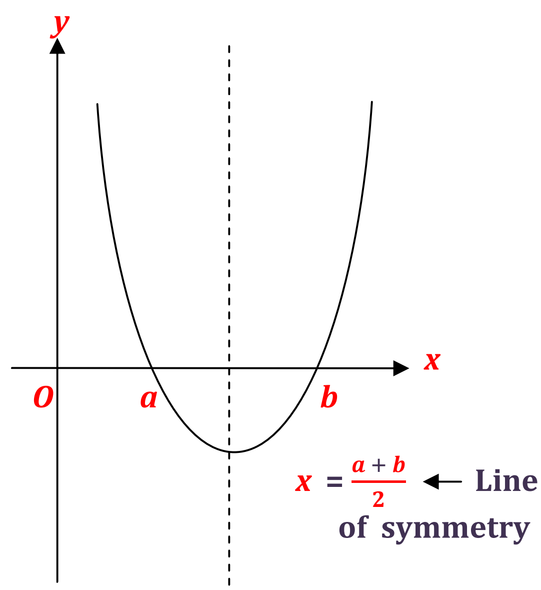
$y = (x - p)^2 + q$



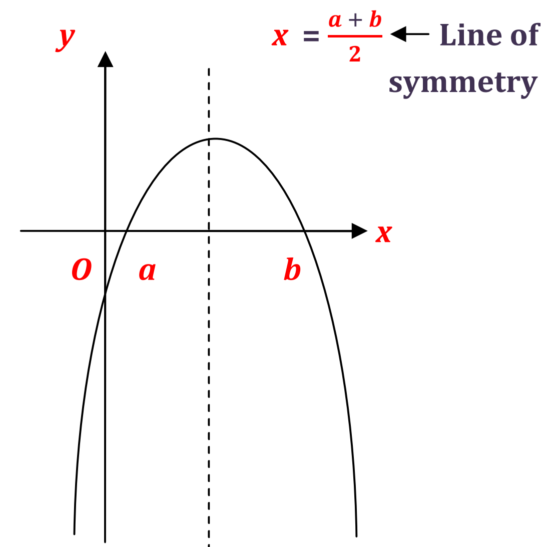
$y = -(x - p)^2 + q$



$y = (x - a)(x - b)$



$y = -(x - a)(x - b)$



EXAMPLE 1:

Sketch the following graphs .

a) $y = (x - 3)^2 + 2$

b) $y = - x^2 + 4x - 3$

SOLUTION :

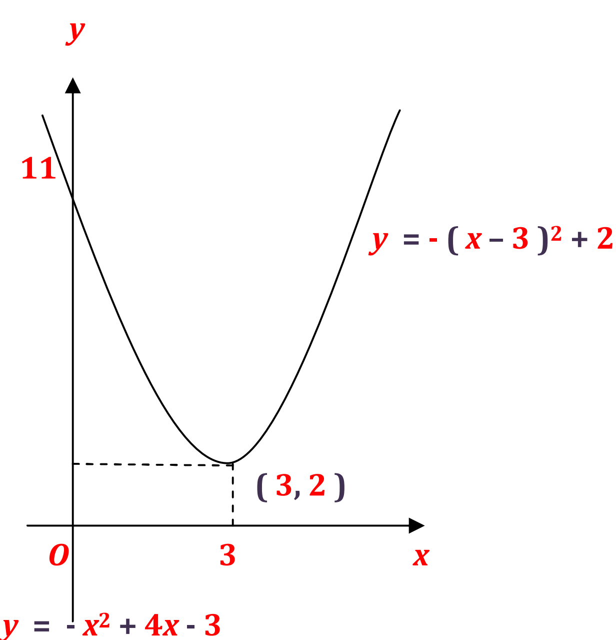
a) $y = - (x - 3)^2 + 2$

Minimum point $= (3, 2)$

At the y **– axis,** $x = 0$,

$y = (0 - 3)^2 + 2$

$= 11$



b) $y = - x^2 + 4x - 3$

$= (x^2 - 4x + 3)$

$= - (x - 1) (x - 3)$

At the x – axis, $y = 0$

$-(x-1)(x-3) = 0$

$\therefore\ x - 1 = 0$ or $x - 3 = 0$

$x = 1$ or $x = 3$

At the Minimum point

$x = \frac{1+3}{2}$

$= 2$

When the $x = 2$,

$y = -2^2 + 4(2) - 3$

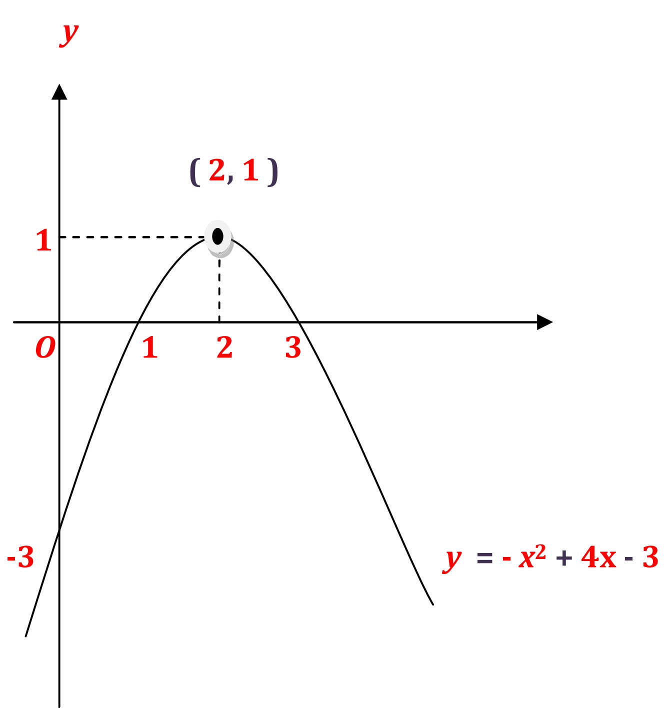
$= 1$

$\therefore$ Minimum point $= (2, 1)$

At the y – axis, $x = 0$

$y = -0^2 + 4(0) - 3$

$= -3$



EXAMPLE2:

a) Sketch the graph of $y = 2x^2 + 3x - 8$.

b) Draw and label the line of symmetry.

SOLUTION

a) $y = 2x^2 + 3x - 8$. ← If the equation cannot be factorised, rewrite it in the form $a = -(x - p)^2 + q$ by completing the square.

$$= 2\left(x^2 + \frac{3}{2}x - 4\right)$$

$$= 2\left[x^2 + \frac{3}{2}x + \left(\frac{3}{4}\right)^2 - 4 - \left(\frac{3}{4}\right)^2\right]$$

$$= 2\left[\left(x + \frac{3}{4}\right)^2 - 4\frac{9}{16}\right]$$

$$= 2\left(x + \frac{3}{2}\right)^2 - 9\frac{1}{8}$$

Minimum point $= \left(-\frac{3}{2}, -9\frac{1}{8}\right)$

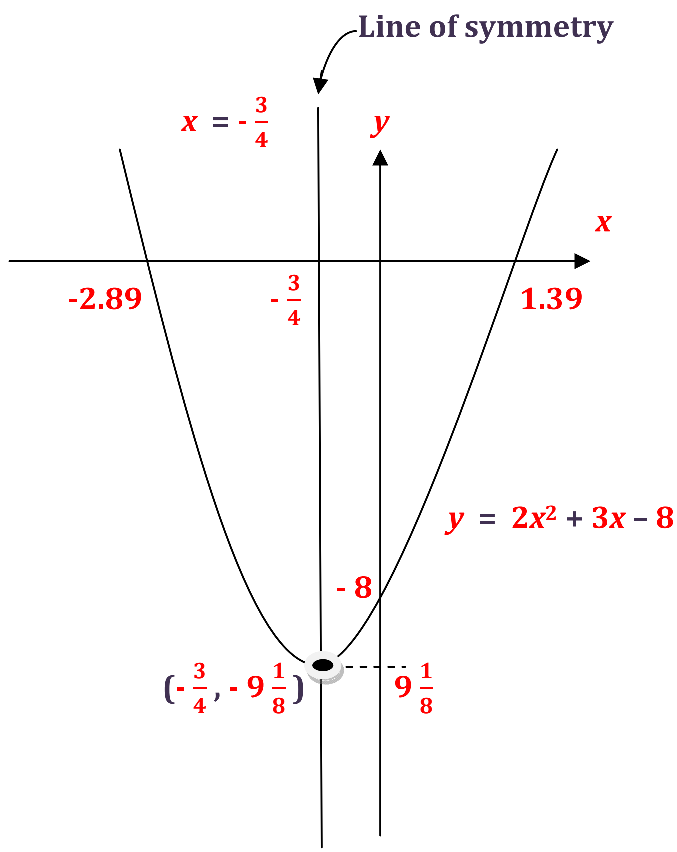
At the y – axis, $x = 0$

$y = 2(0)^2 + 3(0) - 8$

$= -8$

At the x – axis, $y = 0$

$$2\left(x + \frac{3}{4}\right)^2 - 9\frac{1}{8} = 0$$



$$2\left(x + \frac{3}{4}\right)^2 = 9\frac{1}{8}$$

$$2\left(x + \frac{3}{4}\right)^2 = 4\frac{9}{16}$$ ← Divide both sides by 2

$$x + \frac{3}{4} = \pm\sqrt{4\frac{9}{16}}$$

$$\therefore\ x = \sqrt{4\frac{9}{16}} - \frac{3}{4} \quad \text{or} \quad x = -\sqrt{4\frac{9}{16}} - \frac{3}{4}$$

$x \approx 1.39$ or $x \approx -2.89$ (correct to 3 sig . fig .)

b) Equation of line of symmetry : $x = -\frac{3}{4}$ ← The line of symmetry passes through the minimum point of the curve .

EXAMPLE3:

a) Draw the graph of $y = x^2 + x - 2$ for $-4 \leq x \leq 3$.

b) From your graph, find

i) The value of y when $x = 1.5$,

ii) The value of x when $y = 5$,.

iii) The least value of y,

iv) The equation of the line of symmetry.

SOLUTION

a) Construct a table for the corresponding values of x and y.

x	- 4	- 3	- 2	- 1	0	1	2	3
y	10	4	0	- 2	- 2	0	4	10

Plot the points and join them to form a smooth curve.

GRAPH DRAWING

b) From your graph .

i) When $x = 1.5$, $y = 1.7$.

ii) When $y = 5$, $x = 3.2$ or $x \approx 2.2$,

iii) The least value of y is - 2.3,

iv) The equation of the line of symmetry $x = -0.5$.

TIPS FOR STUDENTS:

1. **Do not draw a straight line between (-1, -2) and (0, -2) . The curve dips to its lowest value half – way between the points . To find the minimum point, find y when $x = -0.5$.**

$y = (-0.5)^2 + (-0.5) - 2 = -2.25$

2. The line of symmetry is the vertical line that passes through the vertex (**maximum or minimum point**) of the parabola .

EXAMPLE4 :

The variables x and y are connected by the equation $y = x^2 - x - 5$. Some corresponding values of x and y are given below .

x	- 4	- 3	- 2	- 1	0	1	2	3	4	5
y	15	7	a	- 3	- 5	b	- 3	11	7	15

a) Calculate the values of a and b

b) Taking 2 cm to represent 1 unit on the horizontal x – axis and 1 cm to represent 1 unit on the vertical y – axis , draw the graph of $y = x^3 - 5$ for $-4 \leq x \geq 5$.

c) Use your graph to solve the equations .

i) $x^2 - x - 5 = 0$

ii) $x^2 - x = 8$

iii) $x^2 - x - 5 = 2x + 1$

d) By drawing a tangent, find the gradient of the curve $y = x^2 - x - 5$ at the point (2, -3).

e) Draw and label the line of symmetry of the graph.

SOLUTION:

$y = x^2 - x - 5$

$a = (-2)^2 - (-2) - 5 = 1$

$b = (1)^2 - (1) - 5 = -5$

GRAPH DRAWING

c) i) $x^2 - x - 5 = 0$

$\underbrace{x^2 - x - 5}_{y} = 0$

From the graph ,

When $y = 0$, $x \approx -1.8$ or $x \approx 2.8$.

$\therefore$ The solution are $x \approx -1.8$ or $x \approx 2.8$.

ii) $x^2 - x = 8$ ← Rearrange the equation so that the left – hand side is equal to $x^2 - x - 5$

$x^2 - x - 5 = 8 - 5$ ← Subtract 5 from both sides

$x^2 - x - 5 = 3$

$\underbrace{x^2 - x - 5}_{y} = 3$

Draw the line $y = 3$

From the graph ,

When $y = 3$, $x \approx -2.4$ or $x \approx 3.4$.

$\therefore$ The solution are $x \approx -2.4$ or $x \approx 3.4$.

iii) $\underbrace{x^2 - x - 5}_{y} = 2x + 1$

$y = 2x + 1$

Draw the graph of the straight line $y = 2x + 1$

$y = 2x + 1$

x	-3	0	3
Y	-5	1	7

From the graph the solution are $x \approx -1.4$ or $x \approx 4.4$.

TIPS FOR STUDENTS:

The solution of the equation $x^2 - x - 5 = 2x + 1$ are the x – coordinates of the points of intersection of the curve $y = x^2 - x - 5$ and the straight line $y = 2x + 1$.

d) (1, -6), (3, 0)

Gradient of the curve at the point (2, -3)

$= \frac{0 - (-6)}{3 - 1}$

$= \frac{0 + 6}{2}$

$= \frac{6}{2}$

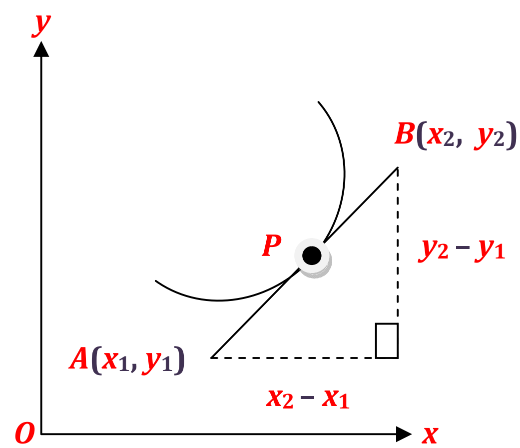
$= 3$

TIPS FOR STUDENTS:

GRADIENT OF THE CURVE

The gradient of the curve at P is the tangent to the curve at P . The tangent to the curve at P is the straight line (AB), that touches the curve at P .

Gradient of the curve at P = $\frac{y_2 - y_1}{x_2 - x_1}$



e) Equation of the line of symmetry is $x = 0.5$.

EXAMPLE 5:

The variables x and y are connected by the equation $y = \frac{1}{2}(8x - x^2)$. Some

corresponding values of x and y are given below .

x	0	1	2	3	4	5	6	7	8
y	0	3.5	6	7.5	a	7.5	6	3.5	0

a) Calculate the values of a .

b) Taking 2 cm to represent 1 unit on each axis, draw the graph of $y = \frac{1}{2}(8x - x^2)$

for $0 \leq x \geq 8$.

c) By drawing a tangent, find the gradient of the curve at the point (6, 6) .

d) On the same axes, draw the graph of the straight line $y = 8 - \frac{2}{3}x$ and use your

graphs to solve the equation $= \frac{1}{2}(8x - x^2) = 8 - \frac{2}{3}x$.

e) From the graphs, find the range of values of x for which $\frac{1}{2}(8x - x^2) \geq 8 - \frac{2}{3}x$.

SOLUTION:

a) $y = \frac{1}{2}(8x - x^2)$

$a = \frac{1}{2}[8(4) - (4)^2] = 8$

b)

GRAPH DRAWING

c) (5, 2, 7, 6), (6.75, 4.5)

Gradient of the curve at the point (6, 6)

$= \frac{4.5 - 7.6}{6.7\ 5 - 5.2}$

$= \frac{-3.1}{1.55}$

$= -2$

d) $y = 8 - \frac{2}{3}x$

x	0	3	6
y	8	6	4

$\frac{1}{2}(8x - x^2) = 8 - \frac{2}{3}x$.

$\therefore$ From the graph the solution are $x \approx 2.25$ or $x \approx 7.1$.

TIPS FOR STUDENTS:

The solutions of the equation $\frac{1}{2}(8x - x^2) = 8 - \frac{2}{3}x$ are the x – coordinates of the points of intersection of the curve $y = \frac{1}{2}(8x - x^2)$ and the straight line $y = 8 - \frac{2}{3}x$.

e) From the graphs, the values of x for which $\frac{1}{2}(8x - x^2) \geq 8 - \frac{2}{3}x$ occurs where the curve is above the line .

The range of values of x is $2.25 \leq x \leq 7.1$.

1.5 GRAPHS AND GRAPHICAL SOLUTION OF OTHER EQUATIONS:

CUBIC GRAPHS

Cubic graphs can be written in the form $y = ax^2 + bx^2 + cx + d$, where a, b, c and d are constants and $a \neq 0$.

e.g. $y = x^3 + x^2 - x + 2$, $y = 2x^3 - 7$ and $y = -3x^3 + 2x - 1$.

EXAMPLE

a) Draw the graph of $y = x^3 - 12x + 6$ for $-4 \leq x \leq 4$.

b) From your graph, find

i) The value of y when $x = 2.5$,

ii) The values of x when $y = 10$.

c) By drawing a tangent, find the gradient of the curve at the point there $x = -1$.

d) On the same axes, draw the graph of the straight line $y = 1\frac{1}{2}x + 9$ and use your graphs to solve the equation $x^3 - 12x + 6 = 1\frac{1}{2}x + 9$

SOLUTION:

a) Construct the table for the corresponding values of x and y .

$y = x^3 - 12x + 6$

x	-4	-3.5	-3	-2	-1	0	1	2	3	3.5	4
y	-10	5.1	1.5	22	17	6	-5	-10	-3	6.9	22

TIPS FOR STUDENTS

Plot extra points at x = -3.5 and x = 3.5 to help you draw the curve because of the large difference between the y values .

GRAPH WORKS

b) **From the graphs,**

i) **When $x = 2.5$, $y \approx -8.25$,**

ii) **When $y = 10$, $x \approx -3.3$, $x \approx -0.35$ or $x \approx 3.6$.**

c) **(-2, 26), (-0.5, 12.5)**

Gradient of the curve at the point where x = -1

$$= \frac{12.5 - 26}{-0.5 - (-2)}$$

$$= \frac{-3.1}{1.55}$$

$$= -9$$

d) $y = 1\frac{1}{2}x + 9$

x	-4	0	4
y	3	9	15

$x^3 - 12x + 6 = 1\frac{1}{2}x + 9$

From the graphs, the solutions are $x \approx -3.55$, $x \approx -0.2$ or $x \approx 3.75$.

TIPS FOR STUDENTS:

The x – coordinates of the points of intersection of the straight line $y = 1\frac{1}{2}x + 9$ and the curve $y = x^3 - 12x + 6$ give the solution to the equation $x^3 - 12x + 6 = 1\frac{1}{2}x + 9$.

RECIPROCAL GRAPHS

A reciprocal graph is written in the form $y = \frac{a}{x + b}$. where a and b are constants and $(x + b) \neq 0$.

e.g. $y = \frac{1}{x}$, $y = \frac{2}{x}$, $y = \frac{8}{x + 1}$ and $y = \frac{5}{2x - 3}$

EXAMPLE : 1

Draw the graph of $y = \frac{8}{x}$ for $-4 \leq x \leq 4$.

SOLUTION

a) $y = \frac{8}{x}$

x	-4	-3	-2	-1	-0.5	-0.3	0.3	0.5	1	2	3	4
y	-2	-2.7	-4	-8	-16	-26.7	26.7	16	8	4	2.7	2

Add more points here to help you draw more accurately

GRAPH DRAWING

TIPS FOR STUDENTS

- ✓ **The graph occurs in two parts .**
- ✓ **Both parts do not touch the *x* – or *y* – axes .**

  (When *x* = 0, the function is undefined .)
- ✓ **Each of the two curves has an asymtote at the *x* – axis and an asymtote at the *y* – axis . Asymtote refers to the situation where the**

curve gets nearer and nearer to the straight line (usually an axis) but never quite touches the line .

✓ The equation of the line of symmetry is $y = x$ and $y = -x$.

EXAMPLE : 2

The variables x and y are connected by the equation $y = \frac{12}{x+1}$. Some corresponding values of x and y are given in the table below .

x	0	1	2	3	4	5	6	7
y	12	6	4	3	a	2	1.7	1.5

a) Calculate the value of a .

b) Using a scale of 2 cm to represent 1 unit on the horizontal x – axis and 1 cm to represent 1 unit on the vertical y - axis, draw the graph of $y = \frac{12}{x+1}$ for $0 \leq x \leq 7$.

c) On the same axes, draw the graph of $y = \frac{3}{4}x + 8$.

From the graph, find

i) The solution of the equation $\frac{12}{x+1} = \frac{3}{4}x + 8$.

ii) The value of x for which the gradient of the curve is equal to $-\frac{3}{4}$.

SOLUTION:

a) $y = \frac{12}{x+1}$

$a = \frac{12}{x+1} = \frac{12}{45} = 2.4$

b)

GRAPH DRAWING

c) $y = \frac{3}{4}x + 8$

x	0	4	6
y	8	5	3.5

i) $\frac{12}{x+1} = \frac{3}{4}x + 8$

From the graphs, the solution is $x \approx$ 0.6 .

ii)

TIPS FOR STUDENTS

To find the value of x for which the gradient of the curve is equal to $-\frac{3}{4}$, draw a tangent to the curve which is parallel to the line $-\frac{3}{4}x + 8$.

From the graph, the gradient of the curve is equal to $-\frac{3}{4}$ at $x = 3$.

GRAPHS OF THE FORM $y = \frac{a}{x^2}$

The graph of $\frac{a}{x^2}$, where a is a constant and $x \neq 0$, lies above the x – axis.

EXAMPLE :

Draw the graph of $y = \frac{1}{x^2}$ for $-3 \leq x \leq -3$.

SOLUTION

$y = \frac{1}{x^2}$

x	-3	-2	-1	-0.5	-0.4	-0.3	0.3	0.4	0.5	1	2	3
y	0.11	0.25	1	4	6.25	11.11	11.11	6.25	4	1	0.25	0.11

GRAPH DRAWING

TIPS FOR STUDENTS

- ✓ **This graph also appears in two parts .**
- ✓ **The curve always lie above the x – axis as the value of y are always positive .**
- ✓ **The graph is undefined at $x = 0$.**
- ✓ **The equation of the line of symmetry is the y = axis or $x = 0$.**

GRAPHS OF SUMS OF POWER FUNCTIONS:

Where we add two power functions together, we get a new function .

e.g. The sum of $y = 2x^2$ and $y = \frac{1}{x}$ gives $y = 2x^2 + \frac{1}{x}$.

EXAMPLE :

The variables x and y are connected by the equation $y = 2x + \frac{6}{x} - 5$. Some corresponding values of x and y are given in the table below .

x	1	1.5	2	2.5	3	3.5	4
y	3	a	2	2.4	3	3.7	b

a) Calculate the value of a and b .

b) Using a scale of 4 cm to represent 1 unit on each axis draw the graph of $y = 2x + \frac{6}{x} - 5$ for $1 \leq x \leq 4$.

c) On the same axes, draw the graph of $y = \frac{1}{4}x + 2$. Find the values of x in the interval $1 \leq x \leq 4$ for which

i) $2x + \frac{6}{x} - 5 = \frac{1}{4}x + 2$.

ii) $2x + \frac{6}{x} - 5 > \frac{1}{4}x + 2$.

d) By drawing a tangent, find the gradient of the curve at the point where $x = 2$.

SOLUTION

a) $y = 2x + \frac{6}{x} - 5$

$a = 2(1.5) + \frac{6}{1.5} - 5 = 2$

$b = 2(4) + \frac{6}{4} - 5 = 4.5$

b)

GRAPH DRAWING:

c) i) $y = \frac{1}{4}x + 2$.

x	1	2	4
y	2.25	2.5	3

$2x + \frac{6}{x} - 5 = \frac{1}{4}x + 2$.

The solutions are the x – coordinates of the points of intersection of the curve $y = 2x + \frac{6}{x} - 5$ and the straight line $y = \frac{1}{4}x + 2$.

From the graphs, the solution are $x \approx 1.25$ or $x \approx 2.75$.

ii) From the graph, the ranges of values for which $2x + \frac{6}{x} - 5 > \frac{1}{4}x + 2$ are $1 < x < 1$ or $2.75 < x < 4$.

d) (1.6, 1.8), (3, 2.5)

Gradient of the curve at the point where x = 2

$= \frac{2.5 - 21.8}{3 - 1.6}$

$= \frac{0.7}{1.4}$

$= 0.5$

GRAPHS OF EXPONENTIAL FUNCTIONS :

The graphs of exponential functions are of the from $y = ka^x$, where $a > 0$ and $a \neq 1$ and k is a constant .

e.g. $y = 2^x$, $y = 8(3^x)$

EXAMPLE

a) Draw the graph of $y = 2^x$ for $-1 \leq x \leq 3$.

b) **From your graph, find**

i) **The value of** y **when** $x = 2.3$

ii) **The value of** x **when** $y = 6.7$

c) **Use your graph to solve the equations**

i) $2^x = 3$,

ii) $2^x = 4 - x$.

SOLUTION

a) $y = 2^x$

x	-1	-0.5	0	0.5	1	1.5	2	2.5	3
y	0.5	0.71	1	1.41	2	2.83	4	5.66	8

GRAPH DRAWING

b) **From your graph**

i) **When** $x = 2.3$, $y = 4.9$

ii) **When** $y = 6.7$, $x = 2.75$.

c) $2^x = 3$

Draw the line $y = 3$.

From the graphs, the solution is $x \approx 1.6$.

The x – coordinate of the point of intersection of the line $y = 3$ and the curve gives the solution .

ii) $2^x = 4 - x$.

x	0	1	2
y	4	3	2

From the graphs, the solution is $x \approx 1.4$

The x – coordinate of the point of intersection of the line $y = 4 - x$ and the curve gives the solution .